Robust Control

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Introduction

• A control system is robust if it remains stable and achieves certain performance criteria in the presence of possible uncertainties.

• The robust design is to find a controller, for a given system, such that the closed-loop system is robust.
Uncertainty Modeling

- Must maintain controllability, observability and stability when there is **uncertainty**:
  - Uncertainty in model of plant
  - Disturbances in the plant system
  - Sensor noise

*Chandrasekharan, P. C., Robust Control of Linear Dynamical Systems, Academic Press, 1996.
*Image: http://www.ece.cmu.edu/~koopman/des_s99/control_theory/#chandra96
Uncertainty Modelling

- Stochastic control assigns probability distributions to each uncertainty to develop new control law.
- In contrast, robust control methods seek to bound the uncertainty rather than express it in the form of a distribution (i.e. **model reduction**).
- Modeling is difficult
  - Imperfect plant data
  - Time varying plants
  - Higher order dynamics
  - Non-linearity
  - Complexity
  - Skill
Example: Two Cart System

- Here the controller is of the following form

\[ C(s) = \frac{100(s + 1)^3}{(0.001s + 1)^3} \]

- Uncertainty:
  - \( k = 1.0 \pm 0.2 \) (20%)
  - \( m_1 = 1.0 \pm 0.2 \) (20%)
  - \( m_2 = 1.0 \pm 0.2 \) (20%)
Two Cart System Diagram

- Cart Models: \( G_1(s) = \frac{1}{m_1s^2}, \ G_2(s) = \frac{1}{m_2s^2} \)

- \( F(s) = \begin{bmatrix} 0 & G_2 \\ G_1 & -G_1 - G_2 \end{bmatrix} \) (applied force)
s = zpk('s'); % The Laplace 's' variable
C = 100*ss((s+1)/(.001*s+1))^3; % triple lead compensator

% set uncertainty parameters
k = ureal('k',1,'percent',20);
m1 = ureal('m1',1,'percent',20);
m2 = ureal('m2',1,'percent',20);
% cart system transfer functions
G1 = 1/s^2/m1;
G2 = 1/s^2/m2;
% Spring-less inner block F(s)
F = [0;G1]*[1 -1]+[1;-1]*[0,G2];
% add spring in feedback
P = lft(F,k);
% u1 = C*(r-y1);
% Uncertain open-loop model is
L = P*C;
Closed Loop Stability

- \[ P_{nom} = \frac{1}{(s^2 + 5.995 \times 10^{-16})(s^2 + 2)} \] (open loop TF)
- Using MATLAB, we close the loop, connecting \( P \), our plant and \( C \), our controller:

\[
\begin{align*}
\text{close the loop} & : T = \text{feedback}(L, 1); \\
\text{compute open loop gain} & \rightarrow \text{not stable} \\
P_{nom} & = \text{zpk}(P.\text{nominal}); \\
\text{compute closed loop gain} & \rightarrow \text{stable} \\
T_{nom} & = \text{zpk}(T.\text{nominal}); \\
\text{maxrealpole} & = \max(\text{real}(\text{pole}(T_{nom}))); \\
\gg \text{maxrealpole} & = -0.8232
\end{align*}
\]
Closed Loop Stability

• We can see that the system is stable in the nominal case.
• MATLAB routine robuststab() can show us how robust this stability is to uncertainty

\[
\begin{align*}
\text{StabilityMargin, Udestab, REPORT} & = \text{robuststab}(T); \\
\text{REPORT} & \\
\text{REPORT} & = \\
\end{align*}
\]

Uncertain system is robustly stable to modeled uncertainty.
-- It can tolerate up to 315% of the modeled uncertainty.
-- A destabilizing combination of 500% of the modeled uncertainty was found.
-- This combination causes an instability at 1.4 rad/seconds.
-- Sensitivity with respect to the uncertain elements are:
  'k' is 20%. Increasing 'k' by 25% leads to a 5% decrease in the margin.
  'm1' is 61%. Increasing 'm1' by 25% leads to a 15% decrease in the margin.
  'm2' is 60%. Increasing 'm2' by 25% leads to a 15% decrease in the margin.
Worst Case Responses

% Compute worst-case gain over specified uncertainty range
[PeakGain,Uwc] = wcgain(T);

PeakGain

% Compute worst-case closed-loop transfer T
Twc = usubs(T,Uwc);

% 4 random samples of uncertain model T
Trand = usample(T,4);
clf

subplot(211), bodemag(Trand,'b',Twc,'r',{10 1000});  % plot Bode response

subplot(212), step(Trand,'b',Twc,'r',0.2);  % plot step response
Uncertainty in Transfer Function
Uncertainty in TF (high $k$, low m’s)
Uncertainty in TF (low k, high m’s)
Theoretical Background

Signals Norms

\[ \|x\|_1 := \int_{-\infty}^{\infty} |x(t)| dt, \quad \text{for} \quad p = 1 \]

\[ \|x\|_p := \left( \int_{-\infty}^{\infty} |x(t)|^p dt \right)^{1/p}, \quad \text{for} \quad 1 < p < \infty \]

\[ \|x\|_\infty := \sup_{t \in \mathbb{R}} |x(t)|, \quad \text{for} \quad p = \infty \]
System Norms

