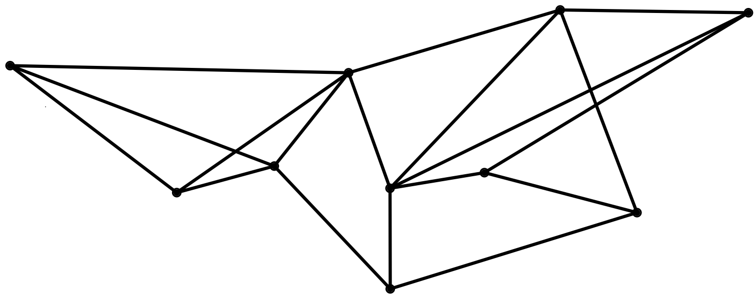
Tri-edge connectivity augmentation in planar straight line graphs

Mashhood Ishaque

Marwan Al-Jubeh, Kristóf Rédei, Diane L. Souvaine, and Csaba D. Tóth.

Edge connectivity

A graph is k-**edge connected** if it remains connected after removing any $k-1$ edges.



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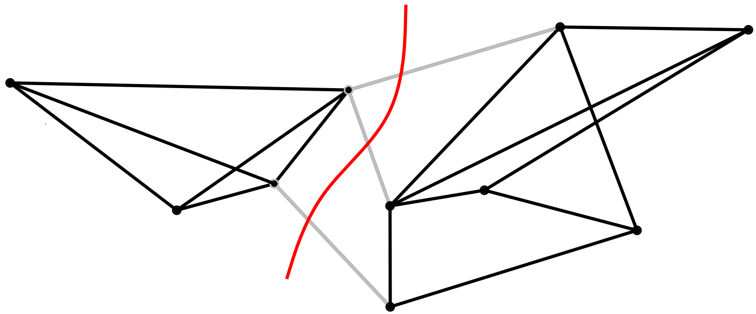

Connectivity augmentation in abstract graphs

Edge-connectivity augmentation problem:

Given an undirected graph $G = (V, E)$ and an integer k, find a minimal set F of new edges such that $(V, E \cup F)$ is k-edge connected.

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For $k = 2$, can be solved in linear time (Eswaran and Tarjan, 1976), (Plesník, 1976).

The edge-connectivity augmentation problem can be solved in polynomial time for every fixed k (Watanabe and Nakamura, 1987).

Planarity preserving connectivity augmentation

Planarity preserving *edge*-connectivity augmentation problem:

Given an undirected planar graph $G = (V, E)$ and an integer k, find a minimal set F of new edges such that $(V, E \cup F)$ is k-edge connected and planar.

Planarity preserving connectivity augmentation

Planarity preserving *edge*-connectivity augmentation problem:
Given an undirected planar graph $G = (V, E)$ and an integer k, find a minimal set F of new edges such that $(V, E \cup F)$ is k-edge connected and planar.

For $k = 2$, it is NP-hard to find the minimum cardinality of F required for augmenting the edge-connectivity (Rutter and Wolff, 2008).

Embedding preserving connectivity augmentation

Embedding preserving *edge*-connectivity augmentation problem:
Given an undirected plane graph $G = (V, E)$ embedded in the plane and an integer k, find a minimal set F of new edges such that $(V, E \cup F)$ is a k-edge connected plane graph where the edges in E have the same embedding as in G

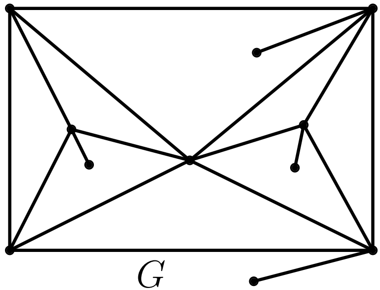
Embedding preserving connectivity augmentation

It is NP-hard to find the minimum cardinality of edges that can augment a *planar straight line graph* (PSLG) to a 2-edge connected PSLG (Rappaport, 1989).

It is NP-hard to find the minimum set of edges that can augment a *planar straight line tree* to a k-edge connected PSLG for $k = 2$, 3, 4, and 5 (Rutter and Wolff, 2008).

No hardness result is known for the embedding preserving connectivity augmentation problem if the new edges may be curvilinear arcs. It might be possible to solve it in polynomial time... (?)

Examples



An augmentation of G to a 2-edge connected graph has to increase the vertex degree to at least 2.

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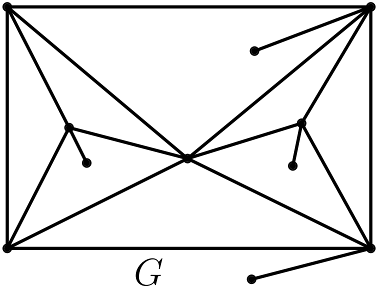

G

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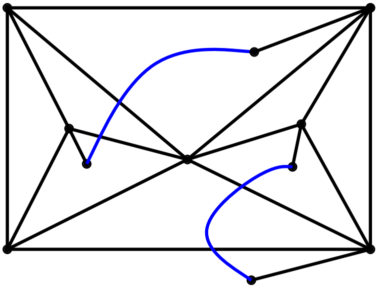

(general) augmentation

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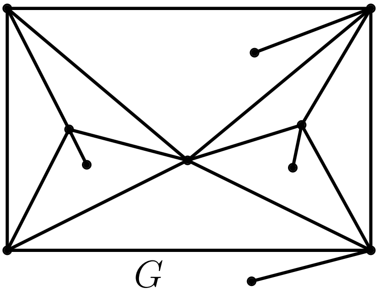

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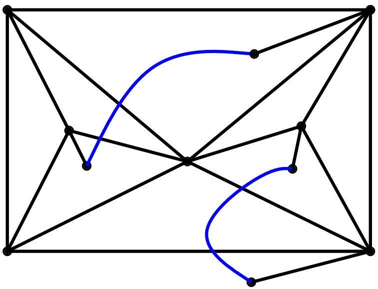

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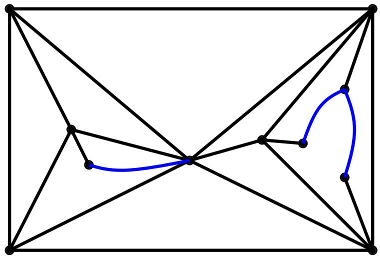

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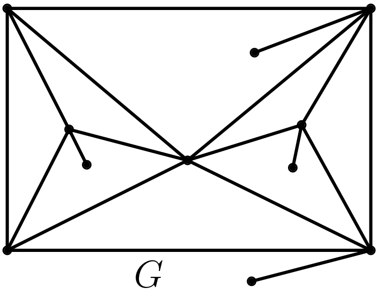

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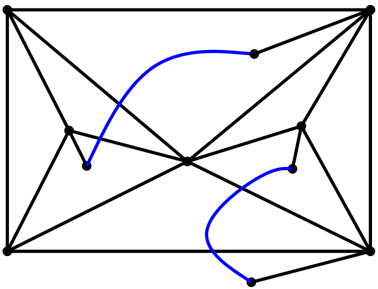

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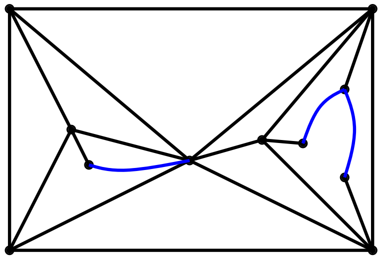

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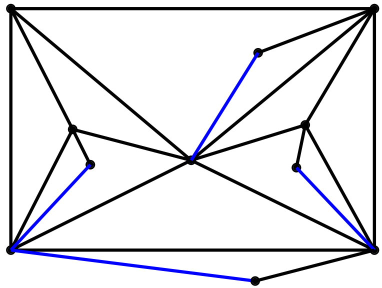

embedding preserving augmentation

Tri-edge connectivity augmentation in planar straight-line graph

Given an undirected plane graph $G = (V, E)$ embedded in the plane with straight-line edges, augment G with a set F of new edges such that $(V, E \cup F)$ is a 3-edge connected planar straight-line graph (PSLG) where the edges in E have the same embedding as in G.

