Consensus on multi-time Scale digraphs

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Abstract

This paper studies the consensus problem in multi-agent network where each agent possesses a local clock that may be misaligned with other agents; we refer to such a network as a Multi-Time Scale (MTS) network. Our first result shows that the consensus on a MTS network is guaranteed when the product of each agent’s local clock skew and in-degree is less than one. The corresponding consensus value depends on both agents’ local clocks and initial states. When considering that the local clocks are corrupted by noise, we propose a stochastic approximation type protocol to ensure consensus in the mean-square and almost-sure sense. We claim that a stochastic MTS network is more tolerant to the local clocks. Furthermore, we design an algorithm based on local-time dependent weights to guarantee average-consensus on a general MTS network. Simulations illustrate our findings.

1. Introduction

Recently, distributed coordination in multi-agent systems has been an attractive research topic in the system and control community. This is mainly due to its wide applications in several emerging areas such as cooperative control of Unmanned Aerial Vehicles (UAVs),\textsuperscript{[1]}, distributed filtering,\textsuperscript{[2]}, distributed computation,\textsuperscript{[3]}, and distributed optimization,\textsuperscript{[4]}. Among these applications, reaching a consensus is one of the widely studied topics, which is to design a distributed protocol to drive all agents to reach an agreement.

In particular, Olfati-Saber and Murray,\textsuperscript{[5]}, proposed a linear distributed protocol for a continuous-time directed information flow. It is shown that a strongly-connected and balanced digraph turns out to be instrumental in guaranteeing average-consensus. The corresponding discrete-time algorithm can be found in\textsuperscript{[6]} and the average-consensus is achieved if the interaction graph is balanced.
and the union of graphs over every bounded time interval is strongly-connected. In order to consider
average-consensus problems on a general digraph, Cai and Ishii in [6] introduced a surplus variable
to record the change of state status at each agent. Using matrix perturbation theory, they show
that if the perturbation is small enough, average-consensus can be achieved even the digraph is not
balanced.

In some applications, consensus is not necessarily on the average. For example, Jadbabaie et
al., [7], provide a theoretical explanation for the observed heading angles of the Vicsek model, [8],
which describes bidirectional information flow among locally visible agents in a group. Even though
the interaction topology between agents may change over time, if the union of interaction graphs
are connected frequently enough, the agents achieve consensus, not necessarily the average. Its
corresponding discrete-time model is proposed by Wei et al. in [9] and the interaction graph is
extended to be directed. It is shown that consensus can be reached if as the system evolves, the
union of directed interaction graphs has a spanning tree frequently enough.

The literature mentioned above assumes that the multi-agent system is not disturbed by uncer-
tain noise. However, real multi-agent systems are often affected by uncertain noise. For instance,
to attenuate the impact of measurement noise, Huang and Manton, [10], introduce decreasing con-
sensus gains, $a(k)$, (where $k$ is the discrete-time step) for a strongly-connected circulant graph and
a connected undirected graph. Kar and Moura, [11], consider average-consensus under a random
interaction topology and communication noise. Necessary and sufficient conditions are reported
in [12] for mean-square average-consensus with measurement noise and fixed topologies. In [13], a
distributed stochastic approximation type protocol is designed to reduce the influence of communi-
cation noise. Furthermore, sufficient conditions for mean-square average-consensus and almost-sure
consensus are derived with time-varying topology.

The time scale in most existing work on distributed coordination is unified, i.e., all agents operate
on identical and completely aligned clocks. This means that each agent acts with a coordinated
global clock or synchronized local clocks. However, in real applications, due to small fabrication
variations or ambient conditions, the local clock at each agent may have its own time-scale, see [14].
A special case of non-identical clocks, often considered, is multi-agent systems with time delays.
For instance, [5] considers the presence of communication time-delay in a proposed linear protocol,
and sufficient and necessary conditions for average-consensus are given therein. The multi-agent
network with different clock skews has been considered in [14], where the aim is to drive all clocks

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with different skews and drifts to achieve an agreement.

In this paper, we focus on the consensus problem in multi-agent system where each agent possesses a local clock that may be misaligned with other agents, as shown in Fig. 1; we refer to such a network as a Multi-Time Scale (MTS) network. Consensus on a MTS network is more challenging since each agent operates according to its own clock. To synchronize the update of state, we introduce an independent third party in the network that directs each agent to update its state. We note that even the instants, at which the agents update, are synchronized by the third party, the agent clocks indicate different times and thus the states are misaligned. As a result, consensus on a MTS network is an agreement, which depends on both initial states and local clock skews. Also, since local clocks may also be affected by noise, we consider the influence of corrupted local clocks on the behaviors of agents. To this end, we propose a stochastic approximation type protocol to deal with corruption in clock skews. We show that the proposed algorithm achieves the mean-square and almost-sure consensus. Finally, we note that the underlying graph may not be balanced due to the misaligned clocks. This is because the state-update at each agent is based on the local clock. In this case, we introduce local-time dependent weights to ensure that average-consensus is achieved on a MTS network.

The remainder of this paper is as follows. In Section 2, some concepts in graph theory and stochastic process are introduced. In Section 3, we formulate the consensus problem on the MTS network and propose a general linear distributed protocol. A deterministic algorithm is discussed and consensus of a MTS network is achieved in Section 4. Also, its corresponding stochastic protocol
is proposed to consider the case of corrupted local clock skews and both mean-square and almost-
sure consensus on a MTS network are investigated. We discuss the average-consensus problem on
a MTS network represented by a general strongly connected digraph in which each agent performs
based on its local clock in Section 5. Section 6 provides numerical examples, and Section 7 concludes
the paper.

**Notation:** Let $1 = [1 \cdots 1]^T \in \mathbb{R}^n$ be the vector of all ones; $I_n$ denote the $n \times n$ identity matrix. Given a matrix, $M$, let $M^T$ denote its the transpose, $\sigma(M)$ denote the set of its eigenvalues and $\rho(M)$ denote its spectral radius. In addition, $\| \cdot \|$ and $\| \cdot \|_{\infty}$ denote the 2−norm and infinity norm of a vector or matrix, respectively. For a complex number $\lambda$, let $|\lambda|$ be the modulus. For a given random variable, $X$, $\mathbb{E}(X)$ and $\text{Var}(X)$ denote its mean and variance, respectively.

2. Preliminaries

Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph (directed graph), where $\mathcal{V} = \{1, \cdots, n\}$ denotes the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, and $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ with nonnegative elements, $a_{ij}$, denotes the weighted adjacency matrix of $G$. A node, $i$, of $G$ represents the $i$th agent in the network. An edge in $G$ is denoted by $e_{ij} = (i, j)$ and $e_{ij} \in \mathcal{E}$ if and only if the $j$th node can send its state information to the $i$th node directly. If the edge $e_{ij} \in \mathcal{E}$, then the adjacency element $a_{ij} > 0$; otherwise $a_{ij} = 0$. Also we assume that $a_{ij} = 0$ for $i = j$. The neighborhood of node, $i$, is denoted by $N_i = \{ j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

The in-degree of node, $i$, is defined as $\text{deg}_{\text{in}}(i) = \sum_{j=1}^{n} a_{ij}$ and the out-degree of node, $i$, is defined as $\text{deg}_{\text{out}}(i) = \sum_{j=1}^{n} a_{ji}$. The Laplacian matrix associated with graph $G$ is defined as $L_G = D - A$ where $D = \text{diag}(\text{deg}_{\text{in}}(1), \cdots, \text{deg}_{\text{in}}(n))$. The digraph $G$ is called balanced if $\text{deg}_{\text{in}}(i) = \text{deg}_{\text{out}}(i)$ for every node $i$. A sequence of edges $(i, i+1), \cdots, (i+k-1, i+k)$ is called a directed path from node, $i$, to node, $i+k$. If for any nodes $i$ and, $j$, there is a directed path from $i$ to $j$, then this digraph, $G$, is called strongly-connected.

