On the genericity properties in distributed estimation: Topology design and sensor placement

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Abstract—In this paper, we consider distributed estimation of linear, discrete-time dynamical systems monitored by a network of agents. We require the agents to exchange information with their neighbors only once per dynamical system time-scale and study the network topology sufficient for distributed observability. To this aim, we provide a novel measurement-based agent classification: Type-α, β, and γ, which leads to the construction of specific graph topologies: $G_α$ and $G_β$. In particular, in $G_α$, every Type-α agent has a direct connection to every other agent, whereas, in $G_β$, every agent has a directed path to every Type-β agent. With the help of these constructs, we formulate an estimator where measurement and predictor-fusion are implemented over $G_α$ and $G_β$, respectively, and show that the proposed scheme leads to distributed observability, i.e., observability of the distributed estimator.

In order to estimate the estimator further, we show that Type-α agents only exist in systems with $S$-rank (maximal rank of zero/non-zero pattern) deficient system matrices. In other words, systems with full $S$-rank matrices only have Type-β agents, and thus, a strongly-connected (agent) network is sufficient for full $S$-rank systems—by the definition of $G_β$ above; however strong-connectivity is not necessary, i.e., there exist weakly-connected networks that result in distributed observability. Furthermore, we show that for $S$-rank deficient systems, measurement-fusion over $G_α$ is required, and predictor-fusion alone is insufficient. The approach taken in this paper is structural, i.e. we use the concept of structured systems theory and generic observability to derive the results. Finally, we provide an iterative method to compute the local estimator gain at each agent once the observability is ensured using the aforementioned construction.

Keywords: Distributed estimation, Observability, Structured system theory, Generic rank

I. INTRODUCTION

Distributed estimation is of interest in a wide range of applications as it requires less communication load at each individual agent, in contrast to the centralized estimation where each agent may require repeated long-distance communication to a center. Applications of distributed estimation include social networks [1] to learn global beliefs based on partial understanding of the state of the society, market, politics, etc., monitoring physical and environmental processes [2], state-estimation in power systems [3]–[5], and multi-agent systems, such as collaborative target tracking and flocking [6].

A variety of solutions exists for distributed estimation from earlier work in [7] performing parallel Kalman filter over complete graphs, to more recent diffusion-based schemes [8], [9] and incremental adaptive distributed strategies [10].

Distributed moving horizon estimation [11], low-cost single-bit data transmission with binary sign of innovations [12], and information-theoretic approach based on consensus over the Kullback-Leibler average of Gaussian PDFs [13], are also proposed. The literature can also be divided into static and dynamic estimation. In static case [8]–[10], the target state to be estimated does not change over time, while related work on dynamic estimation can be found in [3], [13]–[17].

Consensus-based distributed estimation strategies have recently found a lot of interest, where the main focus is to reduce the uncertainty of individual estimates by averaging on the measurements. Early work in [15]–[19] considers a two time-scale method, where consensus is implemented at a time-scale different than the system dynamics (see Fig. 1), where a large number ($\rightarrow \infty$) of data fusion iterations are implemented between every two successive time-scales, $k$ and $k+1$, of the dynamics. In contrast, distributed estimators with the same time-scale of dynamics and agent-communication is studied in [1], [13], [17], [20], among others.

![Fig. 1](left) The traditional two time-scale consensus-based approach. (right) single time-scale approach.

In the two time-scale method, the communication network is irrelevant due to more information exchanges among the individuals (information in a sparsely connected graph is equivalent to the information in a fully connected graph with large number of information exchanges). Therefore, the estimator performance depends only on the data fusion among the agents. However, in the single time-scale scenario, the underlying agent network remains sparse and an arbitrary communication network may not lead to a stable Mean-Squared Estimation Error (MSEE) at each agent (e.g., see [20]–[22]). In this context, the key problem is to design the structure of the agent communication according to the underlying fusion rules in order to recover the distributed observability.

In this paper, we use a variant of the Networked Kalman-type Estimator (NKE) protocol, initially introduced in [20]. To recover the observability of the underlying estimator, we first provide a novel agent classification, based on which we design the network connectivity among the agents possessing those measurements. Contrary to the existing works [23]–[26], we show that the set of crucial measurements required for
centralized observability can be further subdivided into Type-α and Type-β, based on their role in distributed observability. We then determine the connectivity requirements among the agents possessing these measurements by constructing two communication graphs: $G_\alpha$ and $G_\beta$; (i) in $G_\alpha$, every Type-α agent is directly-connected to every other agent; while (ii) in $G_\beta$, every agent has a directed-path to every Type-β agent. We show that the union of these two graphs provides the connectivity to ensure the distributed observability.

We further show that the distributed observability analysis depends on S-rank (maximal rank of zeros and non-zeros) of the system matrix. We partition the set of system matrices into matrices with full S-rank, and matrices that are S-rank deficient. In particular, we show that the agents possessing Type-α measurements only exist in S-rank deficient systems. This implies that for full S-rank systems, the communication network only consists of $G_\beta$, and thus a strongly-connected network is sufficient for stable estimation (since it always satisfy the conditions required by $G_\beta$). However, strong-connectivity may not work in S-rank deficient systems, as the conditions required by $G_\alpha$ may not be satisfied. Combining the above arguments, the main contribution of this paper is to provide general conditions on the underlying communication network. Here, we do not necessitate the network to be strongly connected [8]–[10], [13], [18], [21], i.e. there exist weakly-connected networks that give distributed observability.