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How many edges are necessary?

How many edges are enough?

Which PSLGs are 3-edge augmentable?

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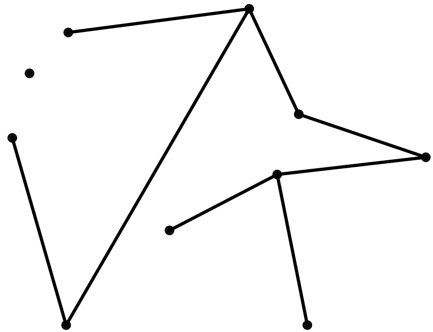
Theorem (Tóth and Valtr, 2009): A PSLG is 3-augmentable if and only if the vertices are not in convex position and there is no edge which is a chord of the convex hull such that all vertices on one side are on the convex hull.



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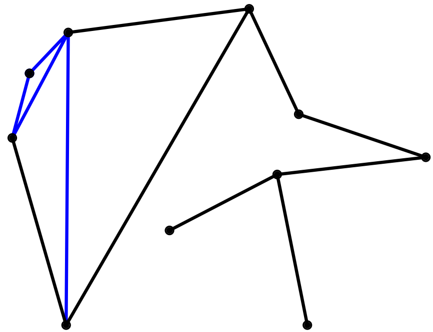

Tri-edge connectivity augmentation in planar straight-line graph

Given an undirected plane graph $G = (V, E)$ embedded in the plane with straight-line edges that is *3-edge augmentable*, augment G with a set F of new edges such that $(V, E \cup F)$ is a 3-edge connected planar straight-line graph (PSLG) where the edges in E have the same embedding as in G.

How many edges are necessary?

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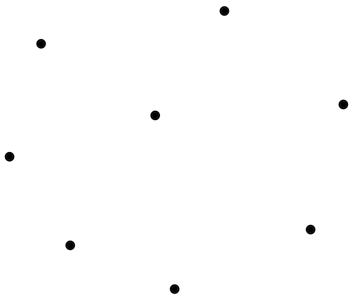
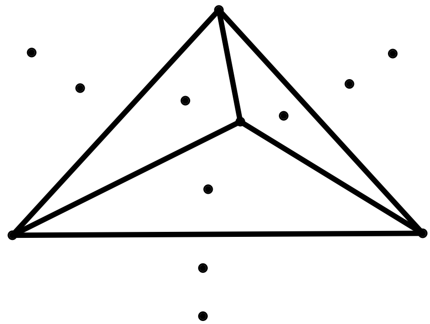
Theorem: There are PSLGs with $n \geq 4$ vertices that require at least $2n-2$ new edges for an embedding preserving augmentation to a 3-edge connected PSLG.





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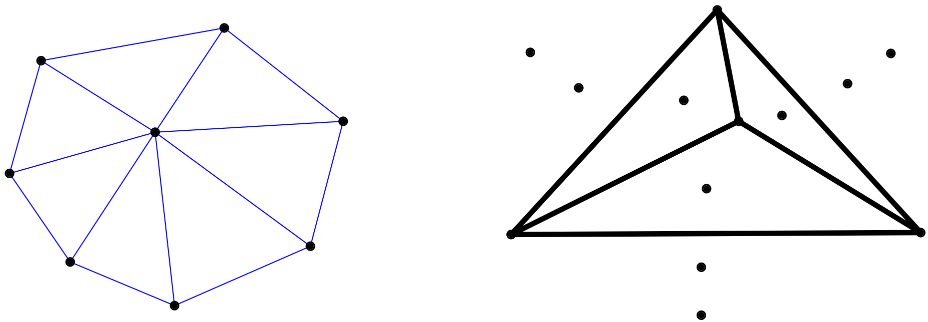

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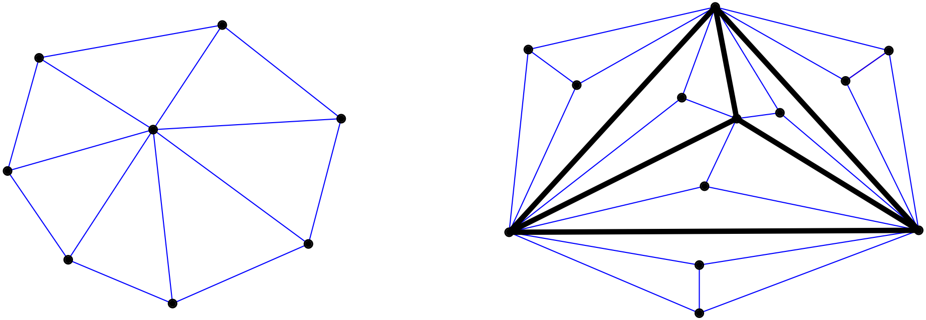

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Theorem: If a PSLG with $n \geq 4$ vertices is 3-edge augmentable, then it has an embedding preserving augmentation to a 3-edge connected PSLG with at most $2n - 2$ new edges.

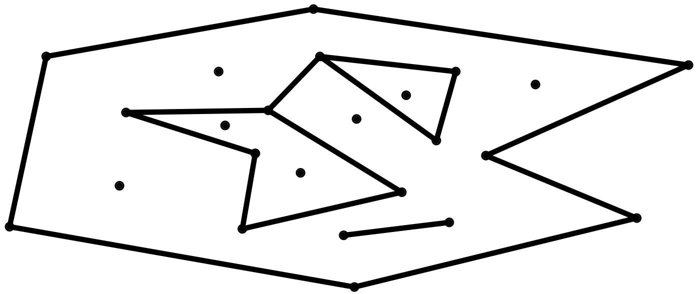
Charging scheme for edge augmentation

Lemma (Souvaine, and Tóth, 2009). Let G be a PSLG with f compact faces and n vertices, r of which are reflex. Then we have $f + r \leq 2n - 2$.

Corollary. Let G be a PSLG with b bridges, c non-singleton components, f compact faces, n vertices, r of which are reflex, and s singletons. Then

$$b + c + f + r + 2s \leq 2n,$$

with equality if and only if G is a forest.