Let $\{X_k \in \mathbb{R}^n\}_{k \in \mathbb{N}}$ be a family of random variables and $\mathcal{F}_k$ be a $\sigma$–algebra generated by random variables $\{X_1, \cdots, X_k\}$.

**Definition 2.1.** [15] A family of random variables $\{X_k \in \mathbb{R}^n\}_{k \in \mathbb{N}}$ is called a martingale if $\mathbb{E}(\|X_k\|) < \infty$ for all $k$ and $\mathbb{E}(X_k|\mathcal{F}_s) = X_s$ for all $k \geq s$ almost-surely (a.s.).
Lemma 2.1. Let a family of random variables \( \{X_k \in \mathbb{R}^n\}_{k \in \mathbb{N}} \) be a martingale and \( \sup_{k \in \mathbb{N}} \mathbb{E}(\|X_k\|^2) < \infty \). Then, this martingale converges to a random variable in mean-square and almost-sure sense.

3. Problem Formulation

Let \( x_i(t_i) \in \mathbb{R} \) denote the state of agent, \( i \), at time \( t_i \), where \( t_i \) is the local clock of node, \( i \). The dynamic of agent, \( i \), according to the local clock, \( t_i \), follows

\[
dx_i(t_i) = u_i(t_i)dt_i, \tag{1}
\]

where \( u_i(t_i) \in \mathbb{R} \) is a local control input and the initial state of agent, \( i \), is \( x_i(0) \). For each agent \( i \), it receives state information \( \hat{x}_j(t_i) \) from its neighbor, \( j \), at its local time \( t_i \). Therefore, we propose a linear local-time protocol

\[
u_i(t_i) = \sum_{j \in N_i} a_{ij} (\hat{x}_j(t_i) - x_i(t_i)). \tag{2}
\]

We assume that the local clock \( t_i \) of agent \( i \) evolves proportional to some reference clock, \( t \), i.e.,

\[
t_i = m_i t, \tag{3}
\]

where \( m_i > 0 \) is the local clock skew. Assume that for different agents, the local clocks are misaligned, i.e., \( m_i \neq m_j \). This also implies that agents in the MTS network possess different update rate. The system consisting of agents, which have misaligned states and different update rate, is complicated. In this paper, we focus on the system, where agents update states at different speed but at same time.

In order to coordinate updates of agents, we introduce an independent third party, which signals all agents to update at periodic ticks of the reference clock. Suppose that the period of this update signal is \( T \). We discretize every local clock, \( t_i \), at reference clock \( t = 0, T, \ldots, kT, \ldots \). Based on this discretization, we have the following local protocol from Eq. (2) and Eq. (3) at \( t = kT \)

\[
u_i(m_i kT) = \sum_{j \in N_i} a_{ij} (\hat{x}_j(m_i kT) - x_i(m_i kT)). \tag{4}
\]

Substituting the discrete-time protocol Eq. (4) into the continuous-time system Eq. (1) and discretizing the updating time domain, we get the following associated update scheme at \( t = kT \)

\[
x_i((k + 1)m_i T) = x_i(km_i T) + \sum_{j \in N_i} a_{ij} (\hat{x}_j(km_i T) - x_i(km_i T))m_i T. \tag{5}
\]
Noticing that although each agent, $i$, evolves based on its local clock and it does not know any information about its neighbors’ local clocks, all agents in the MTS network receive and update their states according to the reference clock. Therefore, the update scheme, Eq. (5), is equivalent to the following update scheme:

$$x_i((k + 1)m_iT) = x_i(km_iT) + \sum_{j \in N_i} a_{ij}(x_j(km_jT) - x_i(km_iT))m_iT. \quad (6)$$

Without loss of generality, we assume that the period of signals is $T = 1$ unit time. Also since the update happens at the same reference time, we let $x_i(m_iT) = X_i(k)$. Thus, the compact form of the update scheme, Eq. (6), is

$$X_i(k + 1) = X_i(k) + \sum_{j \in N_i} m_i a_{ij} (X_j(k) - X_i(k)).$$

By letting $X(k) = [X_1(k), X_2(k), \cdots, X_n(k)]^T \in \mathbb{R}^n$ and $m = \text{diag}(m_1, m_2, \cdots, m_n)$, we have the matrix form:

$$X(k + 1) = (I_n - mL_G)X(k). \quad (7)$$

To illustrate the equivalence between update schemes Eq. (5) and Eq. (6), let us see a two-agents example.

**Example 1.** Suppose $n = 2$ and update processes of agents are as following:

For agent 1 at $t_1 = m_1kT$, it sends its state value $x_1(t_1)$ to agent 2 and receives the state value $\hat{x}_2(t_1)$ of agent 2 at local time $t_1$ and then updates its state value with local step size

$$x_1((k + 1)m_1T) = x_1(km_1T) + a_{12}(\hat{x}_2(km_1T) - x_1(km_1T))m_1T. \quad (8)$$

For agent 2 at $t_2 = m_2kT$, it sends its state information $x_2(m_2kT)$ to agent 1 and receives the state information $\hat{x}_1(m_2kT)$ of agent 1 and then updates the state information as following For agent 2 at $t_2 = m_2kT$, similarly, we get the following update equation with local step size

$$x_2((k + 1)m_2T) = x_2(km_2T) + a_{21}(\hat{x}_1(km_2T) - x_2(km_2T))m_2T. \quad (9)$$

Obviously, for agent 1, the received state value $\hat{x}_2(t_1)$ of agent 2 is equal to $x_2(t_2)$ sending by agent 2 at $t = kT$. Similarly, $\hat{x}_1(t_2) = x_1(t_1)$. Therefore, update equations Eq. (8) and Eq. (9) are equivalent to the following:

$$x_1((k + 1)m_1T) = x_1(km_1T) + a_{12}(x_2(km_2T) - x_1(km_1T))m_1T$$
4. Consensus of the MTS network

In this section, we investigate the consensus problem on a MTS network, which is represented by a strongly-connected and balanced digraph. Particularly, we show in Theorem 4.1 that the consensus is achieved by using algorithm, Eq. (7), when MTS network is free of noise. If the MTS network is disturbed by white noise, we propose a corresponding stochastic approximation type algorithm, Eq. (11). By applying nonnegative matrix theory and stochastic analysis, we state in Theorems 4.2 and 4.5 that the mean-square and almost-sure consensus are achieved, respectively.

4.1. Consensus of the MTS network without noise

We first consider consensus on a MTS with local clocks, which are not corrupted by noise. This means that each agent in the MTS network operates at a different but fixed clock. Theorem 4.1 provides the condition to guarantee the consensus on the MTS network. Also the group decision is given explicitly.