In many practical applications the system matrix is S-rank deficient, e.g., detection/estimation problems [27], Gauss-Markov system models [28], smart grids [3], and Type-C Wind Turbine Generator models [29]. In order to utilize the S-rank, we use the generic approach, i.e. our results are independent of exact system values and rely only on the structure of the underlying system. This leads to a robust estimator design where the analysis is not algebraic, as in the conventional Grammian or PBH observability tests [30], but graph-theoretic [23]–[26], [31]–[33]. The generic approach is also helpful when the system parameters may change depending on the system operating point (e.g. in the linearization of nonlinear systems). Note that, our work is different from existing system operating point (e.g. in the linearization of nonlinear systems). Note that, our work is different from existing system operating point (e.g. in the linearization of nonlinear systems). Note that, our work is different from existing system operating point (e.g. in the linearization of nonlinear systems). Note that, our work is different from existing system operating point (e.g. in the linearization of nonlinear systems).

We consider distributed estimation of the following discrete-time linear dynamics:

$$x_{k+1} = Ax_k + v_k,$$

where $x_k = [x_{1,k}, \ldots, x_{n,k}]^T \in \mathbb{R}^n$ is the state vector, $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ is the system matrix, and $v_k \sim \mathcal{N}(0, V)$ is the Gaussian system noise. We assume that the dynamics, Eq. (1), is monitored by a network of $N$ agents such that each agent $i$ has the following measurement model:

$$y_{1:k} = C_i x_k + r_{1:k},$$

where $y_{1:k} \in \mathbb{R}^p$, $C_i \in \mathbb{R}^{p \times n}$, and $r_{1:k} \sim \mathcal{N}(0, R_i)$ are the measurement vector, matrix, and noise, respectively, at agent $i$. With this notation, the global measurement model is $y_k = C \hat{x}_k + r_k$, where $y_k = [y_{1,k}, \ldots, y_{p,k}]^T \in \mathbb{R}^p$, $C = [C_1^T C_2^T \ldots C_N^T]^T$ and $r_k \sim \mathcal{N}(0, R)$ are the global measurement vector, matrix, and noise, respectively, with $p = p_1 + \ldots + p_N$ and $R = \text{blockdiag}[R_1, \ldots, R_N]$. It is well-known, see [34], that the centralized Kalman filter error,

$$\bar{e}_{k|k} = (A - K_cCA)\bar{e}_{k-1|k-1} + \eta_k,$$

is stable if and only if the system is generically observable, where $K_c$ is the centralized Kalman gain and the vector $\eta_k$ collects the terms independent of $\bar{e}_{k-1|k-1}$. The notion of observability is generic, which is discussed next.

### A. Structured systems theory

Structural analysis deals with system properties that do not depend on the numerical values of the parameters, but only on the underlying structure (zeros and non-zeros) of the system [23]–[26], [31]–[33]. It turns out that if a structural property is true for one admissible choice of non-zero elements as free parameters, it is true for almost all choices of non-zero elements and, therefore, is called a generic property of the system. Furthermore, it can be shown that those particular (non-admissible) choices for which the generic property does not hold, lie on some algebraic variety with zero Lebesgue measure [35]. To introduce generic observability, we need the following graph notations.

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ denote the collection of the individual states, and let $\mathcal{Y} = \{y_1, \ldots, y_p\}$ denote the collection of the measurements. We define the system digraph as $G_A = (\mathcal{V}_A, \mathcal{E})$—the subscript $A$ represents the system, where $\mathcal{V}_A = \mathcal{X} \cup \mathcal{Y}$ is the vertex set, and $\mathcal{E}$ is the edge set. The edge set $\mathcal{E}$ is defined as $\mathcal{E}_A \cup \mathcal{E}_C$, where $\mathcal{E}_A = \{(x_i, x_j) \mid a_{ij} \neq 0\}$, and $\mathcal{E}_C = \{(y_i, x_j) \mid c_{ij} \neq 0\}$. A simple path of length $\ell$ from $v_1 \in \mathcal{V}$ to $v_{\ell} \in \mathcal{V}$ is such that there exists a sequence of vertices, $v_1, v_2, \ldots, v_\ell$ with each subsequent edge $(v_1, v_2), (v_2, v_3), \ldots, (v_{\ell-1}, v_{\ell}) \in \mathcal{E}$; $v_1$ is its begin-vertex and $v_\ell$ is its end-vertex. A path is said to be $\mathcal{V}$-tupled if it ends at a vertex in $\mathcal{Y}$. A digraph is called strongly-connected if there exists a directed path from each vertex to every other vertex. In a not strongly-connected digraph, we define Strongly Connected Components (SCC) as its maximal strongly-connected partitions. A cycle is a simple path where the begin- and end-vertices are the same. Since the nodes in $\mathcal{Y}$ have no outgoing links, nodes included in a cycle all belong to $\mathcal{X}$. As an example, consider Fig. 2 which shows
the system digraph of a dynamical system with \( n = 7 \) states (encircled) and \( N = 3 \) measurements (one measurement per agent) denoted by squares. The following results on generic observability can be found in [24], [31].

![Fig. 2. An example of \((A, C)\)-observable system based on Theorem 1.](image)

**Theorem 1.** A dynamical system is generically observable if and only if in the system digraph:

(i) Every state is the begin-node of a path that ends in an measurement (termed as a \( \mathcal{Y} \)-topped path);

(ii) There exist a disjoint union of \( \mathcal{Y} \)-topped paths and cycles that cover all the state vertices.

**Lemma 1.** The condition (ii) in Theorem 1 on the generic observability of \((A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{p \times n})\) is equivalent to

\[
\text{S-rank} \left( \begin{bmatrix} A \\ C \end{bmatrix} \right) = n.
\]

The proof of Theorem 1 for generic controllability and Lemma 1 is given in [31], where other graph-theoretical conditions to generic controllability (observability) are also defined that we omit here. As an example, consider the system shown in Fig. 2. It can be verified that each state is a begin-vertex of a \( \mathcal{Y} \)-topped path, and \( \{7, 7\}, \{4, 5, 6, 4\}, \{1, 2, 3, a\} \) constitute a disjoint union of cycles and \( \mathcal{Y} \)-topped paths that cover all the state vertices. Thus, the system is observable in generic sense, i.e. for almost all choices of non-zero elements in the system. From the aforementioned SCC definition, note that the state SCCs in Fig. 2 are \( \{7, 7\}, \{1, 2, 1\} \) and \( \{4, 5, 6, 4\} \).