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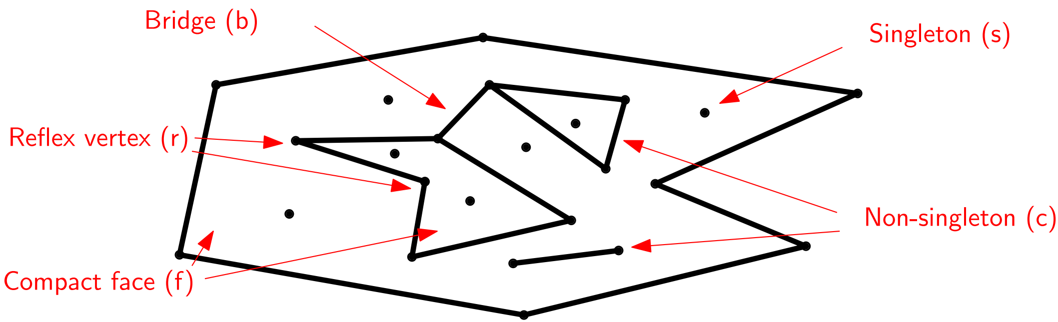


Augmenting the connectivity of a PSLG from two to three

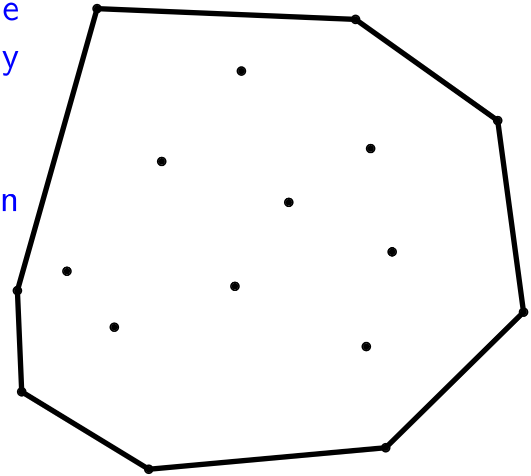
Theorem (Tóth and Valtr, 2009): If a 2-edge connected PSLG with $n \geq 4$ vertices is 3-edge augmentable, then it has an embedding preserving augmentation to a 3-edge connected PSLG with at most $n-2$ new edges.

Augmenting the edge connectivity of a PSLG to three

Theorem: If a PSLG with $n \geq 4$ vertices is 3-edge augmentable, then it has an embedding preserving augmentation to a 3-edge connected PSLG with at most $2n-2$ new edges.

The degree of every singleton has to be raised to at least three, while adding only two new edges per singleton.

This can be done if we have singletons in a *convex* face.



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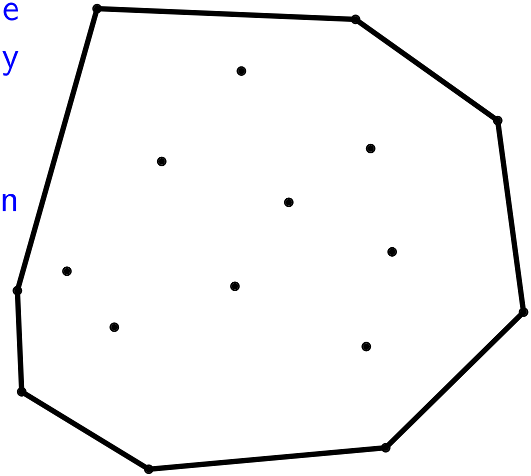
Then apply the result (Tóth and Valtr, 2009) for raising the connectivity from 2 to 3 along the path.



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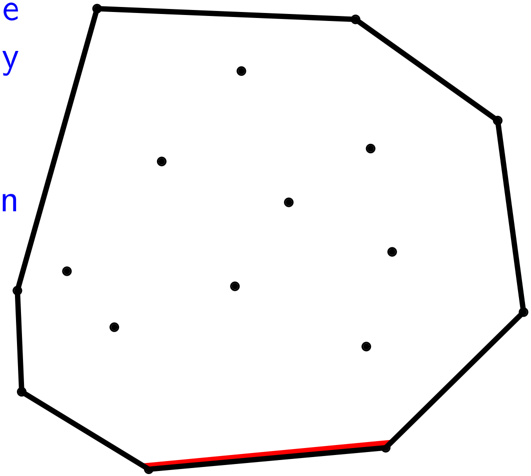

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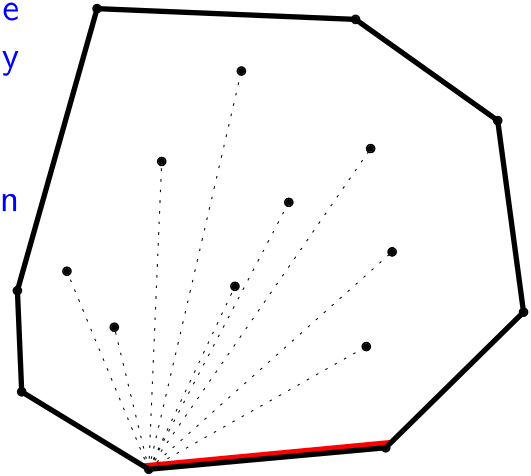

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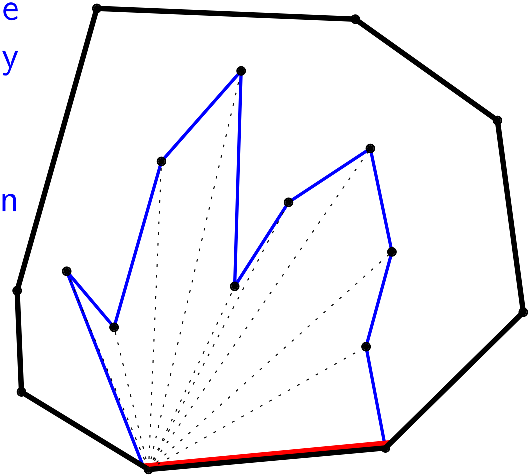

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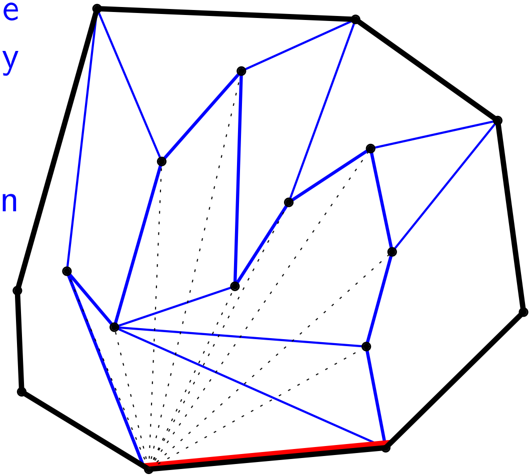

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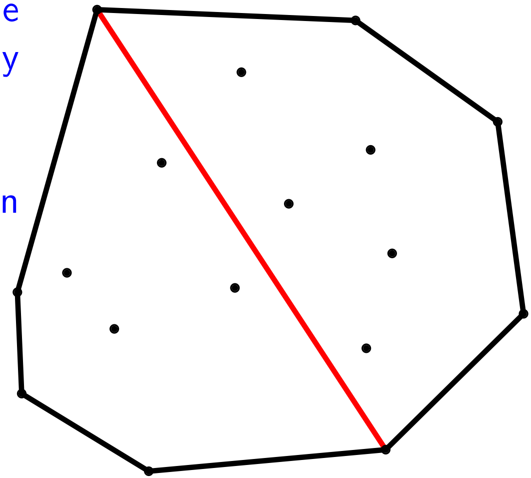

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Add a "deformable" edge (or diagonal) along the convex face, and deform it to a path going through all singletons.
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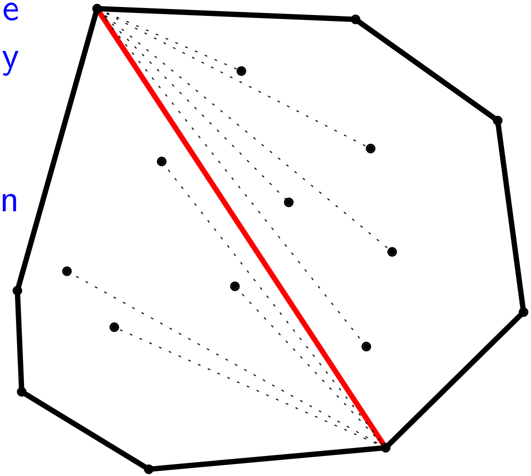