**Theorem 4.1.** Let $G$ be a strongly connected and balanced digraph; $\{X(k)\}$ be the sequence generated by the deterministic algorithm, Eq. (7). When local clocks satisfy $\sum_{j \in N_i} a_{ij} < \frac{1}{m_i}$, for $i = 1, \ldots, n$, the MTS network reaches consensus, i.e., $X(k) \to \mathbf{1}X^*$, as $k \to \infty$. The group decision value, $X^*$, is given by

$$X^* = \frac{\sum_{i=1}^{n} \frac{1}{m_i} X_i(0)}{\sum_{i=1}^{n} \frac{1}{m_i}}.$$

**Proof.** Let $M = I_n - mL_G$, then $M$ is a row stochastic matrix, i.e., every row of $M$ sums up to one. So one is an eigenvalue of $M$ and its corresponding right eigenvector is $\mathbf{1}$. Because of the balanced digraph $G$, we can check that $[\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_n}]^T$ is a left eigenvector of $M$ associated with eigenvalue one. Since for each agent, its in-degree $\sum_{j \in N_i} a_{ij} < \frac{1}{m_i}$, this means $m_i \sum_{j \in N_i} a_{ij} < 1$. Together with nonnegative weights $a_{ij}$, we know that $M$ is a nonnegative matrix. Also since the interaction graph is strongly connected, by Perron-Frobenius theorem (see, e.g., [17]), one is a simple eigenvalue of $M$ and the moduli of other eigenvalues are less than 1. If we denote
by \( w_l = \sum_{i=1}^{n} \frac{1}{m_i} [\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_n}]^T \) the left eigenvector of \( M \) and denote by \( w_r = [1, 1, \cdots, 1]^T \) the right eigenvector of \( M \) corresponding to the eigenvalue one, \( w_l^T w_l = 1 \). Therefore,

\[
X(k) = M^k X(0) \rightarrow w_r w_l^T X(0) = 1 X^*, \text{ as } k \rightarrow \infty.
\]

This completes the proof.

\[ \square \]

**Remark 1.** In practice, in order to satisfy the convergence condition that \( \sum_{j \in N_i} a_{ij} < \frac{1}{m_i} \) for \( i = 1, \cdots, n \), it is possible to incorporate a small enough constant \( C_i \) for agent, \( i \), into its local protocol. As a result, its modified in-degree \( C_i \sum_{j \in N_i} a_{ij} < \frac{1}{m_i} \) is assured and Theorem 4.1 is applicable.

### 4.2. Consensus of a MTS network involved with corrupted noise

We now consider the local clocks with random perturbations. For each local clock, \( t_i \), assume it is corrupted by the white noise \( \eta_i \sim \mathcal{N}(0, s_i) \) and \( \eta_i \) is independent of \( \eta_j \) if \( i \neq j \). Then the corrupted clock skew \( \tilde{m}_i \) becomes

\[
\tilde{m}_i = m_i + \eta_i.
\]

By considering the corrupted local clock skew \( \tilde{m}_i \), the update scheme for agent \( i \), is proposed

\[
X_i(k + 1) = X_i(k) + \beta \sum_{j \in N_i} \tilde{m}_i a_{ij} (X_j(k) - X_i(k)),
\]

whose matrix form is

\[
X(k + 1) = X(k) - \beta m L_G X(k) - \beta \eta(k) L_G X(k),
\]

where \( \beta \) is a positive scalar, called a consensus gain, \( m = \text{diag}(m_1, \cdots, m_n) \), and \( \eta(k) = \text{diag}(\eta_1(k), \cdots, \eta_n(k)) \) are noise at reference time, \( t = kT \). In addition, \( \eta(k) \) is independent of \( X(k) \).

Let us first introduce one of our main results, Theorem 4.2, in which the upper bound for the consensus gain, \( \beta \), is given to assure mean-square consensus. It is also shown in Theorem 4.2 that the group decision is a random variable with finite mean and variance.

**Theorem 4.2.** Let \( G \) be a strongly connected and balanced digraph. Using the stochastic algorithm Eq. (11), the stochastic MTS network achieves mean-square consensus, i.e., for each agent \( i \),

\[
\mathbb{E} \|X_i(k) - X^*\|^2 \rightarrow 0 \text{ as } k \rightarrow \infty,
\]
if the positive consensus gain, $\beta$, satisfies

$$\beta < \frac{2\lambda_2}{\|\sqrt{mL_G}\sqrt{m}\|^2 + \max_{1 \leq i \leq n} \frac{2}{m_i} \|\sqrt{mL_G}\sqrt{m}\|^2}.$$  

(12)

In addition, the agreement, $X^*$, is a random variable which has mean, $\mathbb{E}X^* = \frac{1}{\sum_{i=1}^{n} \frac{1}{m_i}} \sum_{i=1}^{n} \frac{1}{m_i} X_i(0)$, and finite variance.

As shown in the Theorem 4.2 that the mean of the random agreement of the stochastic MTS network is the same as the agreement of the corresponding MTS network without noise. Also, comparing to Theorem 4.1, there is no conditions like $\sum_{j \in N_i} a_{ij} < \frac{1}{m_i}$ for each agent, $i$, in the stochastic MTST network required to reach consensus. This implies that the stochastic MTS network is more tolerant to local clocks.

The proof of Theorem 4.2 is based on results of Theorems 4.3 and 4.4, and the proof of Theorem 4.3 is based on Lemmas 4.1 and 4.2. Now we turn to Lemma 4.1, which deals with properties of a particular matrix, called quasi-Laplacian matrix.

**Lemma 4.1.** Let $G = (V, E, A)$ be a strongly-connected and balanced digraph, and $L_G$ be the weighted Laplacian matrix. Also let $D$ be a diagonal matrix with positive diagonal entries. Then all eigenvalues of the quasi-Laplacian matrix, $\frac{1}{2} D(L_G^T + L_G)D$, are nonnegative. Furthermore, zero is a simple eigenvalue.

**Proof.** First, let $x \in \mathbb{R}^n$ and $x \neq 0$; $\hat{L}_G = \frac{1}{2}(L_G^T + L_G)$ and $D = \text{diag}(d_1, d_2, \cdots, d_n)$ where $d_i > 0$ for all $i = 1, \cdots, n$. Thus, $\frac{1}{2} x^T D(L_G^T + L_G)Dx = (Dx)^T \hat{L}_G(Dx)$. Since $G$ is balanced, from [5], we know that $L_G$ is the Laplacian matrix of the mirror graph, $\hat{G}$, which is undirected. Also noticing that $G$ is strongly-connected means the mirror graph, $\hat{G}$, is strongly-connected. Therefore, we have $0 = \lambda_1(\hat{L}_G) \leq \lambda_2(\hat{L}_G) \leq \cdots \leq \lambda_n(\hat{L}_G)$. Also since $\hat{L}_G$ is symmetric, by Rayleigh quotient theorem, it holds $\frac{1}{2} x^T D(L_G^T + L_G)Dx \geq \lambda_1(\hat{L}_G)||Dx||^2 = 0$. This means that the quasi-Laplacian matrix, $D\hat{L}_G D$, is semi-positive definite and then its eigenvalues are nonnegative $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

On the other hand, since $L_G$ is a Laplacian matrix of the strongly-connected and balanced digraph, $G$, zero is an eigenvalue of quasi-Laplacian matrix, $\frac{1}{2} D(L_G^T + L_G)D$, and its associated eigenvector is $d = [\frac{1}{d_1}, \frac{1}{d_2}, \cdots, \frac{1}{d_n}]^T$. Now we claim that the eigenvalue zero is always simple.