**Definition 1 (Parent/Child SCC).** A state SCC, is called a parent SCC, if it has no outgoing link to any state vertex not belonging to itself. Any SCC that is not a parent is child.

As an example, the SCC containing vertices \( \{4, 5, 6\} \) in Fig. 2 is a parent SCC, since there is no outgoing edges from its states to other states, \( \{1, 2, 3, 7\} \), not included in it. Furthermore, \( \{1, 2\} \) and \( \{7\} \) are child SCCs. More details on parent/child SCCs and efficient algorithms for computing SCCs in a digraph can be found in [22] and [36], respectively.

**B. Results on rank genericity**

Here we state definition and results on the generic S-rank.

**Definition 2 (S-rank).** The generic rank (also called structural rank) is the maximum rank for all numerical values of the non-zero entries of the matrix \( A \). It is, in fact, an upper-bound on the numerical rank of \( A \).

The S-rank as a generic property holds for almost all choices of nonzero parameters of the matrix, \( A \). It is equal to the cardinality of the maximum matching in the bipartite graph associated to the matrix, \( A \), see [23]. In the algebraic sense, this is the maximum number of non-zero elements in distinct rows and columns of \( A \). Details on the generic rank implication in graph theoretic sense and algorithms on maximum matching can be found in [23], [36]. Among other generic properties, controllibility/observability are of interest in the context of this paper. The result below follows from Lemma 1.

**Corollary 1 (Full S-rank).** A matrix, \( A \), is full S-rank (structurally full-rank) if and only if its associated digraph has a disjoint family of cycles covering all vertices.

The proof is well-known and can be found in [31], [37].

**Lemma 2.** For a full S-rank matrix \( W \in \mathbb{R}^{N \times N} \), the matrix, \( W \odot A \), is full S-rank if and only if \( A \in \mathbb{R}^{n \times n} \) is full S-rank. Mathematically,

\[
\text{S-rank}(W \odot A) = Nn \iff \text{S-rank}(A) = n.
\]

**Proof:** For two matrices, \( W \) and \( A \), \( \text{rank}(W \odot A) = \text{rank}(W) \times \text{rank}(A) \). For any full rank matrix \( A_{n \times n} \), we have \( \text{max}(\text{rank}(W \odot A)) = Nn \); this proves the necessity. On the other hand, if \( \text{rank}(A) < n \) for any admissible choice of \( W \), then we have \( \text{max}(\text{rank}(W \odot A)) = \text{S-rank}(W \odot A) < Nn \). This proves the sufficiency.

**C. Assumptions**

In the rest of the paper, we make the following assumptions:

(i) The communication network, \( \mathcal{G}_W \) is static; (ii) For every agent, \( i \), the pairs, \( (A, C_i) \) or \( (A, \sum_{j \in \mathcal{D}_i} C_j^T C_j) \), are not necessarily observable; (iii) The system is generically \( (A, C) \)-observable, i.e. if we collect all the sensor measurements at a center, then the dynamical system is generically observable.

Assumption (ii), in practice, makes the distributed estimation problem more challenging and is where this work becomes significantly different from current approaches, see [6], [26] and references therein. Assumption (iii) is a typical assumption in distributed estimation implying the observability of the centralized estimator; without this, no estimation scheme will work.

**III. MAIN RESULTS**

This section provides the main results of this paper.

**A. Agent Classification**

To describe our approach, we provide a measurement-based agent classification. Since the system is \( (A, C) \)-observable, by Assumption-(iii) in Section II-C, we can enlist a disjoint set of cycles and \( \mathcal{Y} \)-topped paths that covers all the state vertices (existence is ensured from condition (ii)–Theorem 1).

**Definition 3.** In the system digraph, \( \mathcal{G}_A \), we define the maximal set, \( \mathcal{L} \), as the set of disjoint maximal cycles and \( \mathcal{Y} \)-topped paths that covers all the state vertices in \( \mathcal{G}_A \)–the cycles...
are preferred over the $\mathcal{V}$-topped paths and the maximal is with respect to the state-vertices. The elements of $\mathcal{L}$ are such that no two of them can be combined into either a larger cycle or a $\mathcal{V}$-topped path.

From Fig. 2, the set of disjoint cycles and $\mathcal{V}$-topped paths includes $\{(4, 6, 4), (5, b), (1, 2, 1), (3, a)(7, c)\}$, and $\{(4, 5, 6, 4), (1, 2, 3, a), (7, 7)\}$, among others. However, the latter includes the maximal cycles and thus $\mathcal{L} = \{(4, 5, 6, 4), (1, 2, 3, a), (7, 7)\}$. The following classification is with respect to $\mathcal{L}$:

(i) Type-$\alpha$ agent is an agent that appears in the $\mathcal{V}$-topped paths in $\mathcal{L}$; let $\mathcal{A}$ be the set of Type-$\alpha$ agents, e.g., we have agent $a \in \mathcal{A}$ in Fig. 2.
(ii) Type-$\beta$ agent is an agent that measures a state in full $S$-rank parent SCC (parent cycle) in $\mathcal{L}$; let $\mathcal{B}$ be the set of Type-$\beta$ agents, e.g., we have agent $b \in \mathcal{B}$ in Fig. 2.
(iii) Type-$\gamma$ agent is an agent that is neither Type-$\alpha$ nor Type-$\beta$; let $\Gamma$ be the set of Type-$\gamma$ agents, e.g., an agent with no measurement, an agent that measures a state not at the extremity of a $\mathcal{V}$-topped path, or an agent observing a child SCC in $\mathcal{L}$. Per Fig. 2, we have agent $c \in \Gamma$.

The above classification leads to the following results.