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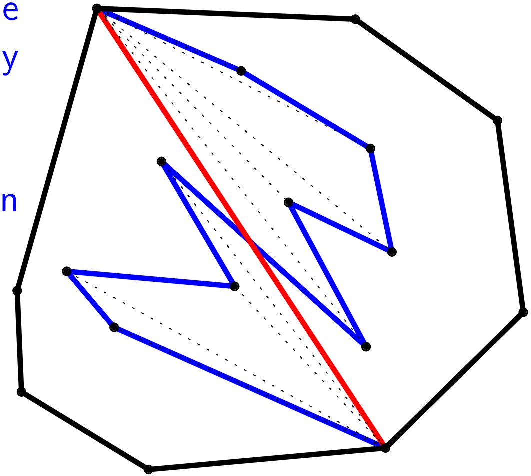
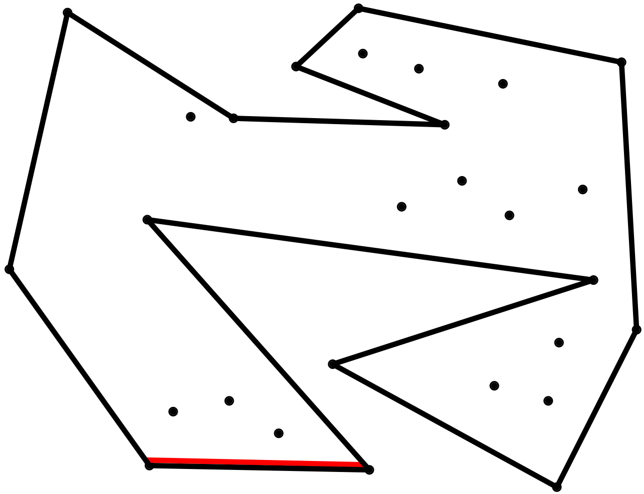


Augmenting the edge connectivity of a PSLG to three

The same argument does not immediately apply to singletons in a non-convex face: an edge of the face may be not be completed to a path through all singletons.



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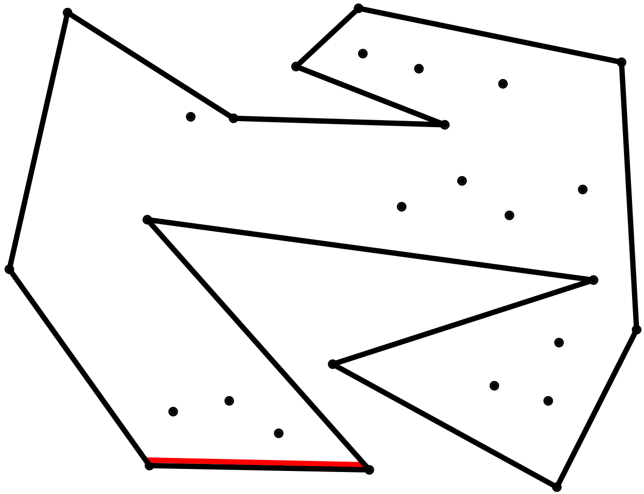
We add a new "deformable" edge at each reflex vertex (we can charge them to the reflex vertices), *and* subdivide the face into convex regions, each responsible for a convex region.



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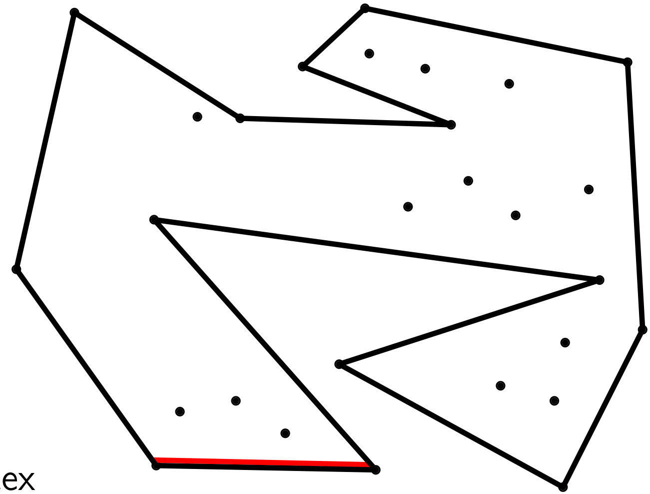

We add a new “deformable” edge at each reflex vertex (we can charge them to the reflex vertices), *and* subdivide the face into convex regions, each responsible for a convex region.

Shoot a ray from each reflex vertex along the angle bisector, and rotate it until it reaches a vertex.
The ray splits the interior of the face into two regions, with fewer reflex angles.

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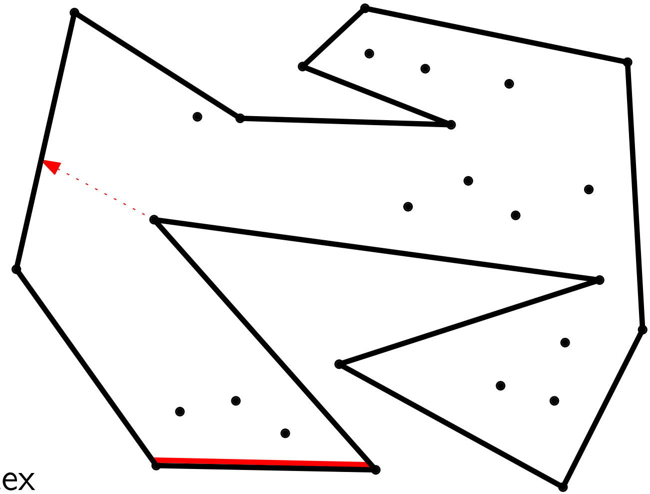

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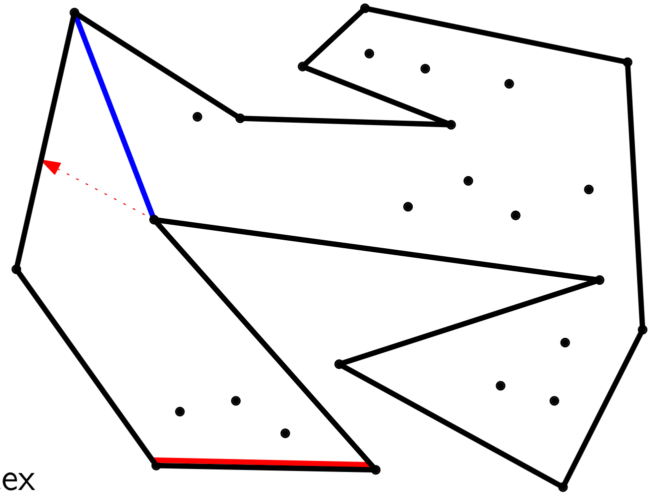

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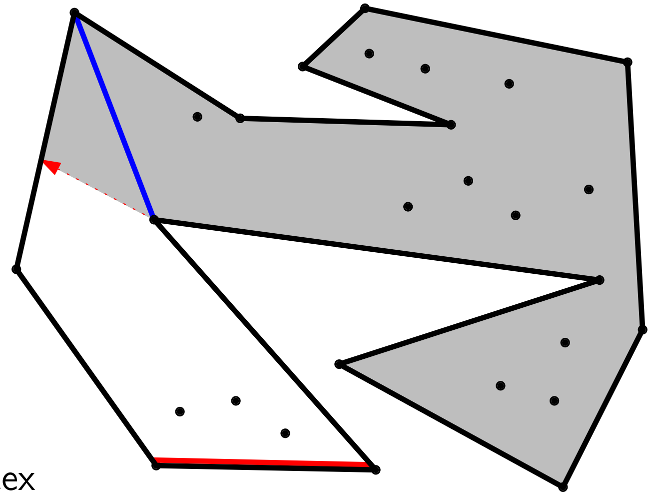