Suppose that the eigenvalue zero of the quasi-Laplacian matrix, $D\hat{L}_G D$, is not simple. This means the algebraic multiplicity of zero is not equal to the geometric multiplicity. However zero
is the simple eigenvalue of the mirror Laplacian matrix, \( \hat{L}_G \); this means \( \text{rank}(\hat{L}_G) = n - 1 \). Also since \( D \) is a full rank diagonal matrix, \( \text{rank}(DL_GD) = n - 1 \). So the geometric multiplicity of eigenvalue zero of the quasi-Laplacian matrix, \( DL_GD \), is 1. That is to say the algebraic multiplicity of eigenvalue zero is at least two. Without loss of generality, we assume the algebraic multiplicity is 2 associated with zero. Therefore, there exists a generalized right eigenvector \( u \in \mathbb{R}^n \) such that

\[
(DL_GD)^2 u = 0 \quad \text{and} \quad (DL_GD)u \neq 0.
\]

Thus \( DL_GDu \) is a right eigenvector of the quasi-Laplacian matrix associated with eigenvalue zero. Since \( d \) is also a right eigenvector of the quasi-Laplacian matrix corresponding to the eigenvalue zero, it must hold

\[
DL_GDu = cd,
\]

where \( c \) is a non-zero constant. By the assumption, \( D \) is invertible and thus we have

\[
\hat{L}_G Du = cd^2.
\]

However, multiplying \( 1^T \) on both sides of Eq. (13), we get

\[
0 = c1^T d^2,
\]

where \( d^2 = \left[ \frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_n} \right]^T \). This is a contradiction. So the desired results are derived.

The next lemma is borrowed from [18]. It establishes a relation between the 2-norm of a matrix and eigenvalues of the corresponding symmetrized matrix.

**Lemma 4.2.** ([18]) Let \( L \in \mathbb{R}^{n \times n} \) and \( \hat{L} = \frac{1}{2}(L^T + L) \), then eigenvalues, \( \lambda_i(\hat{L}) \), of \( \hat{L} \) satisfy \( \lambda_i(\hat{L}) \leq \|L\| \) for all \( i = 1, \cdots, n \).

Before stating Theorem 4.3, we introduce several notations, which are useful to simplify the proof of the Theorem 4.3. Let \( M = \sum_{i=1}^{n} \frac{1}{m_i} \) and \( J = \frac{1}{M} 11^T \frac{1}{M} \), where \( \frac{1}{m_i} = \text{diag}(\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_n}) \).

Define a consensus error of the MTS network

\[
\delta(k) = (I_n - J)X(k),
\]

and a energy function of the MTS network

\[
V_0(k) = \delta^T(k)\delta(k).
\]
Now, we present Theorem 4.3, which claims that the consensus error of the MTS network will decay exponentially in mean-square sense.

**Theorem 4.3.** Let $G$ be a strongly connected and balanced digraph. Applying the algorithm Eq. (11), the energy function, $V_0(k)$, of a MTS network satisfies

$$\mathbb{E}V_0(k) \to 0 \text{ as } k \to \infty,$$

if the consensus gain $\beta$ satisfies condition Eq. (12).

**Proof.** Define a weighted energy function of consensus error (called a weighted Lyapunov function)

$$V(k) = \delta^T(k) \frac{1}{m} \delta(k).$$

From Eq. (11), we compute dynamics of the consensus error

$$\delta(k + 1) = (I_n - J)X(k + 1)$$

$$\delta(k + 1) = (I_n - J)\{X(k) - \beta mL_G X(k) - \beta \eta(k)L_G X(k)\}$$

$$\delta(k + 1) = (I_n - J)X(k) - \beta(I_n - J)mL_G X(k)$$

$$\delta(k + 1) = (I_n - J)X(k) - \beta(I_n - J)\eta(k)L_G X(k)$$

$$\delta(k + 1) = \delta(k) - \beta mL_G \delta(k) - \beta(I_n - J)\eta(k)L_G \delta(k).$$

From Eq. (14), we compute dynamics of the weighted Lyapunov function

$$V(k + 1) = \delta^T(k + 1) \frac{1}{m} \delta(k + 1)$$

$$V(k + 1) = \delta^T(k)(I_n - \beta mL_G)^T \frac{1}{m} (I_n - \beta mL_G) \delta(k)$$

$$V(k + 1) = \delta^T(k)(I_n - \beta mL_G)^T \frac{1}{m} (I_n - J) \eta(k)L_G \delta(k)$$

Based on matrix calculation, we have

$$\eta(k)(I - J) \frac{1}{m} (I - J) \eta(k) = \frac{\eta(k)}{m}(m - \frac{11^T}{M}) \eta(k).$$

(15)
where \( \eta(k) \) is \( \text{diag}(\eta_1(k), \eta_2(k), \ldots, \eta_m(k)) \). Taking expectation on both sides of Eq. (15) by considering the fact that \( \eta_i \sim \mathcal{N}(0, s^2_i) \), we get

\[
\mathbb{E}\{\eta(k)(I_n - J)(I_n - J)\eta(k)\} = \frac{s^2}{m} - \frac{1}{M} s^2 m^2,
\]

where \( \frac{s^2}{m} = \text{diag}(\frac{s^2_1}{m_1}, \ldots, \frac{s^2_m}{m_m}) \) with \( j = 1, 2 \). Also noting the fact that \( \eta(k) \) is independent of \( \delta(k) \), take expectation of \( V(k + 1) \) to obtain

\[
\mathbb{E}V(k + 1) = \mathbb{E}V(k) - \beta \mathbb{E}(\delta^T(k)(L_G^T + L_G)\delta(k)) \\
+ \beta^2 \mathbb{E}(\delta^T(k)L_G^T m L_G \delta(k)) \\
+ \beta^2 \mathbb{E}(\delta^T(k)L_G^T s^2 m (I_n - \frac{1}{M} L_G) \delta(k)).
\]

We consider the above right hand term by term. The first term is well-defined and the second term, \( R_2 \), can be rewritten

\[
R_2 = \mathbb{E}(\delta^T(k) \sqrt{\frac{1}{m}} \sqrt{m} (L_G^T + L_G) \sqrt{\frac{1}{m}} \delta(k)),
\]

where \( \sqrt{\frac{1}{m}} = \text{diag}(\frac{1}{\sqrt{m_1}}, \ldots, \frac{1}{\sqrt{m_m}}) \) and \( \sqrt{\frac{1}{m}} \sqrt{m} = I_n \). From Lemma 4.1, we know that \( \lambda_1 = 0 \) is a simple eigenvalue of the quasi-Laplacian matrix \( \frac{1}{2} \sqrt{m}(L^T + L) \sqrt{m} \) associated with a eigenvector \( e_1 = [\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_m}}]^T \) and \( \lambda_2 > 0 \) is the second smallest eigenvalue. Also we notice that the inner product of the eigenvector \( e_1 \) and \( \sqrt{\frac{1}{m}} \delta(k) \) is

\[
e_1^T \sqrt{\frac{1}{m}} \delta(k) = \left[\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_m}}\right] \sqrt{\frac{1}{m}} \delta(k) = 0,
\]

which indicates that \( \sqrt{\frac{1}{m}} \delta(k) \) is from the perpendicular space of null space of the quasi-Laplacian matrix. By Rayleigh quotient theorem, we have

\[
\delta^T(k) \sqrt{\frac{1}{m}} \sqrt{m} (L_G^T + L_G) \sqrt{\frac{1}{m}} \delta(k) \geq 2 \lambda_2 V(k).
\]

Thus, we have the following lower bound for the second term:

\[
R_2 = \mathbb{E}(\delta^T(k)(L_G^T + L_G)\delta(k)) \geq 2 \lambda_2 \mathbb{E}V(k).
\]

(17)