**Definition 4** (Crucial measurement). A crucial measurement is a measurement such that removing it, renders the dynamical system unobservable.

**Lemma 3.** The following agents are crucial for generic observability: (i) every Type-$\alpha$ agent and (ii) one Type-$\beta$ agent observing a state in every full $S$-rank parent SCC (parent cycle), $K$, in $G_A$. In addition, Type-$\gamma$ agents are non-crucial.

**Proof:** Since the Type-$\beta$ agents monitor the parent cycles with no outgoing link to any other state outside this cycle, having a parent cycle, $K$, with no Type-$\beta$ agent violates condition (i) in Theorem 1. Likewise, removing a Type-$\alpha$ agent violates the condition (ii) in Theorem 1 as the attached state vertex is not included in any $\mathcal{V}$-topped path (nor in any cycle) anymore. Hence, Type-$\alpha$ agents are also crucial. On the other hand, having Type-$\alpha$ and Type-$\beta$ agents satisfies both condition in Theorem 1, and removing a Type-$\gamma$ agent does not affect generic observability.

For example, in Fig. 2, if either agent $a$ or agent $b$ is removed, then the system becomes unobservable, and, hence, $a$ and $b$ are crucial. It can also be verified that agent $c$ is non-crucial. It is worth mentioning that our approach differs from the classification given in [24] for $(A, C)$ observability, as we further subdivide the crucial agents into Type-$\alpha$ and Type-$\beta$ with respect to their role in distributed observability. This classification is strongly tied to the $S$-rank of the underlying system as we show next.

**Lemma 4.** A full $S$-rank system has no Type-$\alpha$ agents.

**Proof:** Since $A$ is full $S$-rank, there exists a disjoint cycle family that covers all the states, from Corollary 1. Hence, the maximal set $\mathcal{L}$ includes no $\mathcal{V}$-topped paths, and the theorem follows.

In the next section, we use the aforementioned agent classification to design different communication topologies that will be used in the distributed estimator.

**B. Topology Design**

In this paper, we deal with two different graph representations: system digraph, $G_A$, representing states of the dynamic system Eqs. (1) and (2), and digraph $G_W$ defining the agent communication network. We now provide the requirements on $G_W$ for the distributed estimator, Eqs. (6)–(7), to be observable. To this aim, let $\mathcal{V}_W$ be the set of $N$ agents and define the following:

**Definition 5.**

(i) Define $\mathcal{E}_0 = \{(i, i) \mid i \in \mathcal{V}_W\}$, and $\mathcal{G}_0 = (\mathcal{V}_W, \mathcal{E}_0)$. Such a graph consists of only self-edges at each vertex (agent).
(ii) Let $\mathcal{E}_a = \{(j, i) \mid j \in \mathcal{V}_W, j \neq i\}$, i.e., there is a direct edge from every Type-$\alpha$ agent to all other agents; where $\mathcal{G}_a = (\mathcal{V}_W, \mathcal{E}_a)$ is the graph with such edges. Let $D_{\alpha}(i)$ be the neighbors of agent $i$ in $\mathcal{G}_a$.
(iii) Define $\mathcal{G}_\beta$ to be a graph among $\mathcal{V}_W$ such that for every parent cycle (full $S$-rank parent SCC) in $A$, say $K$, if agent $i$ does not have measurement of any state in $K$, then in the communication network, $\mathcal{G}_\beta$, there exists a directed path from agent $i$ to an agent $j$, which has a state measurement in $K$. Subsequently, let $\mathcal{E}_b$ be the set of edges induced by $\mathcal{G}_\beta$ and let $D_{\beta}(i)$ be the neighbors of agent $i$ in $\mathcal{G}_\beta$.

**Remark 1.**

(i) The sub-graph of Type-$\alpha$ agents is a complete graph.
(ii) An example of $\mathcal{G}_\beta$ is a strongly-connected graph [21], however, $\mathcal{G}_\beta$ is not necessarily strongly-connected.

The network structure defining the communication among the agents is the union of these three sub-graphs, i.e. $G_W = G_0 \cup G_a \cup G_\beta$. Notice that, unlike existing work in the literature, e.g. [6], [9] among others, we do not constrain $G_W$ to be undirected.

**C. Distributed estimator**

To estimate the dynamics in Eq. (1) with distributed measurements in Eq. (2), we propose the following estimator:

(i) **Predictor-fusion:**

\[
\hat{x}_{k|k-1} = \sum_{j \in \{i\} \cup D_{\alpha}(i)} \mathcal{W}_{ij} A \hat{x}_{k-1|k-1},
\]  

(ii) **Measurement-fusion:**

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \sum_{j \in \{i\} \cup D_{\alpha}(i)} C_j^T \left( y_k - C_j \hat{x}_{k|k-1} \right),
\]

where $\hat{x}_{k|m}$ is agent $i$’s estimate at time $k$ given the measurements up to time $m$ ($m \leq k$), from its neighboring agents, $W = \{w_{ij}\}$ is the weight matrix for predictor-fusion such that $w_{ij} > 0, \forall (i, j) \in \mathcal{E}_0 \cup \mathcal{E}_b$, with $\sum_{j} w_{ij} = 1$ ($W$ is stochastic), and $K_k$ is the local estimator gain at agent $i$. 

\footnote{The computation required for $\mathcal{L}$ is polynomial and is discussed in Section VI.}
Remark 2. Following Corollary 1, \( W \) has a nonzero diagonal \( (w_{ii} \neq 0, \forall i) \), since every agent is in its own extended neighborhood and uses its own information, which immediately gives that \( S\text{-rank}(W) = N \).

Remark 3. Note that since Type-\( \alpha \) and Type-\( \beta \) both belong to crucial agents, the estimator with only measurement-fusion, i.e. Eq. (7), is not observable without predictor-fusion, i.e. Eq. (6), unless there are no Type-\( \beta \) agents. Mathematically, \( (A, \sum_{i,j} E_{\alpha}(i) C_j^T C_j) \) is not necessarily observable in the generic sense.

