Summary

In order to solve the problem of optimizing brownie yield per bake cycle against even heat distribution, we came up with reasonable pan shapes to compare, and solve 2 major problems. Once pan shapes are picked, they must be checked against various oven shapes to see which pans yield the most brownie matter per bake cycle in each oven. In addition to optimizing brownie yield, each pan shape needs to be evaluated at various sizes for evenness and quality of baking. When these data are known, it is a simple matter of evaluating each size shape pan combination for evenness and bake cycle yield for a particular oven, and an optimal pan can be calculated.

We selected 6 pan shapes, including square, 3x2 rectangular, rounded square, hexagonal, rounded hexagonal and circular pans, and evaluated them for yield and evenness. Using geometric methods, we calculated the maximum number of each pan shape and size combination that could fit in an oven of dimensions W x L. In general we found that squares and rectangles fit very well into the oven (have the best space efficiency), hexagons, rounded hexagons, and rounded rectangles fit moderately well, and circles fit poorly.

To calculate how evenly a particular pan shape and size would cook, we modeled a brownie as a collection of discrete points in 3-dimensional space. For each point, we maintain a value for temperature, water content, and cookedness, and create a differential equation with respect to time for each value. The equations for change in water content and cookedness over time are parameterized by temperature, water content, and cookedness, and the equation for change in temperature over time is based on Newton's Law of Cooling, applied to each nearby point in the models, and the interior of the oven and brownie pan for the edges of the model. We step through time discretely in the model, and simulate in 1 second slices the entirety of the cooking process, stopping the simulation when the brownie begins to burn more rapidly than the center cooks.

In this paper we provide a detailed and mathematically rigorous explanation of how we measure and calculate both the space efficiency and heating evenness of an arbitrary brownie pan, and provide a technique for optimizing a pan with respect to both parameters.

A mathematician's Quest in Pursuit of the Ultimate Brownie
On the Subject of the Optimization of Chocolate Based Dessert Cookware

February 3rd, 2013

Abstract

Space efficiency and heating evenness are both essential to consider when one bakes brownies. We develop a mathematical model to determine ultimate shapes for brownies pans of size A when the oven is rectangular shaped with width W and length L.

We present 6 different shapes of brownie pans in the baking oven. Considering two conditions that trades off reciprocally, we build mathematical models and run simulations to illustrate our main conclusion—rectangular pans are best choices for space efficiency; circled pans are the most desired for heating evenness; and shapes in between such as hexagons, rounded squares and rounded hexagons can be chosen when weights of the previous two conditions are assigned.

We selected 6 pan shapes, including square, 3x2 rectangular, rounded square, hexagonal, rounded hexagonal and circular pans, and evaluated them for yield and evenness. Using geometric methods, we calculated the maximum number of each pan shape and size combination that could fit in an oven of dimensions W x L.

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1 Introduction

The brownie is one of the most popular American desserts, but unfortunately, the traditional method of preparation is fatally flawed. Brownies are generally baked in a pan that is approximately a rectangular prism; however, this shape is not optimal with respect to evenness of heating. Heat quickly reaches the corners, and they burn, while the middle remains cool. A true connoisseur of brownies finds these imperfections highly undesirable, so in this paper we present methods to optimize brownie pans with respect to maximum yield per oven cycle and optimally even heat distribution.

2 Model

2.1 Assumptions

Our model has the following assumptions:

1. A width to length ratio of *W/L* for the oven which is rectangular in shape.
2. Each pan must have an area of *A*.
3. Initially two racks in the oven evenly spaced [6].
4. The oven is a conduction oven that blows air around. The raw brownie is put into the oven when the oven's temperature is 350Fº.
5. Baking time is around 30 minutes.
6. Our pans of all shapes are proportionally smaller at the very bottom.
7. In order to solve the problem realistically, we look carefully at a recipe of baking brownies [1]. The recipe indicates that raw brownie mixture is mostly composed of chocolate, butter, flower, and eggs. According to the data, we calculate the proportion of major ingredients to be approximately 40% oil, 40% dry ingredients, and 20% water. Hence, the value of k changes while material changes i.e. water percentage decreases and temperature rises.

2.2 Variables

Our model includes the following variables and parameters (in cm):

W=width of the oven: 50 to 200

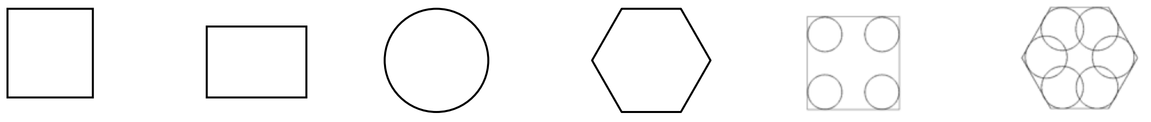
L=length of the oven: 50 to 200

A=area of the each brownies pan: 100 to 1000

t=parameter to determine the radius of rounding circles a: 0 to 1(no units)

2.3 Geometric Model—Space Efficiency

While it is obvious that rectangular pans will be the most space efficient shape of the pans for various W/L, in order to take heating evenness into consideration, we want to consider 6 different shapes of pans--squares, rectangles of width-length ratio 2:3 (similar to baking pans in real life), circles, hexagons, rounded squares, and rounded hexagons as shown below.

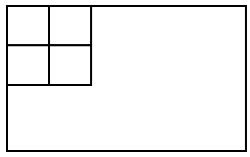


In order to maximize number of pans of size A, Ni, for i=1, 2, 3, 4, 5, 6, that one can fit in the rectangular oven with width-length ratio W:L for the 6 different shapes of pans, we write 6 functions to calculate each Ni.