For the third term, we also rewrite it

\[
R_3 = \mathbb{E}\{\delta^T(k) \sqrt{\frac{1}{m}} \sqrt{m} L_G^T m L_G \sqrt{\frac{1}{m}} \delta(k)\}.
\]

\]
From Lemma 4.2, we have

\[ R_3 = \mathbb{E}(\delta^T(k)L_G^TmL_G\delta(k)) \leq \|\sqrt{m}L_G\sqrt{m}\|^2\mathbb{E}V(k). \]  

(18)

For the last term, we have since \( I_n - \frac{1}{m}m \) is positive definite and all eigenvalues are less than one, we get the following inequality similarly

\[ R_4 \leq \max_{1 \leq i \leq n} \frac{s_i^2}{m_i} \|\sqrt{m}L_G\sqrt{m}\|^2\mathbb{E}V(k). \]  

(19)

Therefore, together with Eqs. (16)-(19), we have

\[ \mathbb{E}V(k+1) \leq (1 - 2\beta\lambda_2^{1/2} + \beta^2\|\sqrt{m}L_G\sqrt{m}\|^2)EV(k) + \beta^2 \max_{0 \leq i \leq n} \frac{s_i^2}{m_i^2} \|\sqrt{m}L_G\sqrt{m}\|^2 EV(k). \]  

(20)

Let

\[ \gamma(\beta) = (1 - 2\beta\lambda_2 + \beta^2\|\sqrt{m}L_G\sqrt{m}\|^2) \]

\[ + \beta^2 \max_{0 \leq i \leq n} \frac{s_i^2}{m_i^2} \|\sqrt{m}L_G\sqrt{m}\|^2. \]

So \( \gamma(\beta) \) is a quadratic function with the minimum value

\[ \gamma^* = 1 - \frac{\lambda_2^2}{\|\sqrt{m}L_G\sqrt{m}\|^2 + \max_{1 \leq i \leq n} \frac{s_i^2}{m_i^2} \|\sqrt{m}L_G\sqrt{m}\|^2} \]

when

\[ \beta^* = \frac{\lambda_2}{\|\sqrt{m}L_G\sqrt{m}\|^2 + \max_{1 \leq i \leq n} \frac{s_i^2}{m_i^2} \|\sqrt{m}L_G\sqrt{m}\|^2}. \]

Since \( \lambda_2 > 0 \) is the second smallest eigenvalue of quasi-Laplacian matrix, \( \frac{1}{2}\sqrt{m}(L_G^T + L_G)\sqrt{m} \), we know from Lemma 4.2 that

\[ \frac{\lambda_2^2}{\|\sqrt{m}L_G\sqrt{m}\|^2} \leq \frac{1}{4}, \]

which implies

\[ 0 < \gamma^* < 1. \]

Therefore,

\[ 0 < \gamma(\beta) < 1 \]  

(21)
when
\[ 0 < \beta < \frac{2\lambda_2}{\|\sqrt{mL_G}\sqrt{m}\|^2 + \max_{1 \leq i \leq n} \frac{\beta^2}{m_i} \|\sqrt{mL_G}\sqrt{m}\|^2}. \]

By fixed point theorem, it gets from Eq. (20) that
\[ EV(k) \to 0 \text{ as } k \to \infty. \]

Also since for the energy function \( V_0(k) \), it is bounded by
\[ EV_0(k) \leq \max_{1 \leq i \leq n} m_i EV(k). \]

This implies
\[ EV_0(k) \to 0 \text{ as } k \to \infty. \]

Theorem 4.3 implies that each agent of the MTS network will approach to a convex combination of states values of all agents. For this convex combination, it is shown to be convergent to a random variable in Theorem 4.4.

**Theorem 4.4.** Let \( G \) be a strongly-connected and balanced digraph. Applying the protocol in Eq. (11) and condition in Eq. (12) for positive consensus gain, \( \beta \), it holds for \( i = 1, \cdots, n \), that
\[ \mathbb{E}\|X_i(k) - X^*\|^2 \to 0 \text{ as } k \to \infty, \]

where
\[ X^* = \frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} X_i(0) - \sum_{k=0}^{\infty} \frac{1}{m}^T \eta(k) L_G X(k). \]

**Proof.** From Eq. (11), we have
\[ \frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} X_i(N) = \frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} X_i(0) - \frac{\beta}{M} \sum_{k=1}^{N-1} \frac{1}{m}^T \eta(k) L_G X(k), \]

where \( \frac{1}{m} = [\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_n}]^T. \)
Let $Y(N) = \sum_{k=0}^{N-1} \frac{1}{m}^T \eta(k) L_G X(k)$. Since $\eta(k)$ is independent of $X(k)$, we have the second moment of $Y(N)$

$$
\mathbb{E} \|Y(N)\|^2 = \sum_{k=0}^{N-1} \mathbb{E} \{X^T(k) L_G^T \frac{s^2}{m^2} L_G X(k)\}
$$

$$
= \sum_{k=0}^{N-1} \mathbb{E} \{X^T(k)(I_n - J^T)\sqrt{\frac{1}{m}}
$$

$$
\times \sqrt{m} L_G^T \frac{s^2}{m^2} L_G \sqrt{m} \left(\frac{1}{m}(I_n - J)X(k)\right)\}
$$

$$
\leq \| \frac{s}{m} L_G \sqrt{m} \|^2 \sum_{k=0}^{N-1} \mathbb{E} V(k) < \infty,
$$

(23)

for all $N$. This is because the sequence of $\mathbb{E} V(k)$ is a geometric series with ratio $0 < r < 1$.

Let $F_{N-1}$ is a $\sigma$–algebra generated by the family of random variables $\{Y(k), k = 0, 1, \cdots, N-2\}$ and compute its conditional expectation

$$
\mathbb{E}(Y(N)|F_{N-1}) = \mathbb{E}(Y(N-1)
$$

$$
+ \frac{1}{m}^T \eta(N-1) L_G X(N-1)|F_{N-1}) \quad (24)
$$

$$
= Y(N-1).
$$

From Eq. (23)-(24), $Y(N)$ is a martingale.

Also since $\mathbb{E} V(k)$ is a geometric series, we have

$$
\sup_{N \geq 0} \mathbb{E} \|Y(N)\|^2 < \infty.
$$

So, by Lemma 2.1, we know that $Y(N)$ converges in mean-square and almost-sure sense to a random variable denoted by $Y^* = \sum_{k=0}^{\infty} \frac{1}{m}^T \eta(k) L_G X(k)$. Therefore, from Eq. (22), we know that

$$
\frac{1}{M} \sum_{i=1}^{\infty} \frac{1}{m_i} X_i(N) \text{ is also convergent in mean-square and almost-sure sense to a random variable denoted by}
$$

$$
X^* = \frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} X_i(0) - \sum_{k=0}^{\infty} \frac{1}{m}^T \eta(k) L_G X(k).
$$

From Theorem 4.3 we know that

$$
\mathbb{E} \|X(k) - JX(k)\|^2 \to 0 \text{ as } k \to \infty.
$$

Thus

$$
\mathbb{E} \|X_i(k) - X^*\|^2 \to 0 \text{ as } k \to \infty
$$
for all $i = 1, 2, \cdots, n$. This means that mean-square consensus is achieved.

Based on pieces built before in this section, we are ready to discuss our main result, Theorem 4.2.