The following is the main result of this paper; we defer its proof to Section VI as it requires the developments on predictor and measurement-fusion from Sections IV-V.

Theorem 2. Let \( (A, C) \) be observable, then the estimator in Eqs. (6)–(7) is distributively observable in the generic sense.

Note that the notion of distributed observability in the generic sense is similar to (centralized) generic observability—see Theorem 1, but extended to a distributed estimator. As we will show later, these two are different, i.e., centralized generic observability is a necessary condition for the distributed case, however it is not sufficient. Theorem 2 states that if the predictor and measurement-fusion are carried out over \( G_\beta \) and \( G_\alpha \), respectively, then the distributed estimator is observable. Let us assume for now that this statement is true, then the following corollaries are immediate:

Corollary 2. Let \( (A, C) \) be observable and let Theorem 2 hold. For a full \( S \)-rank system, a strongly-connected network is sufficient for distributed observability. Moreover, there exist weakly-connected networks that make the system distributedly observable at all agents.

Proof: For a full \( S \)-rank system, there are no Type-\( \alpha \) agents and thus \( D_\alpha(i) = \emptyset, \forall i \), in Eq. (7), by Lemma 4, i.e., we do not have any connectivity requirements on Type-\( \alpha \) agents; whereas, a strongly-connected ensures that the conditions, see Definition 5–(iii), on \( G_\beta \) are satisfied, see Remark 1–(i); however, \( G_\alpha \) can also be weakly-connected.

A direct consequence of the above corollary is that none of the agents need any measurement other than their own when the system is full \( S \)-rank, i.e., measurement-fusion is not required. This is in contrast to the estimator in [26], which only employs measurement-fusion, and thus, requires a direct connection from each Type-\( \beta \) agent to every other agent.

Corollary 3. Let \( (A, C) \) be observable and let Theorem 2 hold. If system, \( A \), is \( S \)-rank deficient, then the estimator (6)–(7) is not distributively observable without measurement-fusion, i.e. when \( D_\alpha(i) = \emptyset, \forall i \).

Proof: Since \( S \)-rank deficiency leads to Type-\( \alpha \) agents, we immediately verify from Theorem 2 that measurement-fusion is required (unlike in the full \( S \)-rank case) as \( D_\alpha(i) \) in Eq. (7) is non-empty. This is true even for strongly-connected networks as they do include \( G_\beta \) as a subgraph, but the connectivity requirements on \( G_\alpha \), see Definition 5–(ii), are not necessarily satisfied.

The above corollary shows that when \( A \) is \( S \)-rank deficient, using predictor-fusion cannot always guarantee the observability of the system and thus, the agents need access to additional measurements to recover observability. Furthermore, we explicitly show that these additional measurements have to come only from Type-\( \alpha \) agents and not from Type-\( \beta \) agents. This result is in contrast with existing work in the literature [24]–[26], because in these works: (i) only fusion in the measurement space is considered; and (ii) crucial agents are classified into a single category without recognizing the different roles the agents may play towards distributed observability, and subsequently, requiring all Type-\( \alpha \) and Type-\( \beta \) agents to be included in measurement-fusion.

D. Error analysis

In order to prove Theorem 2, we analyze the dynamics of the estimation error, which leads to predictor-fusion in Section IV, and measurement-fusion in Sections V. Let the estimation error at agent \( i \) and time \( k \) in the distributed estimator, Eqs. (6)–(7), be \( e_i^k = x_k - \hat{x}_i^k \), and let \( e_{\alpha} = [e_{1}^T, \ldots, e_{N_{\alpha}}^T]^T \) be the distributed estimation error derived in the following:

**Proposition 1.** Define \( q_{\alpha}^k \) to be some function of \( v_{\alpha} \) and \( r_{\alpha}^k \), independent of \( e_{\alpha} \), and let,

\[
K_k = \text{blockdiag} \left[ K_1^{\alpha}, \ldots, K_{N_{\alpha}}^{\alpha} \right],
\]

\[
D_C = \text{blockdiag} \left[ \sum_{j \in \{1\} \cup D_\alpha(i)} C_j^T C_j, \ldots, \sum_{j \in \{N_{\alpha}\} \cup D_\alpha(N)} C_j^T C_j \right],
\]

\[
q_k = \left[ (q_{\alpha}^1)^T, \ldots, (q_{\alpha}^{N_{\alpha}})^T \right]^T.
\]

Then we get the distributed error dynamics as,

\[
e_{\alpha} = (W \otimes A - K_k D_C(W \otimes A)) e_{\alpha} - q_k.
\]

**Remark 4.** (i) The variables \( D_C \) and \( K_k \) are block-diagonal matrices. (ii) Each \( \alpha \)th block diagonal, \( \sum_{j \in \{\alpha\} \cup D_\alpha(i)} C_j^T C_j \), in the matrix \( D_C \), can be thought of as a representation of all of the Type-\( \alpha \) measurements available to the agent \( i \).

The derivation of Eq. (8) requires some straightforward manipulations [21] and is omitted here. Comparing it to Eq. (3), we note that the distributed estimation error, \( e_k \), can be stabilized if and only if the pair \((W \otimes A, D_C)\) is distributively observable in the generic sense\(^3\), implying that a (full) gain matrix\(^4\), \( K_k \), exists such that \( \rho(W \otimes A - K_k D_C(W \otimes A)) < 1 \), for almost all choices of non-zero parameters in the underlying matrices, where \( \rho(\cdot) \) denotes the spectral radius of a matrix.