2.3.1 Square Pans

$$N1 = 2 \cdot \left\lfloor \frac{W}{\sqrt{A}} \right\rfloor \cdot \left\lfloor \frac{L}{\sqrt{A}} \right\rfloor$$

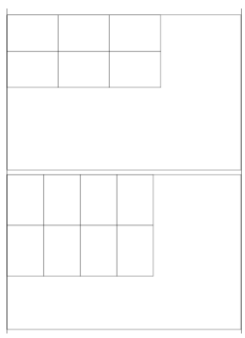
As shown in the following picture, to calculate the maximum number of square pans that can be fit into a W by L oven, we first calculate the side length of the square $\sqrt{A}$, then use the floor function (a function that maps a real number to the greatest integer smaller or equal to the number) to calculate the maximum number of square pans we can fit in horizontal direction of the rectangular oven, n1. Similarly, we can get n2 for vertical direction in the oven. Lastly, N1 is the product of n1 and n2.



2.3.2 Rectangular Pans

$$N2 = max(\left\lfloor \frac{W}{2 \cdot \sqrt{\frac{A}{6}}} \right\rfloor \cdot \left\lfloor \frac{L}{3 \cdot \sqrt{\frac{A}{6}}} \right\rfloor , \left\lfloor \frac{W}{3 \cdot \sqrt{\frac{A}{6}}} \right\rfloor \cdot \left\lfloor \frac{L}{2 \cdot \sqrt{\frac{A}{6}}} \right\rfloor)$$

In Function 2, we consider two ways of compacting pans—first fit the pans horizontally and vertically as shown below. In the first case, floor[W/w] gives the maximum number of vertical pans and floor[L/l] gives the max number of horizontal pans, where w and l are width and length for the pans. The product of them is n1. Conversely, floor[L/w]*floor[W/l] is n2. Consequently, N2=max(n1,n2) gives the maximum number of rectangular pans that can be fit in to a W by L oven.



2.3.3 Circular Pans

N3=m1*m2+n3 where

$$n_3 = 2{\cdot}max(m_1{\cdot}\left\lfloor \frac{(2-\sqrt{3})rm_2 + W - 2rm_2}{2r} \right\rfloor, m_2{\cdot}\left\lfloor \frac{(2-\sqrt{3})rm_1 + W - 2rm_1}{2r} \right\rfloor$$

We first calculate the radius of the circle given A to be $r = \sqrt{\frac{A}{\pi}}$ then we divide the packing into two categories as demonstrated below Figure 2 and 3. Second, consider rectangular packing of circles into a rectangle as shown in Figure 1. Simply, the 1st function can help to calculate the number of circles of radius r--squares of side length 2r can be fit horizontally and vertically to be m1 and m2 respectively. Next, since hexagonal packing of circles into a rectangular is known to be the most efficient packing, we convert n1 into n2 with difference n3. I will detailed explain how we get the difference n3. In each horizontal line of circles packed rectangular, we wasted a tiny bit of vertical distance (2-rad3)r. Since we have m2 lines of circles, floor[(((1-rad3/2)2rm2+W-2rm2)/(2r))]=n4 gives the total distance we wasted by packing circles rectangular. Since diameter is 2r and we have m1 columns of circles, n4*m1 gives the number of circles we can pack besides the rectangular packed circles. Similarly, if we rotate the packing 90 degrees (as shown in graph), then floor [(((1-rad3/2)2rm1+W-2rm1)/(2r))]m2 gives the other maximized number of circle packed. Now, n3 is the greater number of the two products. Hence, the maximum number of circles pans can be packed hexagonally in a rectangle is m1*m2+n3.

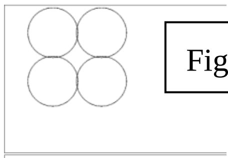


Fig 1

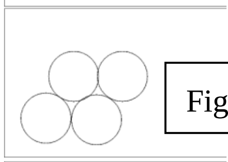


Fig 2

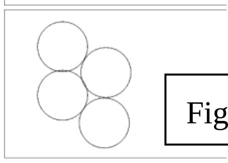
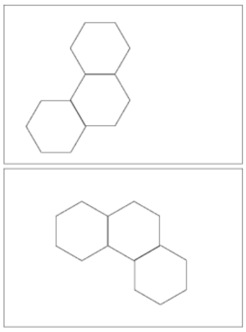


Fig 3

2.3.4 Hexagon Pans

$$N_4 = 2 \cdot max(2 \cdot \left\lfloor \frac{W}{\sqrt{3} \cdot d} \right\rfloor \cdot \left\lfloor \frac{L}{3d} \right\rfloor, 2 \cdot \left\lfloor \frac{W}{3d} \right\rfloor \cdot \left\lfloor \frac{L}{\sqrt{3} \cdot d} \right\rfloor)$$

We first calculate the side length of the hexagon d= sqrt(2A/(rad3*3)). Next, as before, we divide into two directions of packing (as shown). In the first direction, group two attached hexagons as a group as shown. M1=floor [W/rad3*d] gives the max number of hexagons we can fit in the vertical direction. And m2=floor [L/3*d] gives the max number of 2 hexagon groups we can fit horizontally. Hence, n1=m1*m2*2 gives the first direction's max number. Similarly, we can calculate m3=floor [(W/(3*d))] and m4=floor[(L/(rad3*d))], and n2=m3*m4*2. Then, max (n1, n2) would give the maximum of hexagons we can pack in a rectangular pan.