**Proof.** From Theorem 4.3 and 4.4, we know that every agent in the MTS network reaches an agreement in mean-square sense if the consensus gain, $\beta$, satisfies condition, Eq. (12). This means that the stochastic MTS network achieves mean-square consensus. The agreement is the random variable $X^*$ defined in Theorem 4.4. Now, we compute its mean and variance

$$E X^* = \frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} X_i(0)$$

and

$$Var(X^*) = \frac{\beta^2}{M^2} \lim_{N \to \infty} E \|Y(N)\|^2$$

$$\leq \frac{\beta^2}{M^2} \frac{G}{m} \|s^{mL_G} G \|^2 \lim_{N \to \infty} \sum_{k=0}^{N-1} E V(k)$$

$$= \frac{\beta^2}{M^2} \frac{G}{m} \|s^{mL_G} G \|^2 \frac{1}{1 - \gamma(\beta)} V(0) < \infty.$$

Until now, it has shown that if the condition, Eq. (12), is satisfied for consensus gain, $\beta$, the algorithm, Eq. (11), on the stochastic MTS network achieves mean-square consensus. However, in some applications, it is more interested to derive almost-sure consensus since the observation of each experiment is a sample path. To this end, Theorem 4.5 shows that the algorithm, Eq. (11), is also able to drive the MTS network to reach an agreement almost surely if the condition, Eq. (12), is hold.

**Theorem 4.5.** If the positive consensus gain, $\beta$, satisfies the condition, Eq. (12), then applying the protocol, Eq. (11), on the MTS network achieves consensus almost surely.

**Proof.** Let $V(k)$ be a weighted Lyapunov function as defined in Theorem 4.3. From Chebyshev’s inequality, we have

$$\sum_{k=0}^{\infty} P(V(k) > \epsilon) \leq \sum_{k=0}^{\infty} \frac{EV(k)}{\epsilon} < \infty.$$
By Borel-Cantelli Lemma, we have \( P(V(k) > \epsilon \ i.o.) = 0 \). This means \( V(k) \to 0 \ a.s. \). From Theorem 4.4 we know that

\[
\frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} X_i(k) \to X^* \ a.s.,
\]

where \( X^* \) is defined in Theorem 4.4. Therefore, for each agent, \( i, X_i(k) \to X^* \) as \( k \to \infty \) almost surely and this means almost sure consensus is assured.

We have shown that the algorithm, Eq. (7), on the MTS network drives all agents to achieve an agreement, which is related to both local clocks and initial states. If local clocks of agents in the MTS network are affected by white noise, the associated stochastic approximation type algorithm, Eq. (11), is designed. From convergence analysis, we find that there is no need to require agents to satisfy the local clock condition, \( \sum_{j \in N_i} a_{ij} < \frac{1}{m_i} \), for \( i = 1, \cdots, n \). In this sense, stochastic algorithm is more robust to local clocks and it guarantees that the stochastic MTS network reaches consensus in mean-square and almost-sure sense. However, the analysis is not applicable to drive all agents in the MTS network to achieve average-consensus. The next section considers the average-consensus on the MTS network.

5. Average Consensus of the MTS Network

It is discussed in [5] that strongly-connected and balanced graph structure is essential to guarantee average-consensus on a multi-agent system. However, performance of agents according to local clocks destroys the balanced structure of the update matrix. This is the main reason why average-consensus is not guaranteed in Section 4. For a general digraph, Kai and Hideaki in [6] proposed a surplus method to deal with unbalanced digraphs and average consensus is achieved. This method strongly depends on the property of surplus variables, which are able to record state change of associated agents. Unfortunately, this recording property is also destroyed because of local clocks. To fix this problem, we propose a generalized surplus method, called local-time dependent weight surplus method (LTDWS), which is efficient to drive a MTS network to achieve average consensus.

5.1. Model Description

Consider a MTS network represented by a weighted digraph, \( G = (\mathcal{V}, \mathcal{E}, A) \), which is strongly-connected. Also let \( L_G \) be its weighted Laplacian matrix. For each agent, \( i, \) in the network, it has in-neighbors and out-neighbors denoted by \( N_i^+ := \{ j \in \mathcal{V} : (j, i) \in \mathcal{E} \} \) and \( N_i^- := \{ h \in \mathcal{V} : (i, h) \in \mathcal{E} \} \),
respectively. For each agent, $i$, at local time $t_i$, there is an additional variable, $s_i(t_i) \in \mathbb{R}$, called surplus; let $s_i(0) = 0$ for all $i = 1, 2, \cdots, n$. According to signals of reference clock, the performance of agent, $i$, is described as following:

**Step 1:** Agent $i$ reads its local clock $t_i(k) = m_i k T$;

**Step 2:** Agent $i$ sends its state value, $x_i(t_i(k))$, and local-time dependent weighted surplus, $t_i(k) b_{ih} s_i(t_i(k))$, to out-neighbors, $v_h \in N^-_i$, at its local time, $t_i(k)$. The weight, $b_{ih}$, here satisfies that $b_{ih} \in (0,1)$ if $v_h \in N^-_i$, $b_{ih} = 0$ if $v_h \notin N^-_i$ and $\sum_{v_h \in N^-_i} b_{ih} < 1$;

**Step 3:** Agent $i$ receives state value, $\hat{x}_j(t_i(k))$, and local-time dependent weighted surplus, $t_j(k) b_{ji} \hat{s}_j(t_i(k))$, from each in-neighbors, $v_j \in N^+_i$, at its local time, $t_i(k)$. (Remember $\hat{x}_j(t_i(k)) = x_j(t_j(k))$ and $\hat{s}_j(t_i(k)) = s_j(t_j(k))$);

**Step 4:** Agent $i$ updates its state value, $x_i(t_i(k))$, and surplus, $s_i(t_i(k))$, at its local time, $t_i(k)$:

$$x_i(t_i(k + 1)) = x_i(t_i(k)) + \sum_{v_j \in N^+_i} a_{ij}(x_j(t_j(k)) - x_i(t_i(k))) \{t_i(k + 1) - t_i(k)\} + \epsilon s_i(t_i(k)) \{t_i(k + 1) - t_i(k)\} ;$$

$$s_i(t_i(k + 1)) = s_i(t_i(k)) - \frac{1}{t_i(k)} \sum_{v_h \in N^-_i} t_i(k) b_{ih} s_i(t_i(k)) \{t_i(k + 1) - t_i(k)\} + \frac{1}{t_i(k)} \sum_{v_j \in N^+_i} t_j(k) b_{ji} \hat{s}_j(t_i(k)) \{t_i(k + 1) - t_i(k)\} - \{x_i(t_i(k + 1)) - x_i(t_i(k))\} ;$$

where $\epsilon$ is a positive number which indicates the amount of states needed to be recorded.

The major difference between our LTDWS method and the surplus method proposed in [6] is that each agent, $i$, in LTDWS method reads its local clock at every step and embeds it into the weights of its surplus variable. So when the agent, $i$, sends its weighted surplus to its out-neighbors, the local clock information is also embedded. This local-time dependent weighted surplus variable is now capable of recording and preserving state change of agent, $i$, since performance of agent, $i$, is only based on its local clock. Without local clocks involved, the surplus method is unable to record and preserve state change of associated agents.
Similarly, assume that the period of signal, $T = 1$. Let $X_i(k) = x_i(t_i(k))$ and $S_i(k) = s_i(t_i(k))$; $B := [b_{ih}]^T \in \mathbb{R}^{n \times n}$, where $b_{ih}$ is the weight for a surplus variable. Also let $\bar{D} = \text{diag}(\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_n)$ with $\bar{d}_i = \sum_{h=1}^{n} b_{ih}$ and define $P := I_n - \bar{D}\text{diag}(m) + B\text{diag}(m)$. It can be verified that $P$ has a column sum equal to 1. Therefore, we get the compact matrix form

$$
\begin{bmatrix}
X(k+1) \\
S(k+1)
\end{bmatrix}
= M \begin{bmatrix}
X(k) \\
S(k)
\end{bmatrix},
$$

where

$$M = \begin{bmatrix}
I - \text{diag}(m)L & \epsilon \text{diag}(m) \\
\text{diag}(m)L & P - \epsilon \text{diag}(m)
\end{bmatrix}.$$

Based on the LTDWS algorithm, Eq. (25), the following Theorem 5.1 presents the underlying graph structure and local-clock conditions for the MTS network to achieve average-consensus.