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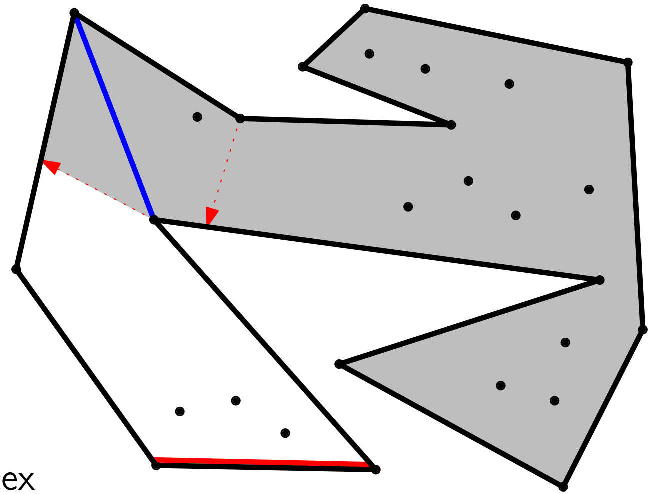

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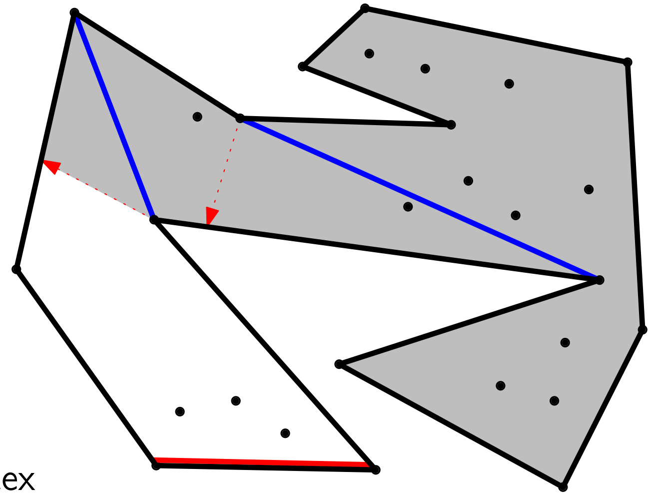

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Augmenting the edge connectivity of a PSLG to three

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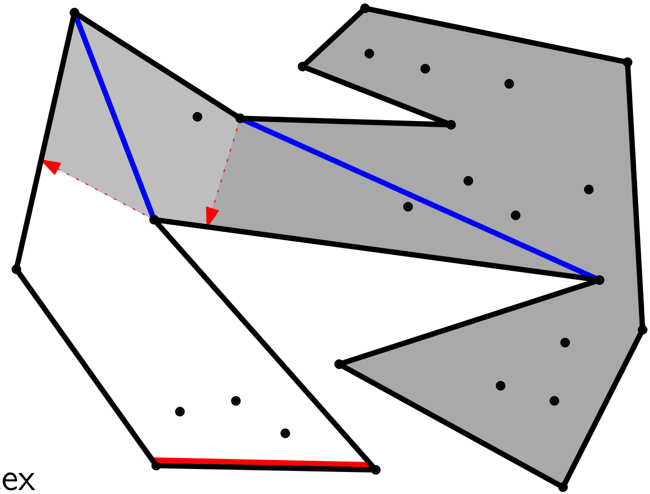

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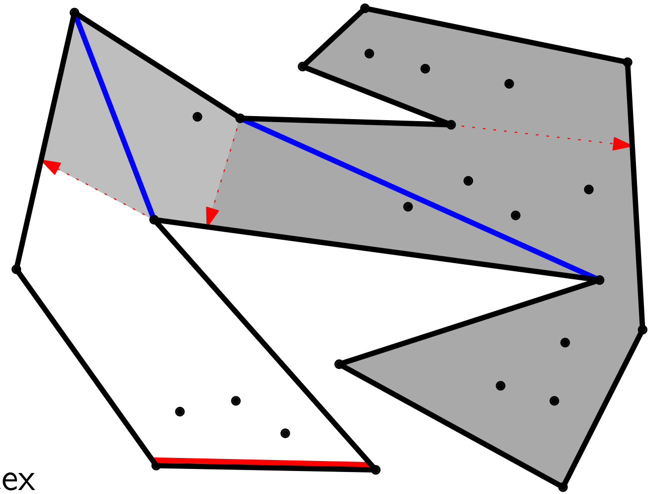

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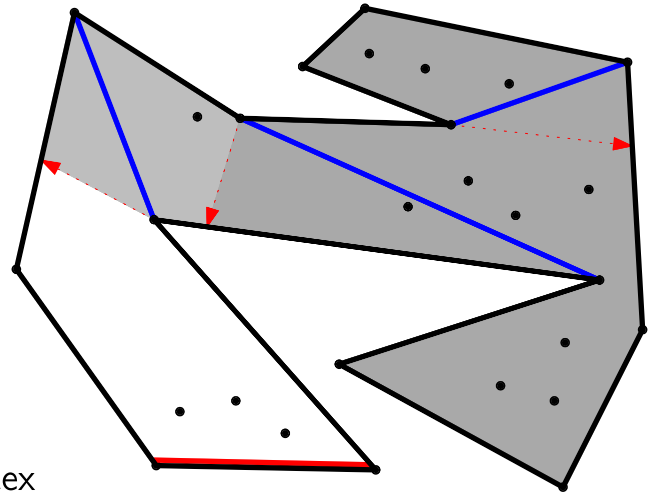

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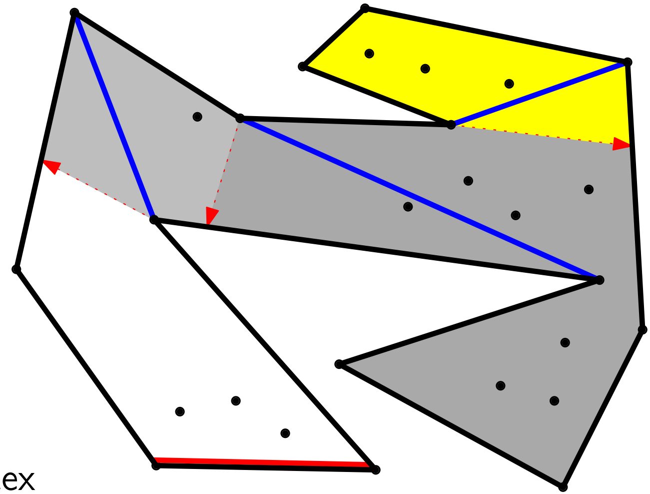

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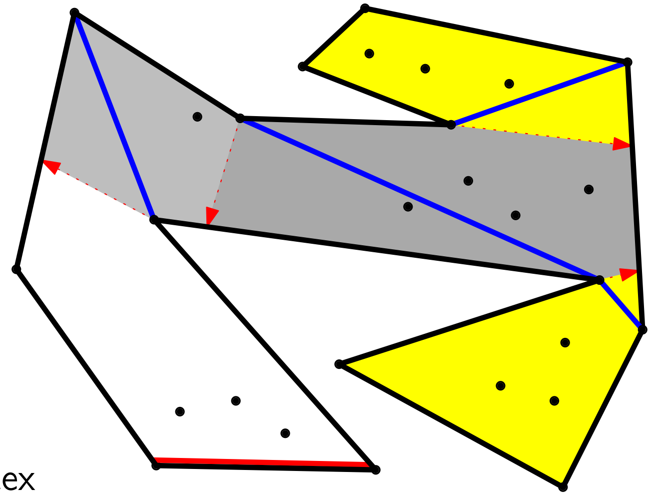