2.3.5 Rounded Square Pans

$$N_5 = 2 \cdot \left\lfloor \frac{W}{\sqrt{A + 4a^2 - \pi a^2}} \right\rfloor \cdot \left\lfloor \frac{L}{\sqrt{A + 4a^2 - \pi a^2}} \right\rfloor$$

Here, t is from 0 to 1 and is used to determine a=t*sqrt(A/pi). Here, as t increase, a increases, hence the rounder a square becomes.

This function uses the idea in calculating N—number of square pans. Packing squares into rectangle idea is the same. The differences include—we round the square with four tangent circles of radius a as shown. In order to keep the area of the pan consistent to be A, the bounding square's side length varies from the d in N1. But it is straightforward calculation wise. The peripheral square's side length d= sqrt(A+4*a*a-pi*a*a); Hence, we can now use the way we calculated N1 to get the desired result.

2.3.6 Rounded Hexagon Pans

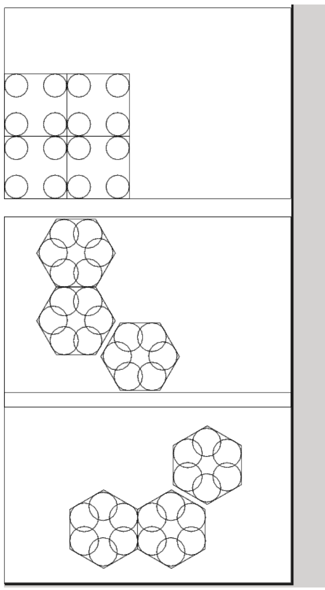
N6 $= 2 \cdot max(2 \cdot \left\lfloor \frac{W}{\sqrt{3} \cdot d} \right\rfloor \cdot \left\lfloor \frac{L}{3d} \right\rfloor, 2 \cdot \left\lfloor \frac{W}{3d} \right\rfloor \cdot \left\lfloor \frac{L}{\sqrt{3} \cdot d} \right\rfloor)$

As one can see, the representation of N6 is identical to that of N4. However, d in N6 differs from N4.

Similar to N5, N6 uses idea from N4.

We set a=t*sqrt(A/pi).

Then we calculate the peripheral hexagon's side length d=(rad3*a+sqrt(2*rad3*A)+(3-2*rad3)*a^2)/3. As shown in the graph—when the rounding circle has radius a, in order for the rounded hexagon to have area A, area 1, 2, and 3 has to sum up to A. Now simply implement the idea in N4 to get the desired result.



2.4 Simulations on Space Efficiency

Our model also includes functions which determine if a point is in a circle/ a hexagon/ a rounded square/ a rounded hexagon.

Having all above functions written and tested, we can do simulations to show which shapes fit best into a W by L rectangular oven.

Pan Area	Oven Width	Oven Length	Square	2x3 Rect	Rounded Square	Hexagon	Rounded Hexagon	Circle
100	50	50	25	24	16	17	17	16
100	50	100	50	48	36	41	33	32
100	50	150	75	72	56	65	57	52
100	100	100	100	96	81	91	65	72
100	100	150	150	144	126	145	113	117
100	150	150	225	216	196	209	169	195
200	50	50	9	8	9	7	7	9
200	50	100	21	20	21	19	19	18
200	50	150	30	32	30	31	31	27
200	100	100	49	40	49	37	37	42
200	100	150	70	64	70	61	61	63
200	150	150	100	96	100	91	91	90
300	50	50	4	6	4	5	5	4
300	50	100	10	14	10	13	11	10
300	50	150	16	21	16	17	17	14
300	100	100	25	28	25	31	21	25
300	100	150	40	49	40	49	41	35
300	150	150	64	70	64	65	57	56
400	50	50	4	6	4	5	5	4
400	50	100	10	12	8	9	9	8
400	50	150	14	18	14	17	13	12
400	100	100	25	24	16	17	17	16
400	100	150	35	36	28	33	25	24
400	150	150	49	54	49	49	37	42
500	50	50	4	2	4	5	5	2
500	50	100	8	6	8	9	9	6
500	50	150	12	10	12	13	13	10
500	100	100	16	15	16	17	17	12
500	100	150	24	25	24	25	25	20
500	150	150	36	40	36	37	37	30
600	50	50	4	2	4	3	3	1
600	50	100	8	6	8	7	7	3
600	50	150	12	10	12	11	11	5
600	100	100	16	15	16	13	13	12
600	100	150	24	25	24	21	21	20
600	150	150	36	35	36	31	31	30
700	50	50	1	2	1	3	3	1
700	50	100	3	6	3	7	7	3
700	50	150	5	8	5	11	11	5
700	100	100	9	12	9	13	13	9
700	100	150	15	18	15	21	21	15
700	150	150	25	24	25	31	31	25
800	50	50	1	2	1	1	1	1
800	50	100	3	4	3	3	3	3
800	50	150	5	8	5	5	5	4
800	100	100	9	8	9	7	7	9
800	100	150	15	16	15	13	13	12
800	150	150	25	24	25	17	17	20
900	50	50	1	2	1	1	1	1
900	50	100	3	4	3	3	3	2
900	50	150	5	8	4	5	5	4
900	100	100	9	8	9	7	7	6
900	100	150	15	16	12	13	13	12
900	150	150	25	24	16	17	17	16

Table 1—Space Efficiency Table

For the simulation that generated above table:
--all data are in cm units
--W and L vary from 50 to 150
--A varies from 100 to 900
--t=0.25 and a= .25*sqrt(A/pi) Notice though the roundness of the square and hexagon pans is fixed for the above table, our model can present results according to different levels of roundness i.e. different values of t and a.
--time—our table also presents the time required to cook the brownies using different pans.

2.5 Observations and Conclusions

Squares outperform the rectangles and other shapes most of the time, with exceptions of a few specific W/L ovens.

