Dynamical Wasserstein Barycenters for Time-series Modeling

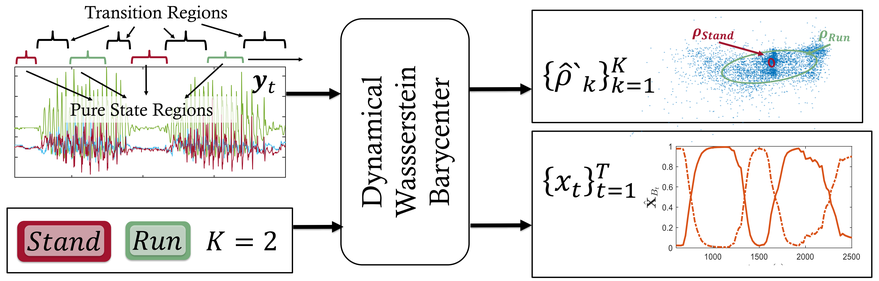
Kevin C. Cheng, Shuchin Aeron, Michael C. Hughes, Eric L. Miller, *Tufts University*

GitHub

Tufts UNIVERSITY

(A) Problem Statement

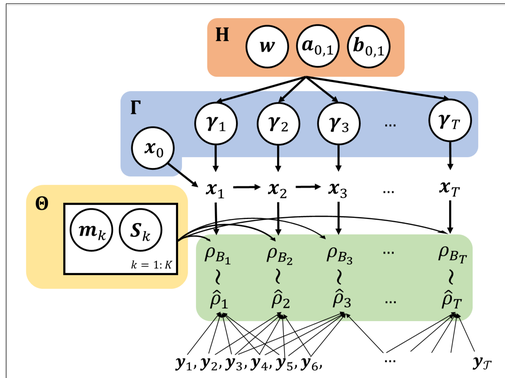
A model for time-series formed from the gradual transition among a set of pure-states



Contributions:

- A new model for time-series consisting of **stationary** and **transitions** periods
- Where **transitions** between pure-states through the **Wasserstein Barycenter**
- A new **random walk on the probability simplex** to model dynamically evolving system state
- Utilize **Wasserstein-Riemannian geometry** of **Gaussian distributions**
- Evaluate on **human activity data**

(B) Dynamical Wasserstein Barycenter

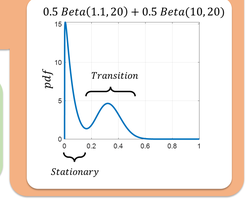


Wasserstein Barycenter

$$\rho_{B_t} = \underset{\rho}{\operatorname{argmin}} \sum_{k=1:K} \boldsymbol{x}[k] \mathcal{W}_2^2(\rho, N(\boldsymbol{m}_k, \boldsymbol{S}_k))$$

Innovation Prior – Beta Mixture

$$\boldsymbol{\gamma}_t[k] = \sum_{i=0}^{1} \boldsymbol{w}_i[k] Beta(\boldsymbol{a}_i[k], \boldsymbol{b}_i[k])$$



Empirical Estimate

Sliding windows of size $(2n+1)$, strided by δ

$$\hat{\rho}_t = \mathcal{N}(\hat{m}_t, \hat{S}_t)$$

Pure State Prior

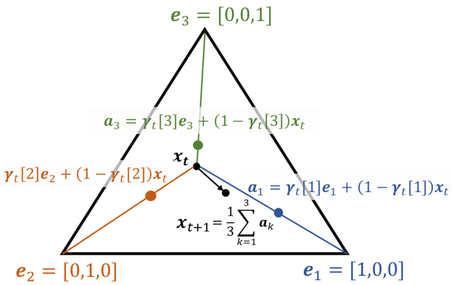
$$p(\boldsymbol{m}_k, \boldsymbol{S}_k) = \kappa(s, \sigma_0) \exp\left(-\frac{1}{2s^2} \mathcal{W}_2^2\left((\boldsymbol{m}_k, \boldsymbol{S}_k), (\boldsymbol{m}_0, \sigma_2^2 I)\right)\right)$$
$$= \kappa(s, \sigma_0) \exp\left(-\frac{1}{2s^2} \|\boldsymbol{q} - \boldsymbol{q}_0\|_2^2\right)$$

Objective Function

$$-\log\left(p_{\boldsymbol{H}}(\boldsymbol{\Gamma}) p(\boldsymbol{\Theta})\right) + \sum_{t=1}^{T} \mathcal{W}_2^2(\rho_t, \rho_{B_t})$$