We refer to \((W \otimes A, D_C)\) as the distributed system and denote its digraph by \( G_{\text{Dist}} \). For better understanding of the structure of the estimator in (6)–(7), we first consider \( W = I \) and \( D_C \) defined as follows:

\[
D_C = \text{blockdiag}[C_1^T C_1, \ldots, C_N^T C_N],
\]

\(^3\)Note the distinction between centralized-\((A, C)\) observability, and distributed-\((W \otimes A, D_C)\) observability.

\(^4\)In order to compute the block-diagonal gain matrix in Eq. (8), we provide an iterative optimization procedure in the Appendix. With this block-diagonal gain matrix, distributed observability leads to a bounded MSEE.
implying no information exchange among the agents. This distributed system, \((I \otimes A, \overline{D}_C)\), can be thought of as \(N\) decoupled subsystems (as shown in Fig. 3). In the matrix representation, each of these subsystems is associated to an \(n \times n\) block diagonal, see Fig. 4 (Left).

Now consider \(W\) to have some non-zero off-diagonal entries. As it is shown in Fig. 4 (Right), these entries define the intra-connections among the subsystems. For example, adding edges from the parent SCC, \(\{4,5,6\}\), in agent \(c\)'s subsystem to the same parent SCC in agent \(a\)'s subsystem. Similarly, \(b \leftarrow a\) adds edges from the parent SCC, \(\{4,5,6\}\), in agent \(a\)'s subsystem to the same SCC in agent \(b\)'s subsystem. Since \(b\) is Type-\(\beta\), this parent SCC has a measurement from at least one of its states, \(x_3\), in this example; this entire setup allows the parent SCC without measurement at agent \(c\) to have a \(\mathcal{Y}\)-topped path, and thus, recovers its observability.

We discuss, in detail, the role of predictor-fusion (role of matrix \(W\) and graph \(G_B\)) in Section IV, and then the role of measurement-fusion (role of matrix \(D_C\) and graph \(G_a\)) in Section V. The description of predictor and measurement-fusion is summarized in Table I. The reason to separate solutions for predictor and measurement-fusion is only to give more intuition; obviously, in real applications if two agents are linked together they share all of their information, including both their measurement and predictor-estimates, to maximally improve their future estimates.

### IV. Predictor-Fusion

In this section, according to Table I, we analyze the structure of \(G_B\) for \((W \otimes A, \overline{D}_C)\) observability. First, we consider the system matrix, \(A\), to be full \(S\)-rank. This is the case, for example, in discretization of continuous-time systems where the system matrix almost always has non-zero diagonal entries (e.g., in [26]).

**Theorem 3.** With a full \(S\)-rank system, \(A\), the pair \((W \otimes A, \overline{D}_C)\) is generically observable over \(G_B\).

**Proof:** For \(W \otimes A\) to be generically observable, the system digraph, \(G_{Dist}\), should follow (i) and (ii) in Theorem 2. From Remark 2 and Lemma 2, \(W \otimes A\) is full \(S\)-rank, which ensures condition (ii) in Theorem 1. To satisfy condition (i), according to Lemma 3–(ii), every parent SCC in every subsystem of \(G_{Dist}\) has to be \(\mathcal{Y}\)-connected, i.e., has a path ending in \(\mathcal{Y}\). Let us assume that agent \(i\) has no measurement of parent SCC \(K_i\) in its subsystem. Then, according to \(G_B\), there is a path from \(i\) to another agent \(j\) measuring a state in SCC \(K_j\) (counterpart of SCC \(K_i\)). This implies that every SCC in subsystem of agent \(i\) has a path to its counterpart SCC in subsystem of agent \(j\), that implies \(K_i \rightarrow K_j\) (see Fig. 5). Therefore, every state in SCC \(K_i\) is also connected to \(y_j\), and thus, \(K_i\) is \(\mathcal{Y}\)-connected. Having this for every parent SCC in every subsystem, all SCCs in \((W \otimes A, \overline{D}_C)\) are \(\mathcal{Y}\)-connected and the theorem follows. \(\blacksquare\)

For example, consider again the system in Fig. 3. Having vertices \(\{4,5,6\}\) as parent SCC, agent \(b\) is Type-\(\beta\). According to the above theorem any other agent without any measurement in \(\{4,5,6\}\), like agent \(c\), must have a path to agent \(b\). This provides a connection from SCC \(\{4,5,6\}\) in subsystem of \(c\)
to its counterpart SCC in subsystem of $b$ in graph $\mathcal{G}_{Dist}$ and, in turn, its $\mathcal{Y}$-connectivity.

**Theorem 4.** Let $(A, C)$ be observable. If the system, $A$, is $S$-rank deficient, then $(W \otimes A, D_C)$ is not distributively observable in the generic sense, i.e. when $D_\alpha(i) = 0, \forall i$.

**Proof:** Let $i$ be an agent for which condition (i) in Theorem 1 does not hold, i.e.

$$S\text{-rank}\left(\begin{bmatrix} A & A \\ C_i^T C_i & C_i^T C_i \end{bmatrix}\right) < n.$$  \hspace{1cm} (10)

Such an agent always exists because: (i) based on the assumption (ii) in Section II-C, the entire system is not observable at any agent; and (ii) the matrix $A$ is not full $S$-rank. Now consider $(W \otimes A, D_C)$ for the best-case scenario where $\mathcal{G}_\beta$ is a complete graph, and thus, $W$ has all non-zero elements. Let $W_i$ be the $i$th column of $W$. Obviously, $W_i \otimes A$ is the $i$th block column of $(W \otimes A)$, and contains block matrices $w_{ij}A$, $j = 1, \ldots, N$. It follows that,

$$S\text{-rank}\left(\begin{bmatrix} w_{ij}A & A \\ C_i^T C_i & C_i^T C_i \end{bmatrix}\right) < n.$$  \hspace{1cm} (11)

for all $j = 1, \ldots, N$, as $w_{ij} \neq 0$ and scalar multiplication does not change the structure and the $S$-rank (maximum possible rank over all values). Since $A$ is not full $S$-rank, $W_i \otimes A$ has rank less than $n$ as stacking matrices with the same structure on top of each other (see Fig. 6 (Left)) does not improve the $S$-rank. This immediately results in,

$$S\text{-rank}\left(\begin{bmatrix} W_i \otimes A & A \\ C_i^T C_i & C_i^T C_i \end{bmatrix}\right) < n.$$  \hspace{1cm} (12)

Consequently, according to Fig. 6, the structure of the matrix $W \otimes A$ is given as the side-by-side concatenation of the matrices $W_i \otimes A$. Thus we have,

$$S\text{-rank}\left(\begin{bmatrix} W \otimes A & A \\ D_C & D_C \end{bmatrix}\right) < Nn.$$  \hspace{1cm} (13)

This holds for all choices of non-zero elements in full matrix $W$. Therefore, condition (ii) in Theorem 1 is violated and the theorem follows.