Circles perform the worst space-efficiency wise with exceptions of small oven when they perform better than rounded hexagons.

Rounded squares and hexagons perform closely well. They are more space efficient than circles and less space efficient than rectangles and squares.

Speaking of space efficiency, the ranking from most to least efficient is (most of the case with exceptions of some unusual W/L value) square, rectangle, rounded square, hexagon, rounded hexagon, circle.

2.5 Heating Evenness

2.5.1 Theory

We modeled each of the above pan shapes at various sizes to determine which pan shapes cook the most evenly for various sizes.

To model the various shapes of brownie pan:

In addition to optimizing pan choice for space, a brownie pan must allow the brownie to cook evenly. In order to determine how even the heat distribution for each pan is, we model the brownie as it cooks in the oven as a set of discrete points arranged on a cubic grid in 3 dimensional space. Create models of each pan shape, we define a discrete valued function pan((a, b) ELEM_OF R^2) → {0, 1} that evaluates to 1 when the point is within the shape, and 0 when it is not. We then initialize the model with the set of points {(a, b, c) : (dx, dy, dz), x, y, z ELEM_OF Z and pan(a, b, c) = 1 and 0 LESSTHAT_OR_EQUAL z LESSTHAT_OR_EQUAL h} for some d, h ELEM_OF R > 0. Intuitively, this is each point on a cubic grid with spacing d that lies within a prism of height h.

From this set, we then remove each point with z coordinate 0 for which the set has fewer than 5 additional points within distance d. To put it less formally, we round off the edges of the bottom of the pan. This is ubiquitous in commercially available pans, and helps to prevent the bottom corners and edges of brownies from burning and sticking to the pan. Our model mimics this behavior, producing brownies that are more evenly cooked when the bottom has been rounded off.

The primary problem in the creation of an ideal brownie pan with respect to evenness is modeling the way that heat moves through an oven and into a brownie. The “burnt corners” defect occurs only because the center of the brownie is consistently cooler than the edges and corners, and thus cooks more slowly. To understand the way heat flows through the brownie, we modeled the temperature, “cookedness,” and water content of each of the discrete points on our model through time. We make use of each of these values frequently in our model, and for any model, a discrete timestep algorithm can be used to calculate them, so let

temp(m, p, t) = temperature of brownie m at position p, time t
cookedness(m, p, t) = cookedness of brownie m at position p, time t
water(m, p, t) = water content fraction of brownie m at position p, time t

We modeled the oven as a convecting oven with a constant air temperature of 350 F^o. We then took a discrete time slice at each second of the brownie's cook time, and modeled the flow of heat over time. Conduction through the pan and convection through the air and the brownie itself are all significant methods of heat transfer inside an oven, so we took each into account separately. Though radiative heating from does occur inside of an oven, particularly through radiation emitted from the heating coils, we decided to leave radiation out of our model, as the primary heat source in a convection oven is still convection.

We modeled conduction from the pan into the brownie using Newton's Law of Cooling [2].

2.5.2 Formula

Similarly, we modeled convection from the air into the brownie, using a smaller k value (because convection through air is a slower process than conduction through metal). We assume our pan is made of aluminum, a common ovenware metal [2]: The k value of aluminum is approximately 1000 times greater than that of air [2]: so in our model, we assume that once the air heats the metal, it can almost immediately heat the brownie in contact with the metal. Since the metal has more contact with the air than with the brownie, it is as if the brownie touching the metal has more contact with the air, so we used a k value approximately 1.5 times greater for heat transfer through the pan than in the air.

In addition to heat input from the oven, we modeled convection within the brownie; however, unlike the air and the pan, the brownie's thermodynamic properties change significantly during cooking. To model this change, we define a function bk(c, w, t) to determine the k value of brownie material for cookedness c, water content fraction w, and temperature t. bk is continuous (it is interpolated between discrete values at discrete points), so we can assume that for nearby points in the brownie, k values will be similar. For each discrete point p within the brownie, we assume that heat must move from each adjacent discrete point in the model (including diagonals), through the brownie material, and into or out of p. For this calculation, we use Newton's Law of Cooling, and model heat transfer between point each point (x0, y0, z0), and each of the points in a 3x3 cube surrounding the point, ie, each point in the set of: discrete coordinate (x, y, z) elem {x, y, z | x elem [x0 - 1, x0 + 1], y elem [y0 - 1, y0 + 1, z elem[z0 - 1, z0 + 1], (x, y, z) != (x0, y0, z0}.

We determine the distance d between the discrete point p and the 26 adjacent points, aSUBi for I = 0 to 26, and hold the area of contact to be constant for all of these points (this parameter of Newton's Law of cooling does not really apply in a homogenous brownie environment since everything s essentially in contact), and of course obtain a deltaT of temp(p, t) – temp(a SUB i, t) for each a SUB i, at time t. We then have all the parameters needed to use Newton's Law of Cooling, and we get the desired dTemp/dt. In our discrete time model, we essentially apply Euler's Method for each point to determine heat flow over time, and add the dT/dt value multiplied by a small timestep constant to the temperature at the point in question. We repeat this process for each point in the brownie for each timestep, giving us a picture of the heat distribution in the brownie at any time during the interval the simulation runs over.