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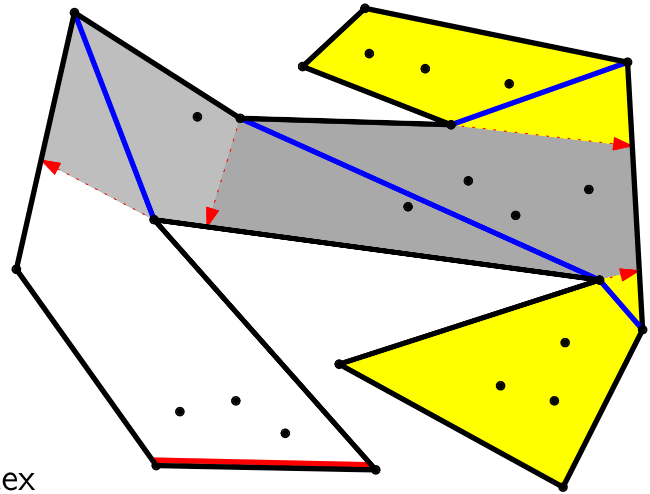

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We can apply the previous observation in each convex region.

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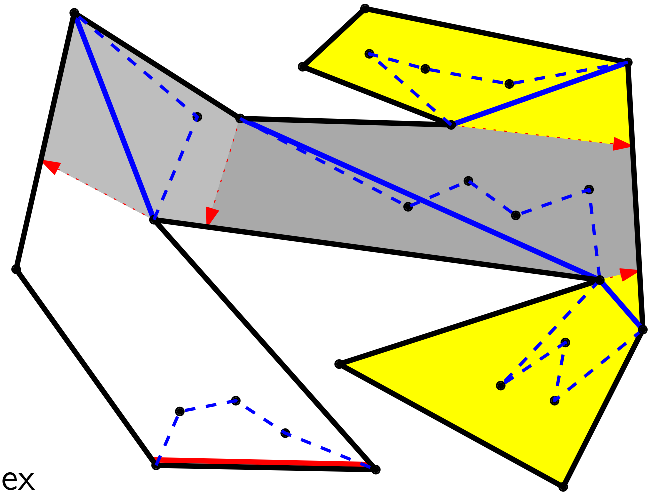

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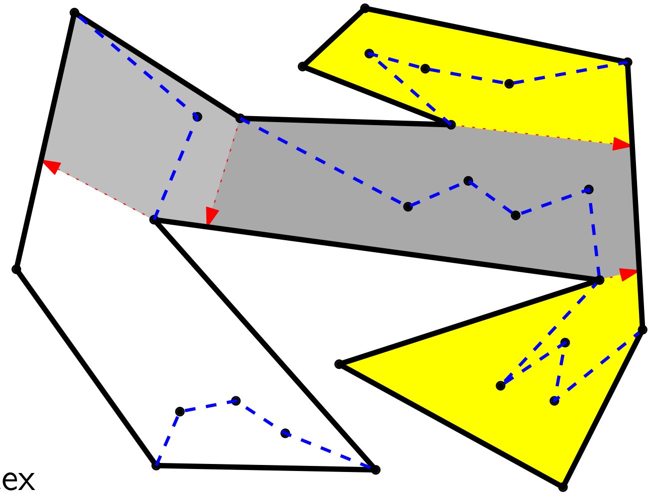

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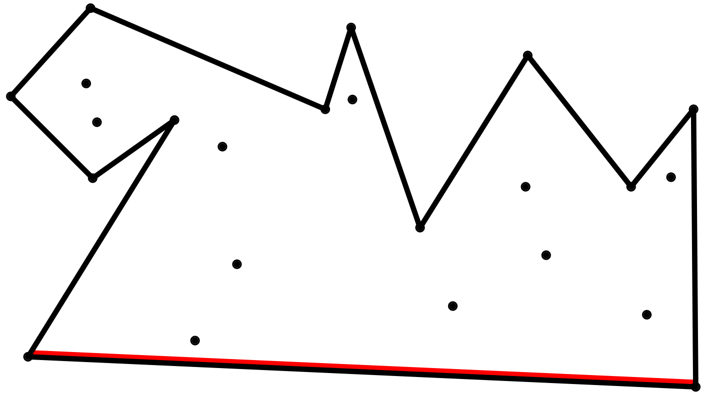

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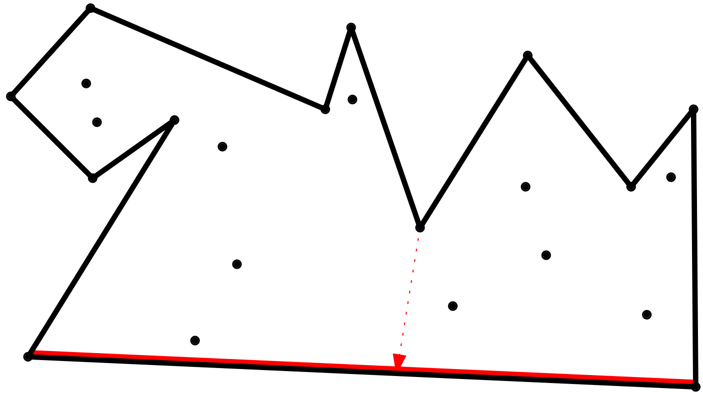

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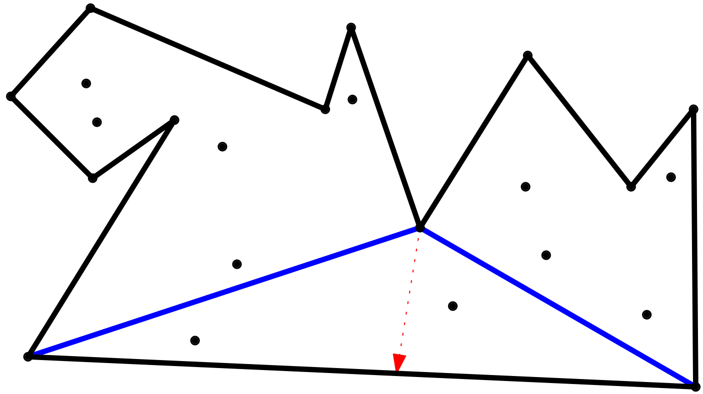

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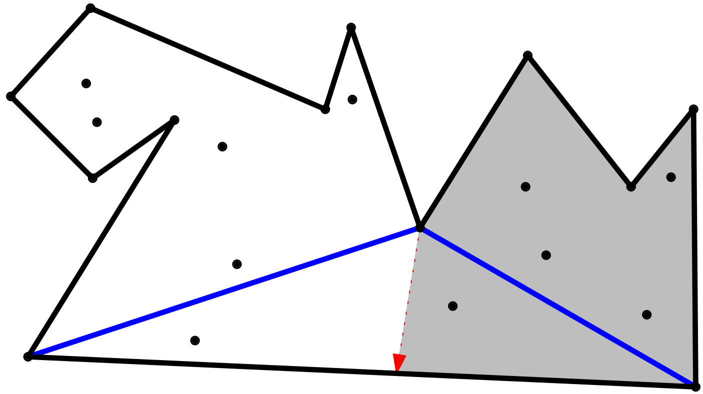