The above theorem shows that when $A$ is $S$-rank deficient, then using predictor-fusion alone cannot guarantee the distributed observability of the system, and thus, the agents need access to more measurement data to recover their observability, which is discussed next.

**V. MEASUREMENT-FUSION**

In this section, we discuss the second update level given in Eq. (7). Each agent, $i$, shares its measurement with its direct neighbors and implements this as an innovation to update its prediction. According to Table I, measurement-fusion alone is related to distributed observability of $(I \otimes A, D_C)$. Based on the definition of $D_C$, in the distributed system graph $\mathcal{G}_{Dist}$, this is equivalent to adding all the neighboring measurements of agent $i$ to its subsystem. However, with no predictor-fusion, the only way to include a Type-$\beta$ agent (crucial measurement) is to directly access its measurement as we do for Type-$\alpha$ agents, and thus, here we need to define a new communication graph for Type-$\beta$ agents as follows:

**Definition 6.** Define $\mathcal{G}_\alpha^*$ to be a graph among $\mathcal{V}_W$ such that for every parent cycle in $A$, say $K$, if agent $i$ does not have a measurement of a state in $K$, then it has a direct link from any agent $j$ with state measurement in $K$. Subsequently, $\mathcal{E}_\alpha^*$ is the set of edges induced by $\mathcal{G}_\alpha^*$ and let $D_\alpha^*(i)$ be set of $i$’s neighbors in $\mathcal{G}_\alpha^*$.

With this definition we now provide the main result on measurement-fusion.

**Theorem 5.** Let the assumptions in Section II-C hold. The system $(I \otimes A, D_C)$ is distributively observable in generic sense over $\{\mathcal{G}_0 \cup \mathcal{G}_\alpha \cup \mathcal{G}_\alpha^*\}$.

**Proof:** Sufficiency: With the given conditions (i) and (ii), each agent has access to all crucial measurements. This makes every agent generically observable. Necessity: If agent, $i$, is not connected to a crucial agent, then it is missing a crucial measurement and the statement follows.$\blacksquare$

Notice that, $\mathcal{G}_\alpha^*$ contains all the Type-$\beta$ agents but with a stringent connectivity requirement as compared to $\mathcal{G}_\beta$. In $\mathcal{G}_\alpha^*$, every Type-$\beta$ agent is directly connected to all other agents; a restriction imposed by only considering measurement fusion, see [26] for related work. Clearly, this requires stronger connectivity as compared to the directed path condition in $\mathcal{G}_\beta$.

In this paper, we combine both measurement- and predictor-fusion to obtain connectivity conditions, where we need direct
links only from the Type-$\alpha$ agents. This is particularly of interest in resource constraints applications, where we cannot afford possibly long-distance links in the network.

VI. PROOF OF THEOREM 2

Finally, the developments of Section IV and V lead to the following proof of Theorem 2.

Proof: The proof of Theorem 2 is a direct consequence of the Theorems 3, 4, and 5 stated in previous sections. ■

Recall that Theorem 3 sets the condition for predictor-fusion for full $S$-rank systems, i.e. conditions for $(W \otimes A, D_C)$ generic observability, while Theorem 4 states that for general $S$-rank deficient systems distributed observability cannot be achieved via the predictor-fusion alone. Measurement-fusion, i.e. generic observability of $(I \otimes A, D_C)$, is discussed in Theorem 5. Combining these results, the proof for generic distributed observability of the distributed estimator in Eqs. (6)–(7) is immediate. Loosely speaking, in our approach, predictor-fusion is applied to the full $S$-rank part of the system, while measurement-fusion is applied to the $S$-rank deficient part. We provide some additional comments next.

(1) In the case of Type-$\beta$ agents, every agent requires either a directed path to each Type-$\beta$ agent (as stated in the Theorem 2) or a direct link from each Type-$\beta$ agent (as stated in the Theorem 5); either one of these is sufficient for observability. Notice that, the first strategy may exploit nearest-neighbor or similar communication topologies, while the second may require avoidable long-distance communication.

(2) An agent may have no measurement of the system and still be able to estimate the global system states via the proposed strategies. Such agents, for example, may play a role to provide and maintain connectivity of the communication network [38], or assist in providing directed paths to Type-$\beta$ agents in $\mathcal{G}_3$.

(3) Computational complexity: The determination of maximal set $\mathcal{L}$ can be done via combinatorial algorithms to find the maximal matching in associated bipartite graph of the system and output; e.g. Hopcroft-Karp algorithm with running time of $O((n + N)^{2.5})$ [39]. The size of the maximum matching of the system bipartite graph, further, defines the $S$-rank($A$). Moreover, the parent/child SCC classification can be performed using DFS algorithm in $O((n + N)^2)$ [36].

(4) If system is not $(A, C)$ observable (violating Assumption (iii) in Section II-C) then even using a fully-connected communication network does not recover observability irrespective of any estimation strategy. Clearly, the only way to recover observability is by increasing the number of state measurements and recovering the $(A, C)$ observability [25].