In addition to modeling the heat flow and temperature of each point in the brownie, we model the water content, and a third parameter, which we have deemed "cookedness," that represents how well cooked the brownie is at a certain point. A cookedness value of 0 represents a completely raw brownie, or uncooked brownie batter, a value of 1 represents a brownie that has been in the oven just long enough to be deemed cooked, and a value of 2 represents a brownie that is a bit crisper, just below the cusp of being considered burnt. For each discrete point p in the brownie model, at each timestep, we determine the rate of cooking at that point with the function:

dC/dt(temperature, water content, cookedness),

In reality, the process of cooking is incredibly complicated, so we have modeled it as a rate in order to approximate it as best we can. Our research suggests that the chemical reactions that occur inside the brownie that we consider "cooking" occur at significant rates starting at 325 F°, and the rate of cooking increases approximately linearly until the brownie reaches 350F° (the brownie will never get hotter than this). At these temperatures, much heat energy goes to evaporating water when water is present, so we wanted the function to produce lower values when water content was high, so we multiply by 1/(1 + water content). As the brownie cooks, the rate of cooking slows, as the reactants for the cooking reactions are consumed: Once a brownie is entirely carbonized, it can cook no further in a 350 degree oven. We model this reduced rate of cooking for well cooked brownies by making dC/dt proportional to 1/(1 + cookedness). Finally, the rate is multiplied by a cooking speed constant, denoted cc. Thus we have

$$\frac{dC}{dt}(temperature, watercontent, cookedness) = \begin{cases} 0 & temperature < 325 \\ \frac{cc\cdot(temperature-325)}{(1+watercontent)(1+cookedness)} & temperature > 325 \end{cases}$$

This function models the rate of cooking of the brownie in a complex and accurate manner, taking many of the properties of cooking we consider to be significant into account. For any model m, dC/dt can be calculated at point p, time t, as dC/dt(temp(m, p, t), water(m, p, t), cookedness(m, p, t)). As such, we could just as easily define dC/dt as a function of (m, p, t) for a model, a point within the model, and a time parameter, however the function defined as above is far more useful to use within the discrete timestep model, as such a model is needed to evaluate the functions temp, water, and cookedness. For this reason, we prefer to think of dC/dt as function of temperature, water content, and cookedness.

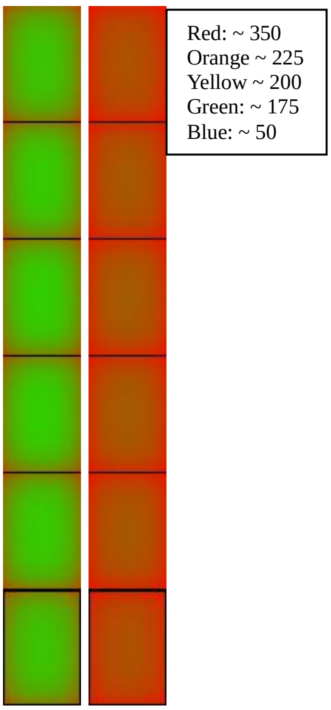
In much the same way as we evaluate the rate of cooking, we calculate the rate of water evaporation at each discrete timestep. We define the function dW/dt as below:

$$\frac{dW}{dt}(temperature, water\,fraction) = \begin{cases} 0 & \text{temperature } > 211 + 4(1 - water\,fraction) \\ -b(a - (211 + 4(1-b))) \cdot wc & \text{temperature} < 211 + 4(1 - water\,fraction) \end{cases}$$

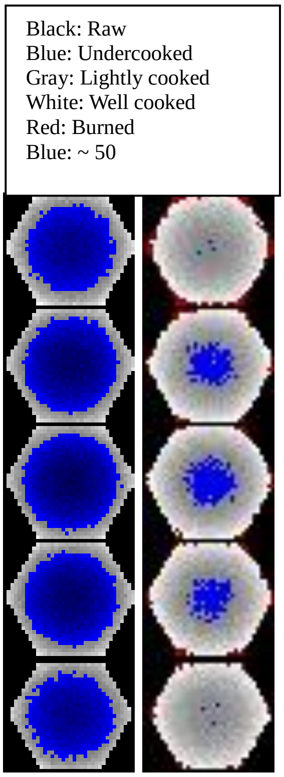
Here we are modeling water as a fluid that does not evaporate until its boiling point is reached. We are initially setting the boiling point to slightly above 212 F° (similar to pure water), but raising it as the water content drops, because the boiling point of water increases with the concentration of solute. Additionally, the rate of evaporation is proportional to the amount of water present, and to a positive rate constant wc.

By calculating the heat flow, change in cookedness, and change in water concentration at each discrete point in each timestep of our model, we are able to look at the heat distribution, water distribution, and cookedness at each point in the brownie at any time during the baking process. These three parameters affect each other, and ultimately heat distribution and evenness, in a complex manner, in much the same way the changing properties of a real brownie in an oven affect the final product. Below are maps of horizontal slices of the temperature, cookedness, and water content functions of brownie models we have simulated.

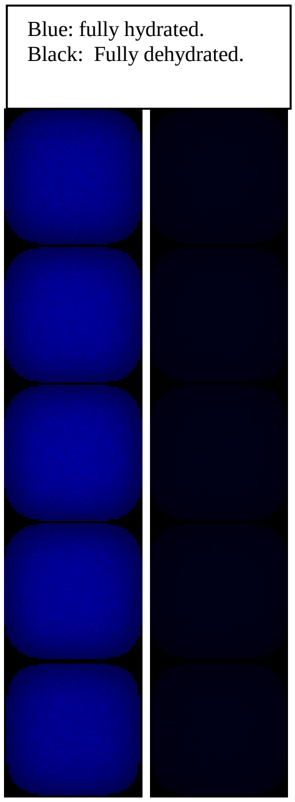
2.5.3 Images



Heat distribution of a 23 x 34.5 cm rectangle after 10 and 15 minutes



Cookedness distribution after 20 and 22 minutes



Water map after 8 and 16 minutes.

The brownie quickly heats up and loses water, although it does not begin to cook until much later. 3 different shapes are shown for visualization purposes.