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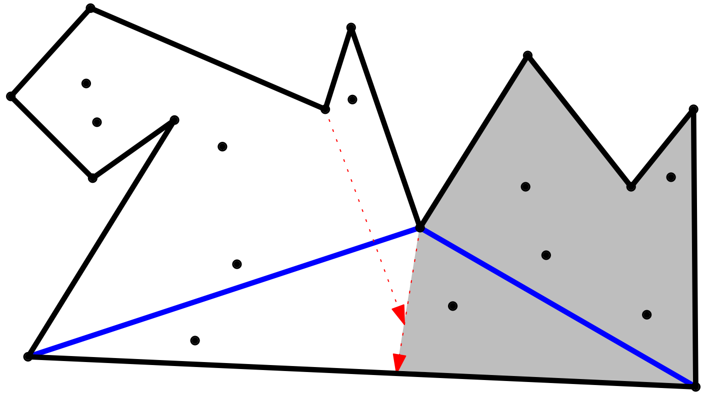

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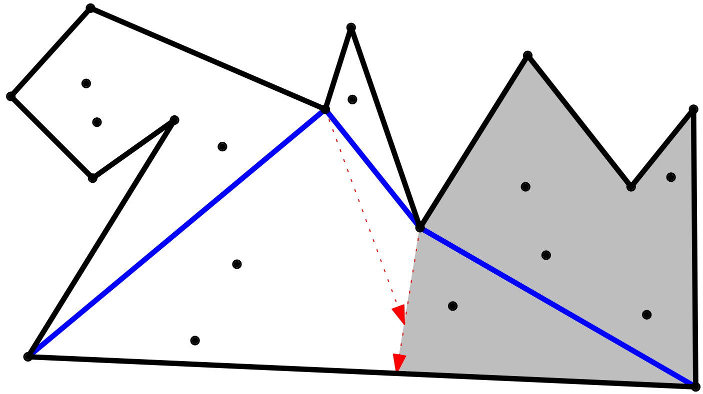

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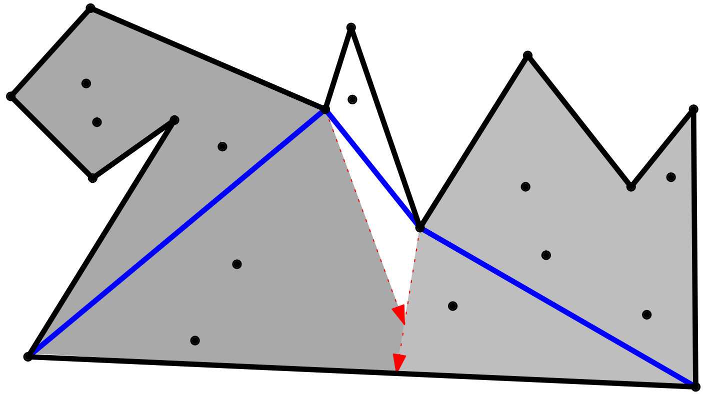

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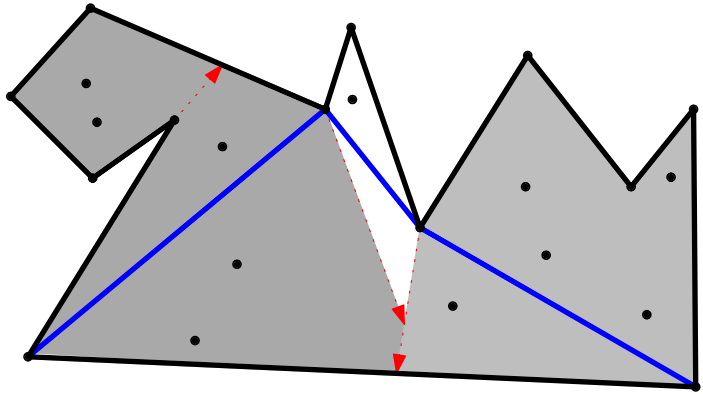

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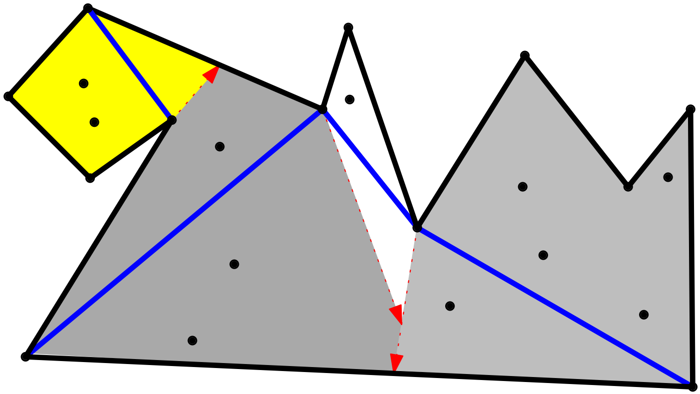

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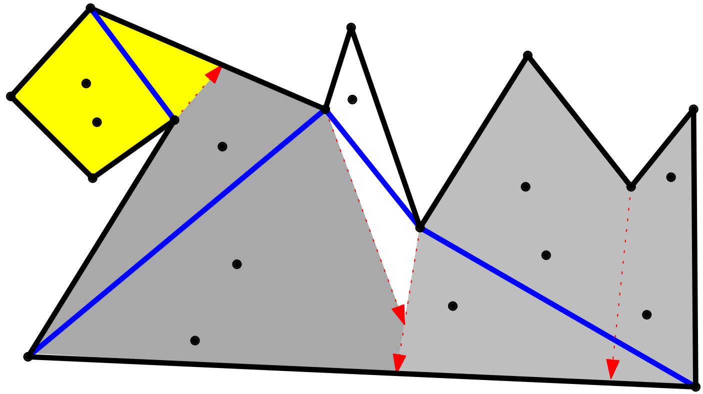

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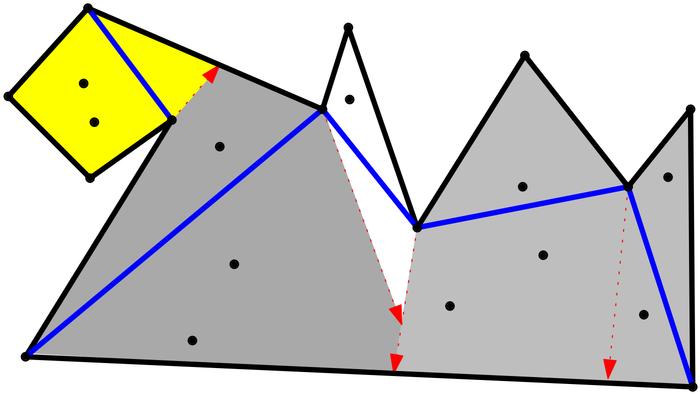

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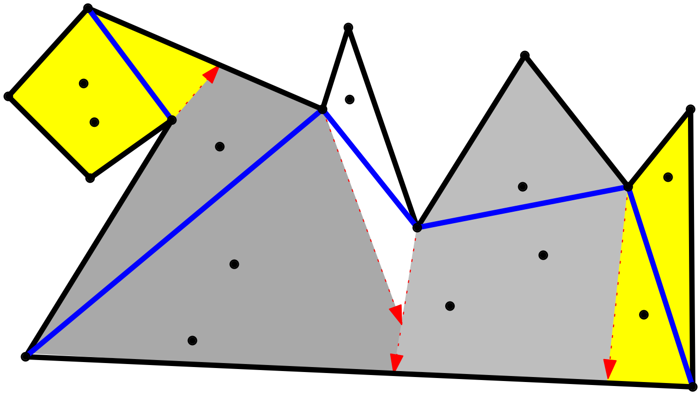