- System norms are actually the input-output gains of the system.
- For a LTI stable system the $\infty$-norm is decided by the peak value of the largest singular value of the frequency response matrix over the whole frequency axis:

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2$$
Theoretical Background

Internal Stability

• An interconnected system is internally stable if the subsystems of all input-output pairs are asymptotically stable.

\[ T_{yr} = GK(I + GK)^{-1} \]
\[ T_{yd} = G(I + KG)^{-1} \]
\[ T_{ur} = K(I + GK)^{-1} \]
\[ T_{ud} = -KG(I + KG)^{-1} \]
Robust Design Specifications

Small-gain Theorem

- Important theorem in the derivation of many stability tests
- Provides only a sufficient condition for stability
Robust Design Specifications

Small-gain Theorem

• If $G_1(s)$ and $G_2(s)$ are stable then the closed-loop system is internally stable if and only if

$$\|G_1G_2\|_\infty < 1 \quad \& \quad \|G_2G_1\|_\infty < 1$$
Robust Design Specifications

• Additive perturbation configuration, where $\Delta(s)$ is the perturbation which is unknown but stable

• It can be worked out that the transfer function from the signal $v$ to $u$ is $T_{uv} = -K(I + GK)^{-1}$
Robust Design Specifications

• K is a stabilising controller for the nominal plant $G$, since we always assume that the perturbation set includes zero (no perturbation)

• Hence, from the Small-Gain theorem, for stable $\Delta(s)$, the closed-loop system is robustly stable if $K(s)$ stabilises the nominal plant and the following holds:

$$\|\Delta K(1 + GK)^{-1}\|_\infty < 1$$

and

$$\|K(1 + GK)^{-1}\Delta\|_\infty < 1$$

or,

$$\|K(1 + GK)^{-1}\|_\infty < \frac{1}{\|\Delta\|_\infty}$$
Sensitivity Matrix

• S: transfer function from measurement noise to process output
  \[ S = (I + GK)^{-1} \]

• Typically we want to minimize not only the sensitivity of the system to noise, but also maintain nominal performance, robust stabilization, etc. w.r.t. additive perturbation.

• This is formulated as a multiple cost function minimization problem
Cost Functions involving Sensitivity

\[
\min_{K_{stabilising}} \left\| \frac{(I + GK)^{-1}}{K(I + GK)^{-1}} \right\|_{\infty}
\]
$H_\infty$ Design

- An optimisation approach which is effective and efficient robust design method for LTI control systems
- In the $H_\infty$ approach, the designer from the outset specifies a model of system uncertainty, such as additive perturbation and/or output disturbance
Standard $H_\infty$ Configuration

- external inputs denoted by $w$ (inputs and disturbances)
- $z$ denotes the output signals to be minimised/penalised (e.g. error) that includes both performance and robustness measures
- $y$ is the vector of measurements available to the controller $K$
- $u$ the vector of control signals.
Standard $H_\infty$ Configuration

- The objective is to find a stabilising controller $K$ (less than or equal to one) to minimise the output, $z$, in the sense of energy, for all $w$. This is equivalent to minimising the $H_\infty$-norm of the transfer function from $w$ to $z$. 
The problem can be formulated as:

\[
\begin{bmatrix}
Z \\
y
\end{bmatrix} = P(s) \begin{bmatrix}
w \\
u
\end{bmatrix} = \begin{bmatrix}
P_{11}(s) & P_{12}(s) \\
P_{21}(s) & P_{22}(s)
\end{bmatrix} \begin{bmatrix}
w \\
u
\end{bmatrix}
\]

\[u = K(s)y\]

and it can be obtained directly that

\[z = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]w =: F_l(P, K)w\]

This is known as the lower linear fractional transformation.
$H_\infty$ Optimization Problem

- We want to minimize this transform w.r.t. the $H$ infinity norm:
  \[
  \|F_l(P, K)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}[F_l(P, K)(j\omega)]
  \]
- Here, $\bar{\sigma}$ represents the maximum singular value of $F_l(P, K)$ for a given frequency.
- Thus, the infinity norm is the supremum of this function over all frequencies.
- Finally, the design problem is the following
  \[
  \min_{K_{stabilizing}} \|F_l(P, K)\|_\infty
  \]
Mu-Synthesis Design

• Used to achieve both robust stability (RS) and robust performance (RP) if there is structured uncertainty
• The system is robustly stable if $M(s)$ is stable and $\mu_\Delta(M(s)) < 1$. 
Structured Singular Values

- $\mu_\Delta$: Smallest "size" of the uncertainty that makes $I - M(j\omega)\Delta(j\omega)$ singular at some frequency.