2.5.4 How to determine if a brownie is cooked / cook time

In order to compare the heat distribution of various pan shapes and sizes, one of the questions that must be addressed is what qualifies as an evenly cooked brownie? Furthermore, what makes one brownie more evenly cooked than another? To do this, we create a quality metric Q such that Q(m, t) for model m and a time t parameters gives the "quality" of a brownie, with respect to evenness. We decided arbitrarily that quality is a value on the interval of [0, POSITIVE_INFINITY), such that 0 is a perfectly cooked brownie, and higher values represent more poorly cooked brownies.

Initially, for our quality metric, we took the sum of squares of the difference between the cookedness of the brownie at each discretely modeled point and 1 (recall that a cookedness value of 1 represents a minimally cooked brownie), then divided by the degrees of freedom, and finally took the square root of that result (much like a standard deviation) $\sqrt{\frac{\sum_0^i (cookedness(i,t)-1)^2}{(i-1)}}$,where I is the number of points in the model.

This model worked reasonably well, but tended to stop cooking early, while the edges were a bit crisp, and the middle was still quite underdone. We needed a model in which the cost of having a bit of overdone or burnt brownie was less than the cost of having a raw center. Since any value of cookedness on the interval [1, 2] is considered "properly cooked," we needed to not penalize overcooked edges until they reached a cookedness value of > 2. We modified our quality metric to take the sum of the minimum difference between each cookedness value in the model and the interval [1, 2], then divide by the degrees of freedom and take the square root of the result (as before).

$$\sqrt{\frac{\sum_0^i (distance(cookedness(i,t),[1,2]))^2}{i-1}}$$

Under this metric, cookedness values on the interval [1, 2] are not penalized, and a theoretical brownie for which each discrete point's cookedness value is fully on that interval would have a perfect score of 0.

With a quality metric that accurately reflects the quality of a real brownie, we were able to assess the quality of a brownie at each discrete timestep. In order to evaluate the quality of a model in general (and thus evaluate the quality of the shape of the pan for an arbitrary size value), we need to simulate the cooking of each model, and stop each simulation after the brownie has been cooked for an optimal amount of time.

Each pan shape x size combination has a different optimal cook time, and no simple function can model this optimal time (such a function would require solving the very differential equations our model is designed to evaluate). However, our choice of metric results in an interesting property: Quality values start at ~1 (as brownies start completely uncooked), and do not begin to increase until the edges reach a temperature high enough for the cooking process to begin. This occurs when t reaches a threshold such that dCookedness/dt(t, w, c) > 0. When this occurs, Q(m, t) begins to decrease as the cookedness values of edge points begin to rise. Eventually, the cookedness values of edge points rise above 1, but the interior is cooking as well, so the overall change in quality is still negative. When the rate that the overcooked exterior is burning exceeds the rate that the undercooked interior is cooking, the quality has reached a minimum (the burning rate will continue to rise as the cooking rate drops, so q(m, t) will never be greater for a larger value of t), and the simulation ceases for the model in question: we have found the optimal cook time. Thus we can define a optimal cook time function, OT(m), that gives the optimal time to cook a specific model brownie m.

With an optimal cook time function and a clearly defined quality metric, we can now assess a model's evenness quality and compare it fairly with other models of different shapes and/or sizes. Any model m has an overall quality of Q(m, OT(m)), where Q is the quality metric, taken at time OT(m), the optimal cook time for the brownie model in question.

2.5.5 Conclusions on Heating Evenness

For constant brownie volume, in general the surface area to volume ratio dictated both the optimal cooking time and the result of the quality metric. Round objects with low surface area to volume ratios, such as cylinders and hexagonal prisms, cook more evenly than models with sharp corners, such as rectangular prisms, because the heat can more quickly reach and cook the center. Smaller brownies cooked more evenly (and more quickly), because heat is able to quickly reach the center and cook the brownie before the edges had time to burn.
In our model, a 5cm radius circular pan cooked a brownie perfectly (minimum Q value was 0), and the worst shape we looked at was a 30 by 30square pan (Q value of 0.197757).

For constant area, we found that the pan shapes ranked circle, rounded hexagon, hexagon, rounded square, and rectangle, square. The chart below summarizes our results for various pan sizes and shapes:

Area	Shape	Time	Mean	Standard Deviation	Quality
100	10.0cm square	22:40	1.438919	0.552561	0.056749
100	8.0cm x 12.0cm rectangle	22:20	1.42861	0.540057	0.051072
100	6.0cm leg hexagon	20:00	1.452103	0.518492	0.013429
100	10.0cm rounded square	20:20	1.453436	0.521043	0.011491
100	6.5cm leg rounded hexagon	20:25	1.451737	0.524915	0.013707
100	5.5cm radius circle	20:15	1.480794	0.540804	0.007373
200	14.0cm square	24:40	1.45213	0.617997	0.111307
200	11.5cm x 17.5cm rectangle	24:35	1.451197	0.615488	0.108594
200	9.0cm leg hexagon	22:00	1.457464	0.567754	0.041731
200	14.5cm rounded square	22:05	1.463192	0.565139	0.033111
200	9.5cm leg rounded hexagon	22:10	1.452355	0.566884	0.042692
200	8.0cm radius circle	22:00	1.472861	0.569695	0.030233
300	17.5cm square	25:45	1.443859	0.649336	0.147038
300	14.0cm x 21.0cm rectangle	25:35	1.451371	0.64553	0.140679
300	10.5cm leg hexagon	22:35	1.4505	0.57906	0.05746
300	17.5cm rounded square	22:45	1.465366	0.590314	0.056941
300	11.0cm leg rounded hexagon	22:45	1.459718	0.590068	0.060044
300	10.0cm radius circle	22:50	1.46287	0.589007	0.053388
400	20.0cm square	26:20	1.440741	0.664763	0.165582
400	16.5cm x 24.5cm rectangle	26:15	1.437475	0.658514	0.161855
400	12.5cm leg hexagon	23:10	1.451106	0.595955	0.074141
400	20.0cm rounded square	23:10	1.453023	0.593438	0.068871
400	13.0cm leg rounded hexagon	23:15	1.440981	0.591421	0.075294
400	11.5cm radius circle	23:15	1.448959	0.590396	0.06445
500	22.5cm square	26:50	1.446271	0.680403	0.178777
500	18.5cm x 27.5cm rectangle	26:45	1.441186	0.671698	0.172991
500	14.0cm leg hexagon	23:25	1.434003	0.59515	0.085636
500	22.5cm rounded square	23:25	1.431429	0.584843	0.075209
500	14.0cm leg rounded hexagon	23:30	1.444612	0.59593	0.077469
500	12.5cm radius circle	23:30	1.450283	0.597816	0.072787
600	24.5cm square	27:05	1.426476	0.678338	0.189295
600	20.0cm x 30.0cm rectangle	27:00	1.426357	0.668691	0.179552
600	15.0cm leg hexagon	23:35	1.430026	0.596616	0.090408
600	24.5cm rounded square	23:40	1.431831	0.58903	0.079107
600	15.5cm leg rounded hexagon	23:40	1.419936	0.586287	0.084924
600	14.0cm radius circle	23:45	1.429904	0.589893	0.078
700	26.5cm square	27:25	1.439261	0.690066	0.194365
700	21.5cm x 32.5cm rectangle	27:15	1.419718	0.671433	0.186906
700	16.5cm leg hexagon	23:50	1.425006	0.597543	0.096052
700	26.5cm rounded square	23:50	1.426283	0.58983	0.08504
700	16.5cm leg rounded hexagon	23:50	1.418468	0.587655	0.088025
700	15.0cm radius circle	23:55	1.431075	0.59249	0.082001
800	28.5cm square	27:35	1.418777	0.68078	0.197071
800	23.0cm x 34.5cm rectangle	27:30	1.42963	0.680379	0.190387
800	17.5cm leg hexagon	23:55	1.408653	0.586373	0.096447
800	28.5cm rounded square	24:00	1.420188	0.58479	0.084646
800	17.5cm leg rounded hexagon	23:55	1.407519	0.582641	0.090894
800	16.0cm radius circle	24:00	1.412154	0.583048	0.086078
900	30.0cm square	27:45	1.421599	0.682535	0.197757
900	24.5cm x 36.5cm rectangle	27:40	1.418669	0.675644	0.192806
900	18.5cm leg hexagon	24:00	1.400961	0.585163	0.101215
900	30.5cm rounded square	24:05	1.408086	0.580173	0.08914
900	18.5cm leg rounded hexagon	24:05	1.409932	0.585739	0.093535
900	17.0cm radius circle	24:05	1.399352	0.575296	0.088243
1000	31.5cm square	27:50	1.406171	0.67677	0.20083
1000	26.0cm x 38.5cm rectangle	27:50	1.418559	0.677147	0.195263
1000	19.5cm leg hexagon	24:05	1.391426	0.576547	0.098947
1000	32.0cm rounded square	24:10	1.395768	0.570175	0.087743
1000	19.0cm leg rounded hexagon	24:05	1.398889	0.579176	0.094104
1000	18.0cm radius circle	24:10	1.389022	0.567306	0.087367

Table 2—Heating Evenness Table

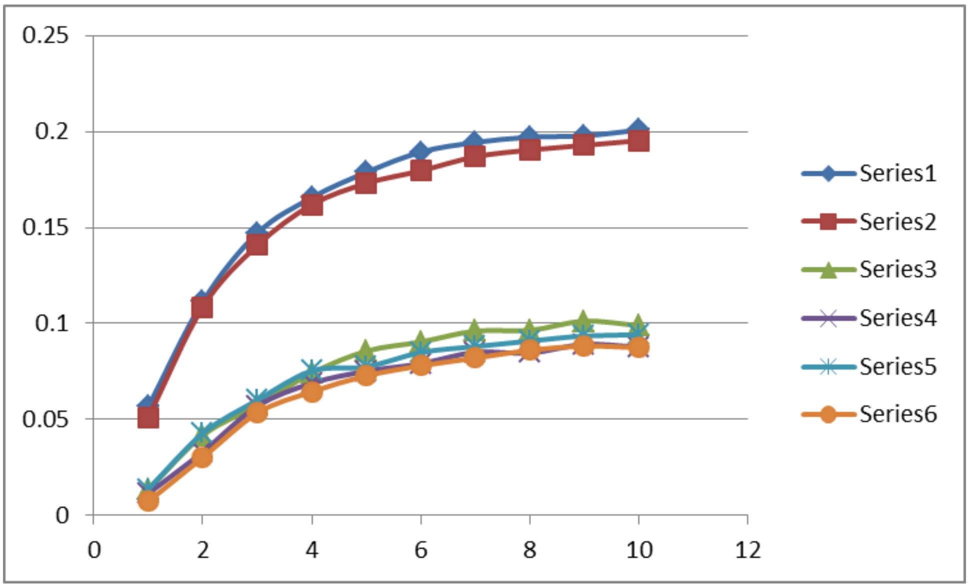
3 Discussion

3.1 Quality Chart

Area	Square	2*3 Rec	Hex	Circle	Rdd Square	Rdd Hexagon
100	0.056749	0.051072	0.013429	0.011491	0.013707	0.007373
200	0.111307	0.108594	0.041731	0.033111	0.042692	0.030233
300	0.147038	0.140679	0.05746	0.056941	0.060044	0.053388
400	0.165582	0.161855	0.074141	0.068871	0.075294	0.06445
500	0.178777	0.172991	0.085636	0.075209	0.077469	0.072787
600	0.189295	0.179552	0.090408	0.079107	0.084924	0.078
700	0.194365	0.186906	0.096052	0.08504	0.088025	0.082001
800	0.197071	0.190387	0.096447	0.084646	0.090894	0.086078
900	0.197757	0.192806	0.101215	0.08914	0.093535	0.088243
1000	0.20083	0.195263	0.098947	0.087743	0.094104	0.087367

In the quality chart above, heating quality (evenness) is better when the data (deviation) is smaller.