VII. ILLUSTRATIVE EXAMPLE AND SIMULATION

Consider the system, $(A, C)$, given in Fig. 2. Recall that system is globally observable by collecting each measurement at a center. The state partitioning with maximal cycles covering all the states (in light of condition (i)–Theorem 1) is $\mathcal{L} = \{(4, 5, 6, 4), (7, 7), (1, 2, 3, a)\}$. By Def. 5, $a \in A$, $b \in B$, $c \in \Gamma$. Note that the distributed system with no information fusion, i.e. the graph associated to $(I \otimes A, D_C)$, is not observable at any individual agent (Fig. 3). Furthermore, the system with the predictor-fusion alone, i.e. $(W \otimes A, D_C)$, is not observable as it can be verified to be $S$-rank deficient.

To make the distributed system observable at each agent, we propose the following communication matrices $W_1$ and $W_2$, and their associated graphs $\mathcal{G}_W$ in Fig. 7.

$$W_1 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}, \quad W_2 = \begin{bmatrix} \times & 0 & \times \\ \times & \times & 0 \\ 0 & \times & \times \end{bmatrix}.$$ (14)

The graph $W_1$ is based on Theorem 5. In this network crucial agents, $\{a, b\}$, are directly connected among each other, and both have a direct link to the non-crucial agent, $c$. The second network $W_2$ is based on Theorem 2; there is a direct link from agent $a$ (Type-$\alpha$) to all other agents, and there is a path from every other agent to agent $b$ (Type-$\beta$). It can be verified that for both topologies $(W \otimes A, D_C)$ is generically observable. Note that the solution for the network design problem is not unique, and there maybe other examples of communication network satisfying the conditions in the last section.

VIII. CONCLUSIONS

In this paper, we study the role of agent communication towards the error stability in distributed estimation and show that the proposed estimator is able to track potentially unstable
systems. We note that strong-connectivity among the agents, in general, is not required and weakly-connected networks may result in bounded Mean-Squared Estimation Error (MSEE). In this regard, we provide sufficient network connectivity leading to an efficient communication infrastructure design. We define three types of agents/measurements where two are crucial for observability, and among these two, the subgraph of Type-$\alpha$ agents has stringent connectivity requirements than the subgraph of Type-$\beta$ agents. We provide two methods for recovering distributed observability: (i) predictor-fusion, and (ii) measurement-fusion. We determine dynamical systems (S-rank deficient) for which no predictor-fusion gives bounded MSEE and one has to rely further on measurement-fusion. The results are generic, i.e. they are independent of any particular fusion rule, e.g., Metropolis-Hastings [40] among others, chosen for the protocol. We illustrate our methodology by a simple academic example; however, the algorithms are of polynomial-time and feasible for large-scale systems.

APPENDIX: DESIGN OF LOCAL ESTIMATOR GAIN

We now provide the estimator gain, $K_k$, computation; we assume it to be independent of time, $k$, and denote it by $K$. Having $(W \otimes A, D_C)$ observable guarantees existence of a full matrix, $K$, such that $\rho(\tilde{A}) < 1$, where $\tilde{A} = W \otimes A - K D_C (W \otimes A)$. However, according to Remark 4, we need the gain matrix to be block-diagonal. It is known that such $K$ is the solution of the following Linear Matrix Inequality (LMI):

$$ \begin{bmatrix} X & A^T X \\ X A & X \end{bmatrix} > 0 \Rightarrow \begin{bmatrix} X & A^T \\ A & Y \end{bmatrix} > 0, \quad (15) $$

with $X > 0$, where `\(\succ\)` (\(\succeq\)) denotes positive-(semi) definiteness. Since the L.H.S. in the above equation is nonlinear in $K$, the equivalent solution is proposed in [41], [42] as the R.H.S. in the above with $X = Y^{-1}$. The above LMI is now linear in $K$ but the constraint involved $X = Y^{-1}$ is non-convex; however, it can be approximated with a linear function [43]. In particular, the matrices, $X, Y > 0$, satisfy $X = Y^{-1}$ as the optimal point of the following optimization problem [43]:

$$ \min \text{trace}(XY) \text{ subject to } \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0, \quad (16) $$

with $X, Y > 0$. We summarize the above in the following:

**Lemma 5.** Having $(W \otimes A, D_C)$ observable, the gain matrix, $K$, is the solution of the following optimization problem:

$$ \min \text{trace}(XY) $$

subject to $X, Y > 0$,

$$ \begin{bmatrix} X & A^T \\ A & Y \end{bmatrix} > 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad (17) $$

$K$ is block-diagonal.

Notice that, the solution to the second LMI is equivalent to $X = Y^{-1}$, which gives the minimum trace (optimal value) as $nN$. Furthermore, $\text{trace}(XY)$ can be replaced with the linear approximation $\phi_{lin}(X, Y) = \text{trace}(Y_0 X + X_0 Y)/2$ [43], and an iterative algorithm can be utilized to minimize this under the given constraints:

**Algorithm 1** Iterative calculation of local gain estimator, $K$.

1: Find feasible points $X_0, Y_0, K$. If no such points exist, Terminate.

2: At iteration $t \geq 0$ minimize $\text{trace}(Y_t X + X_t Y)$ under the constraints given in Eq. (17) and find $X, Y, K$.

3: If $\rho(A) < 1$ terminate, otherwise set $Y_{t+1} = X, X_{t+1} = X$ and run the step (2) for next iteration $t = t + 1$.

It is shown in [43] that $\text{trace}(Y_t X + X_t Y)$ is a non-increasing sequence that converges to $2nN$. In this regard, a stopping criterion in step (3) of the algorithm is established as reaching within $2nN + c$ of the trace objective. The given algorithm is centralized, however, the center has to implement this process only once, off-line; then it transmits the estimator gains to each agent and plays no further role. We refer interested readers to [21], [42]–[44] for more details.

**REFERENCES**