$$
\mu^{-1}_\Delta(M) := \min_{\Delta \in \Delta} \{ \sigma(\Delta) : \det(I - M\Delta) = 0 \}
$$

$$
\mu_\Delta(M(s)) := \sup_{\omega \in \mathcal{R}} \mu_\Delta(M(j\omega))
$$

- Here, $\Delta$ is the block uncertainty, and bold $\Delta$ is the set of structured uncertainties.
Computing $\mu(M)$

• It can be shown that $\mu(M)$ is bounded by

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M)$$

• Later we will need to minimize $\mu(M)$.
• The gap between the spectral radius and the max singular values could be very large, hard to compute
• We can transform $M$ to narrow the range, making the minimization over $\mu(M)$ easier to compute.
• We define $U$ and $D$ matrices that match the structure of bold $\Delta$ (block diagonal).
Computing $\mu(M)$

• From the structure of U and D, we can derive the following transformation to tighten the bounds on $\mu(M)$:

$$\max_{U \in \mathcal{U}} \rho(MU) \leq \mu(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

• In many cases this reduces to

$$\mu(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

• Minimizing w.r.t. the upper bound in this way is preferred because it is a convex problem, but the lower bound is not.
Mu-Synthesis

- We can find the system output, $z$, w.r.t. perturbations, $\Delta$.
  $$ z = [M_{22} + M_{21} \Delta (I - M_{11} \Delta)^{-1} M_{12}]w $$
  $$ z = F_u(M, \Delta)w $$
- For stability
  $$ \|F_u(M, \Delta)\|_\infty < 1 $$
- We can derive the following conditions:
  1. RP: $\|M\|_\mu < 1$
  2. RS: $\|M_{11}\|_\mu < 1$
  3. NP: $\|M_{22}\|_\infty < 1$
  4. NS: $M$ is internally stable
D-K Iteration Method

• For the optimal RSRP design, we want to solve for $K$ s.t.

$$\inf_{K(s)} \sup_{\omega \in \mathcal{R}} \mu[M(P, K)(j\omega)]$$

• A stabilizing controller is found s.t.

$$\sup_{\omega \in \mathcal{R}} \inf_{D \in \mathcal{D}} \bar{\sigma} \left[ DM(P, K)D^{-1}(j\omega) \right] < 1$$

• If $D$ is constant, this is simply an $H_\infty$ optimization problem for $K$

• If $K(s)$ is fixed, and $D$ varies, this is a convex optimization problem over all frequencies $\omega$
D-K Iteration Method

Step 1: Start with an initial guess for $D$, usually set $D = I$.
Step 2: Fix $D$ and solve the $\mathcal{H}_\infty$-optimisation for $K$,

$$K = \arg\inf_K \| F_I(\tilde{P}, K) \|_\infty$$

Step 3: Fix $K$ and solve the following convex optimisation problem for $D$ at each frequency over a selected frequency range,

$$D(j\omega) = \arg\inf_{D \in \mathcal{D}} \sigma[DF_I(P, K)D^{-1}(j\omega)]$$

Step 4: Curve fit $D(j\omega)$ to get a stable, minimum-phase $D(s)$; goto Step 2 and repeat, until a prespecified convergence tolerance or (6.10) is achieved, or a prespecified maximum iteration number is reached.
Example: Two Cart System

- Design goal: attenuate effect of disturbance $f_2$ on position of mass $m_2$.
- Performance goal: attenuate the disturbance on mass m2 by a factor of 80 below 0.1 rad/s.
Uncertainty Modeling

- Uncertainty in $k_1$ -> same as before, use `ureal()`
- Time delay between command and application of actuator force, $f_1$. The error from this is bounded by a high pass filter transfer function

$$W_{delay} = \frac{2.6s}{s + 40}$$
Error from Time Delay

Multiplicative Time-Delay Error: Actual vs. Bound

Frequency (rad/s)
Magnitude (dB)
Plant Model

\[ A = \begin{bmatrix}
0 & 0 & \frac{k_1}{m_1} & \frac{k_1}{m_2} \\
0 & 0 & \frac{k_1}{m_1} & \frac{k_1 + k_2}{m_2} \\
0 & 0 & \frac{b_1}{m_1} & \frac{b_1}{m_2} \\
1 & 0 & \frac{b_1}{m_1} & \frac{b_1 + b_2}{m_2}
\end{bmatrix} \]
Uncertainty in Transfer Function
Controller Design

- $k_1$ is uncertain due to sensor noise, $W_n$.
- Controller will measure noisy $\Delta x$ of $m_2$ and apply $f_1$, which acts on $m_2$ through uncertain $k_1$.
- Actuation is penalized by a filter, $W_u$.
- Disturbance is filtered by $W_{dist}$.
Synthesized Controller Loop Gain (high uncertainty in k)
Synthesized Controller Loop Gain (low uncertainty in k)
Disturbance Rejection

Red: high uncertainty in $k_1$
Blue: low uncertainty in $k_1$
Pros and Cons of Robust Control

Advantages
• Allows control in the face of uncertainties
• Applicable to multivariable problems

Disadvantages
• Dimensionality reduction of model and/or controller often necessary
References

• Chandrasekharan, P., C., Robust Control of Linear Dynamical Systems, Academic Press, 1996.