3.2 Optimization Graph



In the above optimization graph, series 1 to 6 respectively represent rectangle, square, hexagon, rounded hexagon, rounded square, and circle pan shapes, where the roundedness t is constant 0.25.

3.3 Compare Pan Shapes

We investigated several shapes in our attempts to optimize the heat distribution (measured by the quality metric) and oven space efficiency. Out of the shapes we investigated, we had decent performance in both respects with 3 by 2 rectangles, squares, rounded squares, hexagons, rounded hexagons, and circles. Depending on the relative importance of evenly cooked brownies and oven space efficiency, different shapes are optimal. Briefly we will enumerate the pros and cons of each:

3x2 rectangle: Very ineffective in heating evenness, the corners and short ends tend to burn. We did not expect this shape to heat well, and chose it for its similarity to existing baking pans.

Square: Like the rectangle, the corners and edges burn frequently, though sometimes less than the rectangle. Space efficiency is high, exceeded only by the rectangle in 3x2 ovens

Rounded Square: Cooks more evenly than squares or rectangles, but is less space efficient. Rounded squares are a good compromise shape.

Hexagon: Hexagonal pans heat quite evenly, and achieve excellent space efficiency in large ovens, as they tile with no gaps. They perform poorly in small ovens because of space inefficiencies at the edges.

Rounded Hexagon: Rounded hexagons tile well but are not quite as space efficient as regular hexagons. Rounded hexagons have better heat distribution.

Circle: Circular pans have the best heat distribution, but the poorest oven space efficiency. Pans tile with large gaps on a hexagonal grid, and are very inefficient around the edges, which is a huge problem for smaller ovens.

In conclusion, though no exact and simple function exists to optimize heating evenness against space efficiency, our model allows audience to choose the most suitable pan shape when considering different weights of space efficiency and heating evenness, p, and dimensions of the oven, W/L. Looking at Table 1 and 2 is most likely sufficient, but our program can offer more detailed results when specific demands of p, W/L and t, coefficient of roundedness, are provided.

4. References

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Advertising Sheet for Brownie Gourmet Magazine

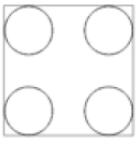
This is what you get if you use regular pans...

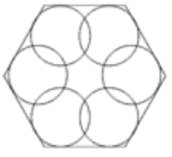
Well, circular pans are better...

But they take up a lot of space in your oven!

Though you can resolve the hard-edge problem diligently like this seven-step baking preparation before you bake [4], there is a smarter way!

Look at our cool mathematically designed pans.

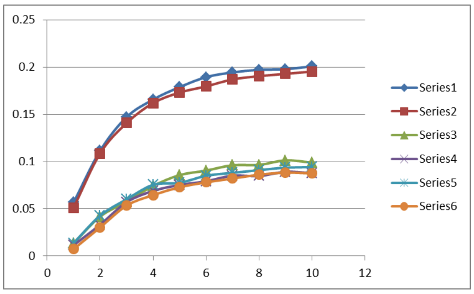




Consider these rounded regular polygons—they can save more space for your oven then circular pans; they also heat your brownie more evenly so that the edges never get hard!

Here is SOME evidence:

This is our Quality Optimization Graph, where smaller value of deviation represents better evenness during heating process. Series 1 through 6 respectively represent rectangle, square, hexagon, rounded hexagon, rounded square, and circle pan shapes.



This is our Space Efficiency Table. You can see which shapes can be packed more efficiently into your oven.

Pan Area	Oven Width	Oven Length	Square	2x3 Rect	Rounded Square	Hex	Rounded Hex	Circle
300	50	50	4	6	4	5	5	4
300	50	100	10	14	10	13	11	10
300	50	150	16	21	16	17	17	14
300	100	100	25	28	25	31	21	25
300	100	150	40	49	40	49	41	35
300	150	150	64	70	64	65	57	56
400	50	50	4	6	4	5	5	4
400	50	100	10	12	8	9	9	8
400	50	150	14	18	14	17	13	12
400	100	100	25	24	16	17	17	16
400	100	150	35	36	28	33	25	24
400	150	150	49	54	49	49	37	42
500	50	50	4	2	4	5	5	2
500	50	100	8	6	8	9	9	6
500	50	150	12	10	12	13	13	10
500	100	100	16	15	16	17	17	12
500	100	150	24	25	24	25	25	20
500	150	150	36	40	36	37	37	30

We have numerically shown that rounded square pans and rounded hexagonal pans not only heat evenly well but also tile together so well that they do not waste your precious baking space. So if you want perfectly baked brownies OR want to teach your lovely children something about geometry before school age, you should try our freshly designed BROWNIE PANS!!

For more details, please read our report:

A mathematician's Quest in Pursuit of the Ultimate Brownie

On the Subject of the Optimization of Chocolate Based Dessert Cookware