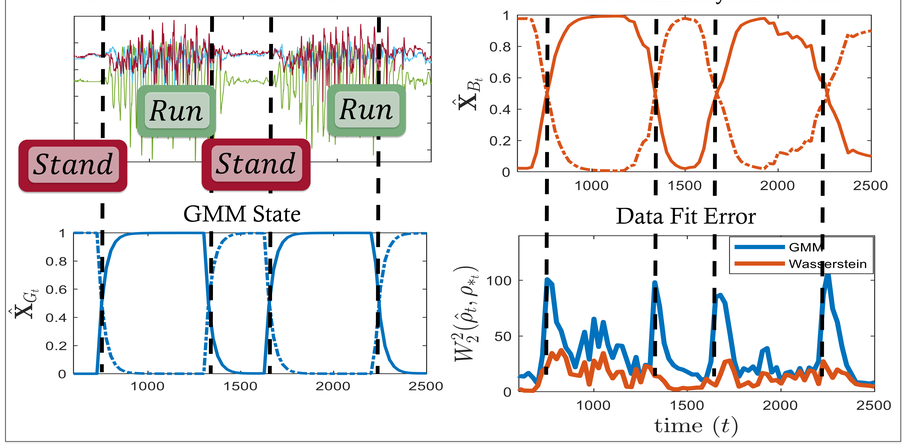
State Innovation Update

$$\boldsymbol{x}_t = \left(1 - \frac{1}{K}\sum_{k=1}^{K} \boldsymbol{\gamma}_t[k]\right) \boldsymbol{x}_{t-1} + \frac{1}{K} \boldsymbol{\gamma}_t$$

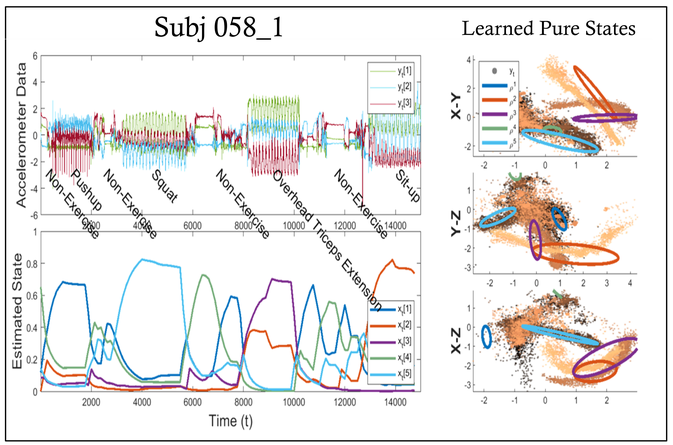


(F) Results

Beep Test: Run-Stop *proprietary*



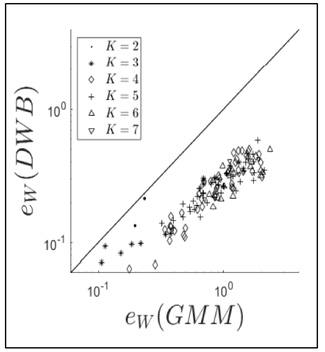
Gym Exercise Data (MSR) *(Morris 2014)*



Benchmark

DSS: *Krishnan 2016*

	DWB $p = 91$	DSS $p = 2(94)$	DSS $p \approx 2(88k)$
e_W	**0.27**	4.34	3.07
e_{nll}	1.02	1.49	**-0.508**



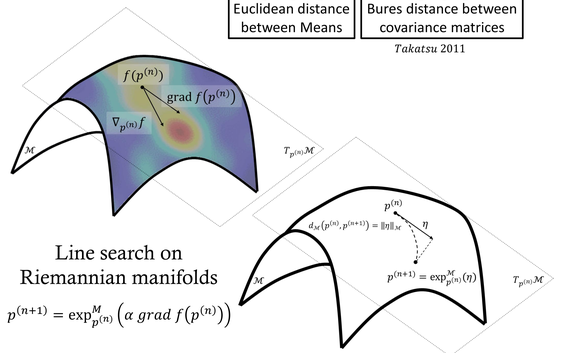
(E) Gaussian-Wasserstein Geometry

$$\mathcal{W}_2^2(\rho_1(\boldsymbol{m}_1, \boldsymbol{S}_1), \rho_2(\boldsymbol{m}_2, \boldsymbol{S}_2) = \underbrace{\|\boldsymbol{m}_1 - \boldsymbol{m}_2\|_2^2}_{\mathcal{E}_2^2(\boldsymbol{m}_1, \boldsymbol{m}_2)} + \underbrace{tr(\boldsymbol{S}_1 + \boldsymbol{S}_2 - 2\left(\boldsymbol{S}_1^{\frac{1}{2}} \boldsymbol{S}_2 \boldsymbol{S}_1^{\frac{1}{2}}\right)}_{\mathcal{B}_2^2(\boldsymbol{S}_1, \boldsymbol{S}_2)}$$

Euclidean distance between Means

Bures distance between covariance matrices

Takatsu 2011



Line search on Riemannian manifolds

$$p^{(n+1)} = \exp_{p^{(n)}}^{\mathcal{M}}\left(\alpha\ grad\ f(p^{(n)})\right)$$