**Theorem 5.1.** Suppose parameter, $\epsilon$, is sufficiently small and for each agent, $i$, its local clock skew, $m_i$, satisfies $\sum_{j=1}^{n} a_{ij} < \frac{1}{m_i}$ and surplus weights, $\sum_{h=1}^{n} b_{ih} < \frac{1}{m_i}$. Applying algorithm Eq. (25), the MTS network achieves average-consensus if the interaction topology is strongly connected.

**Proof.** First, let

$$M_0 := \begin{bmatrix}
I - \text{diag}(m)L & 0 \\
\text{diag}(m)L & P
\end{bmatrix} \quad \text{and} \quad F := \begin{bmatrix}
0 & \text{diag}(m) \\
0 & -\text{diag}(m)
\end{bmatrix}.$$

Then $M = M_0 + \epsilon F$, which is obtained from the matrix $M_0$ corrupted by $\epsilon F$. Since $M_0$ is the block lower triangular, its spectrum is a combination of spectrum of $I_n - \text{diag}(m)L_G$ and spectrum of $P$. For each agent, $i$, since its local clock skew, $m_i$, satisfies $\sum_{j=1}^{n} a_{ij} < \frac{1}{m_i}$ and $\sum_{h=1}^{n} b_{ih} < \frac{1}{m_i}$, it implies that $I - \text{diag}(m)L$ and $P = I - \bar{D}\text{diag}(m) + B\text{diag}(m)$ are both nonnegative; Also it is checked that $I_n - \text{diag}(m)L_G$ is row stochastic, $P$ is column stochastic and then both of their spectral radii equal to one, i.e., $\rho(I_n - \text{diag}(m)L_G) = \rho(P) = 1$. Due to the fact that $G$ is strongly connected, $L_G$ and $P$ are irreducible. Therefore, by Perron-Frobenius theorem [17], we know $\rho(I_n - \text{diag}(m)L_G) = \rho(P) = 1$ are both simple eigenvalues of $I_n - \text{diag}(m)L_G$ and $P$. This means that all eigenvalues of $M_0$ can be ordered as

$$|\lambda_{2n}| \leq \cdots |\lambda_3| < \lambda_2 = \lambda_1 = 1.$$
Let \( \lambda_{2n}(\epsilon), \cdots \lambda_3(\epsilon), \lambda_2(\epsilon), \lambda_1(\epsilon) \) be eigenvalues of \( M \) corresponding to the eigenvalues of unper-
turbed matrix \( M_0 \). Now, from the matrix perturbation theory \[19\] and the fact that eigenvalues are continuously depending on matrix entries, we find that for sufficiently small \( \epsilon \), \( \lambda_1(\epsilon) = 1 \) keeps stay while \( \lambda_2(\epsilon) = 1 \) moves to the left along the real axis; also for all \( i = 3, 4, \cdots, 2n, |\lambda_i(\epsilon)| < 1 \). This implies that the eigenvalue 1 of \( M \) is simple and other eigenvalues have moduli smaller than one. Therefore,

\[
\begin{bmatrix}
X(k) \\
S(k)
\end{bmatrix} = M^k \begin{bmatrix}
X(0) \\
S(0)
\end{bmatrix} \to w_r w^T_l \begin{bmatrix}
X(0) \\
S(0)
\end{bmatrix}, \quad \text{as } k \to \infty.
\]

where \( w_r = [1^T 0]^T \) is the right eigenvector associated with the eigenvalue 1 of \( M \); \( w_l = (1_n [1^T 1^T]^T) \) is the left eigenvector corresponding to the eigenvalue 1 of \( M \). This means

\[
\begin{bmatrix}
X(k) \\
S(k)
\end{bmatrix} \to \begin{bmatrix}
\frac{1}{n} 11^T X(0) \\
0
\end{bmatrix}, \quad \text{as } k \to \infty.
\]

\[\square\]

**Remark 2.** For the details of movements of eigenvalues of matrix \( M \), we refer the reader to [6]. The convergence of the power of \( M \) is related Jordan canonical decomposition. Actually, similar to [6], it can be qualified that the strongly connected digraph \( G \) is also a necessary condition for the multi-time scale network to achieve an average consensus under the algorithm [25].

The perturbation parameter, \( \epsilon \), in Theorem 5.1 is small and there is a sufficient bound for \( \epsilon \) given by \( \bar{\epsilon} = \frac{(1-|\lambda_3|)^n}{(24+8 \sum_{i=1}^{n} m_i + 4 \max_{1 \leq i \leq n} m_i)^n} \) to guarantee the average consensus of LTDWS algorithm, Eq. (25), on the MTS network. As stated in Theorem 5.1, the required underlying graph structure of average-consensus on the MTS network is strong connectivity. But conditions \( \sum_{j=1}^{n} a_{ij} < \frac{1}{m_i} \) and \( \sum_{h=1}^{n} b_{ih} < \frac{1}{m_i} \) of local clocks are needed to guarantee average-consensus.

Now let’s discuss the range of parameter \( \epsilon \) in Theorem 5.1 to guarantee the average consensus on a MTS network. The following lemma provides a sufficient bound for perturbation parameter \( \epsilon \) to guarantee the average consensus of algorithm Eq. (25).

**Lemma 5.1.** Suppose for each agent \( i \), its local clock skew \( m_i \) satisfies \( \sum_{j=1}^{n} a_{ij} < \frac{1}{m_i} \) and \( \sum_{h=1}^{n} b_{ih} < \frac{1}{m_i} \). If the parameter \( \epsilon \in (0, \bar{\epsilon}) \) where \( \bar{\epsilon} = \frac{(1-|\lambda_3|)^n}{(24+8 \sum_{i=1}^{n} m_i + 4 \max_{1 \leq i \leq n} m_i)^n} \), then \( |\lambda_2(\epsilon)|, |\lambda_4(\epsilon)|, \cdots, |\lambda_{2n}(\epsilon)| < 1 \).
The proof of Lemma 5.1 is depending on the following optimal matching metric and lemmas. We first introduce a metric for the distance between the spectrums of $M_0$ and $M$ where $M = M_0 + \epsilon F$. Let $\sigma(M_0) := \{\lambda_1, \cdots, \lambda_n\}$ and $\sigma(M) := \{\lambda_1(\epsilon), \cdots, \lambda_n(\epsilon)\}$. The optimal matching distance $d(\sigma(M_0), \sigma(M))$ is defined by

$$d(\sigma(M_0), \sigma(M)) := \min_\pi \max_{1 \leq i \leq 2n} |\lambda_i - \lambda_{\pi(i)}(\epsilon)|,$$

where $\pi$ is taken over all permutations of $\{1, 2, \cdots, 2n\}$. Based on the above definition, $d(\sigma(M_0), \sigma(M))$ is the smallest radius for any circle centering at any $\lambda_i$, $i = 1, \cdots, 2n$ such that all $\lambda_1(\epsilon), \cdots, \lambda_{2n}(\epsilon)$ are falling into the circle. Since for two stochastic matrices, one is always an eigenvalue for them, we propose a modified optimal matching distance between

$$d_0(\sigma(M_0), \sigma(M)) := \min_\pi \max_{2 \leq i \leq 2n} |\lambda_i - \lambda_{\pi(i)}(\epsilon)|,$$

where $\pi$ is taken over all permutations of $\{2, 2, \cdots, 2n\}$.

For an optimal matching distance between $M_0$ and $M = M_0 + \epsilon F$, there is an upper bound.

**Lemma 5.2.**

$$d(\sigma(M_0), \sigma(M)) \leq 4\left(\|M_0\|_\infty + \|M\|_\infty\right)^{1 - \frac{1}{n}} \|\epsilon F\|_\infty^{\frac{1}{n}}.$$

Since $M$ is column stochastic, one is always its eigenvalue, say $\lambda_1(\epsilon) = 1$. Next, based on the modified optimal matching distance, we claim that the remaining $\lambda_2(\epsilon), \cdots, \lambda_{2n}(\epsilon)$ have moduli less than one for small $\epsilon$.

**Proof.** Since $L_G = D - A$ is the Laplacian of a digraph $G$ and local clock skew $m_i$ satisfies $\sum_{j=1}^n a_{ij} < \frac{1}{m_i}$, it holds

$$\|\text{diag}(m)L_G\|_\infty = 2 \max_{1 \leq i \leq n} \frac{m_i}{\sum_{j=1}^n a_{ij}} < 2;$$

Also $P = I_n - \dot{D}\text{diag}(m) + B\text{diag}(m)$ and local clock skew $m_i$ satisfies $\sum_{h=1}^n b_{ih} < \frac{1}{m_i}$, we compute

$$\|P\|_\infty = \|I_n - \dot{D}\text{diag}(m) + B\text{diag}(m)\|_\infty = \max_{1 \leq i \leq n} \left\{ (1 - m_i \sum_{h=1}^n b_{ih}) + \sum_{j=1}^n b_{ij}m_j \right\} \leq 1 + \sum_{j=1}^n m_j.$$
Furthermore, we have \( \|M_0\|_{\infty} \leq \|\text{diag}(m)L_G\|_{\infty} + \|p\|_{\infty} \) and \( \|F\|_{\infty} \leq \max_{1 \leq i \leq n} m_i \). On the other hand, according to the definitions of optimal matching distance and modified optimal matching definition, we compute

\[
d_0(\sigma(M_0), \sigma(M)) \leq d(\sigma(M_0), \sigma(M))
\]

\[
\leq 4(2\|M_0\|_{\infty} + \epsilon\|F\|_{\infty})^{1-\frac{1}{n}} (\epsilon\|F\|_{\infty})^{\frac{1}{n}}
\]

\[
\leq 4(2(3 + \sum_{i=1}^{n} m_i) + \epsilon \max_{1 \leq i \leq n} m_i)^{1-\frac{1}{n}} \times (\max_{1 \leq i \leq n} m_i)^{\frac{1}{n}}
\]

\[
\leq (24 + 8\sum_{i=1}^{n} m_i + 4\epsilon \max_{1 \leq i \leq n} m_i)\epsilon^\frac{1}{n}
\]

\[
< 1 - |\lambda_3|.
\]

From the proof of Theorem 5.1, it is checked that eigenvalues \( \lambda_2, \lambda_3, \cdots, \lambda_{2n} \) of \( M_0 \) satisfies

\[
|\lambda_{2n}| \leq \cdots |\lambda_3| < \lambda_2 = 1.
\]

Thus, for its corresponding perturbing eigenvalues, we have

\[
|\lambda_i(\epsilon)| \leq |\lambda_i(\epsilon) - \lambda_3| + |\lambda_3| \leq d(\sigma(M_0), \sigma(M)) + |\lambda_3| < 1
\]

for all \( 2 \leq i \leq 2n \). This implies \( |\lambda_2(\epsilon)|, |\lambda_3(\epsilon)|, \cdots, |\lambda_{2n}(\epsilon)| < 1 \).

6. Simulations

In this section, we provide numerical examples to illustrate the convergence of algorithms, Eqs. (7), (11), and (25), on a MTS network represented by strongly-connected digraphs, Fig. 2, where \( G_a \) is balanced but \( G_b \) is unbalanced. The digraph, \( G_a \), has weight \( \frac{1}{3} \) for each edge and \( G_b \) has the uniform weights as described in [6].

![Figure 2: The MTS network described by strongly-connected digraphs: \( G_a \) is balanced while \( G_b \) is unbalanced.](image)

Fig. 3 shows the convergence of the algorithm, Eq. (7), on the strongly-connected and balanced digraph, \( G_a \), with uncorrupted local clocks. The top figure belongs to a MTS network with local
clocks satisfying the local clock condition, $m_i \sum_{j \in N_i} a_{ij} < 1$, for all $i = 1, \cdots, n$. The bottom figure is associated with the same MTS network but incorporates a small constant, 0.05, to adjust local clocks. So the adjusted MTS network evolves slower than the original MTS network. This property is also revealed by comparing the convergence speed in Fig. 3.

**Figure 3:** Convergence path of states: are obtained by applying the distributed algorithm, Eq. (7), on digraph, $G_a$, with uncorrupted local clocks.

Fig. 4 shows the convergence of the algorithm, Eq. (11), on the strongly connected and balanced digraph, $G_a$, with local clocks corrupted by standard Gaussian noise. The top figure is associated with a MTS network, which clock skews satisfy the local clock condition, $m_i \sum_{j \in N_i} a_{ij} < 1$, for all $i = 1, \cdots, n$. However, the bottom figure is associated with the MTS network, where local clocks do not follow such condition. Both MTS networks can achieve consensus, which implies that stochastic MTS network is robust to local clocks. Also, we can find that the MTS network satisfying the local clock condition achieves consensus faster.

Fig. 5 shows the comparison between the convergence of the LTDWS algorithm, Eq. (25), and the algorithm proposed in [6] on the strongly connected but unbalanced digraph, $G_b$, with local clocks. The average-consensus displayed in the top figure is from the LTDWS algorithm; the consensus produced by algorithm in [6] in the bottom figure is not average. In this sense, the LTDWS algorithm is more general.
Figure 4: Sample path convergence of states: are obtained by applying the distributed algorithm, Eq. (11), on digraph, $G_a$, with corrupted local clocks.

Figure 5: Convergence of states: are obtained by applying the LTDWS algorithm, Eq. (25), and the algorithm proposed in [6] on digraph, $G_b$, with uncorrupted local clocks.
7. Conclusion

In this paper, the consensus problem on the MTS network has been investigated in different situations. For an interaction topology which is strongly connected and balanced, both deterministic and stochastic algorithms have been proposed for the MTS network to achieve an agreement. Consensus of the deterministic algorithm depends on both local clocks and initial states; By taking into account corrupted local clocks, consensus of the stochastic algorithm is a random variable which has a finite mean and variance. In addition, by nonnegative matrix theory and the martingale convergence theorem, consensus on the MTS network with corrupted local clocks is achieved in mean square and almost surely. For an interaction topology which is a strongly connected digraph without balance, we have proposed a local-clock dependent algorithm to guarantee average-consensus on a MTS network with local clocks, which is more general than those reported in the existing literature for an unbalanced digraph.

References


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