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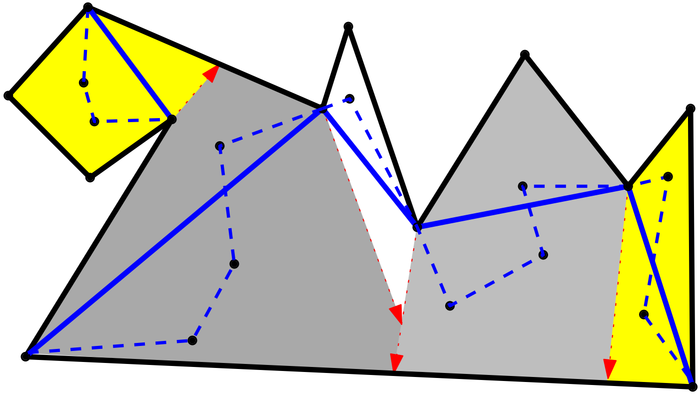

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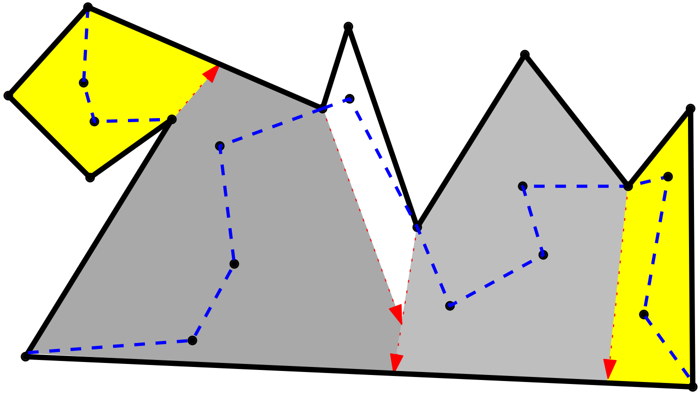

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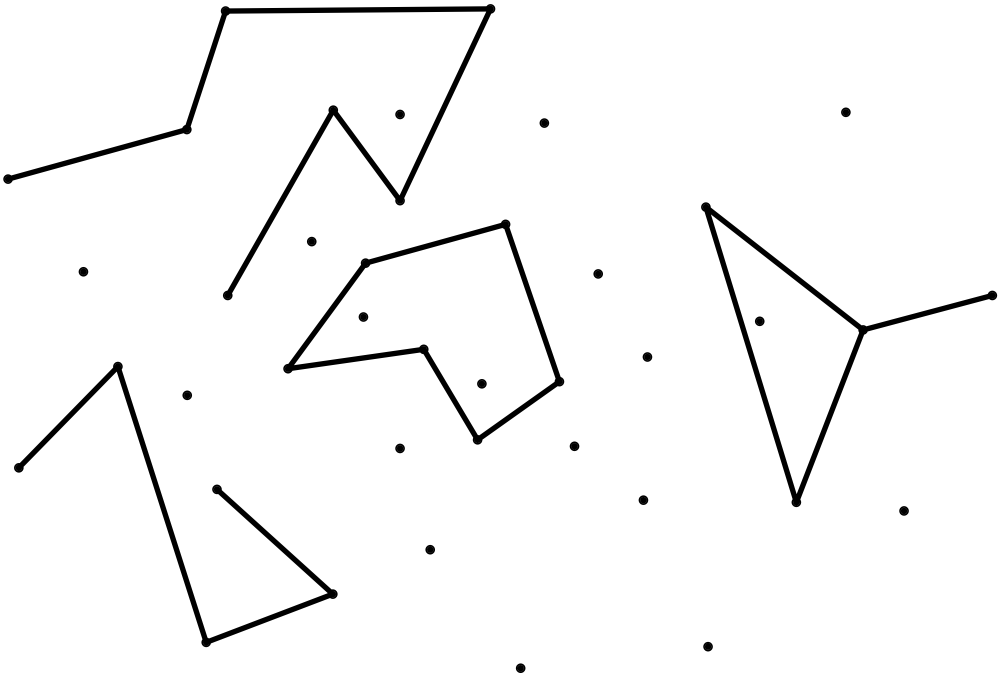
Augmenting the edge connectivity of a PSLG to three

Theorem: If a PSLG with $n \geq 4$ vertices is 3-edge augmentable, then it has an embedding preserving augmentation to a 3-edge connected PSLG with at most $2n - 2$ new edges.

0. Create a deformable edge for each compact face.
1. Add all convex hull edges
   [all hull edges are in one connected component].
2. Connect all non-singleton components
   [one big component and singletons].
3. Eliminate bridges
   [a big 2-edge connected component and singletons].
4. Add a new edge at each reflex vertex in the interior of $\mathrm{ch}(G)$
   [a big 2-edge connected component, all 2-bridges are along the outer face, and singletons].
5. Connect singletons lying in the interior of $\mathrm{ch}(G)$
   [a 2-edge connected PSLG].
6. Eliminate 2-bridges [a 3-edge connected PSLG].

Augmenting the edge connectivity of a PSLG to three



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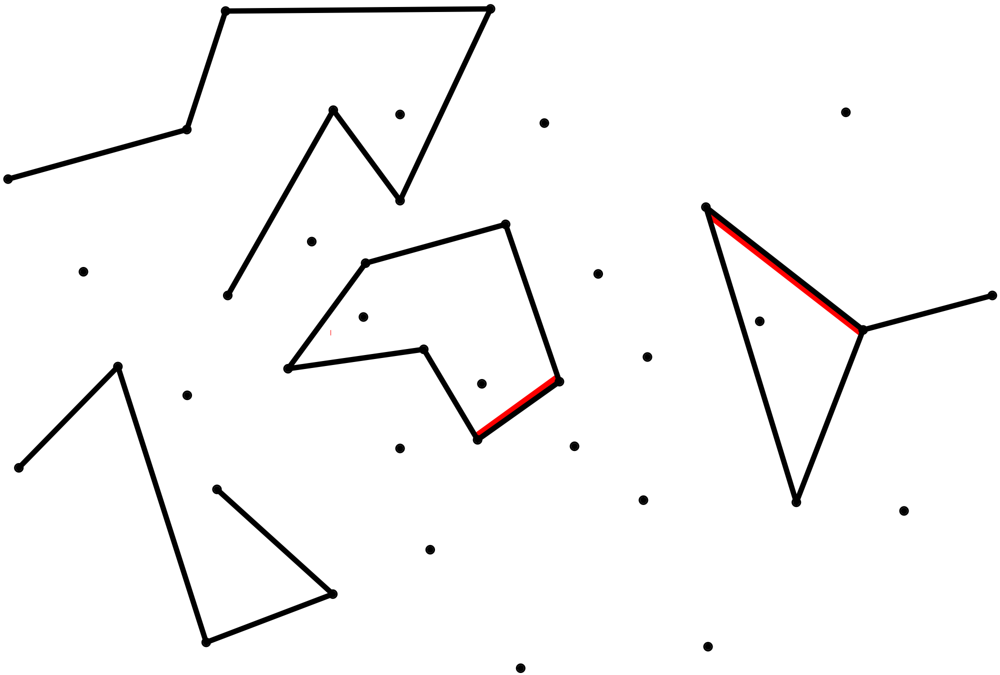

Augmenting the edge connectivity of a PSLG to three

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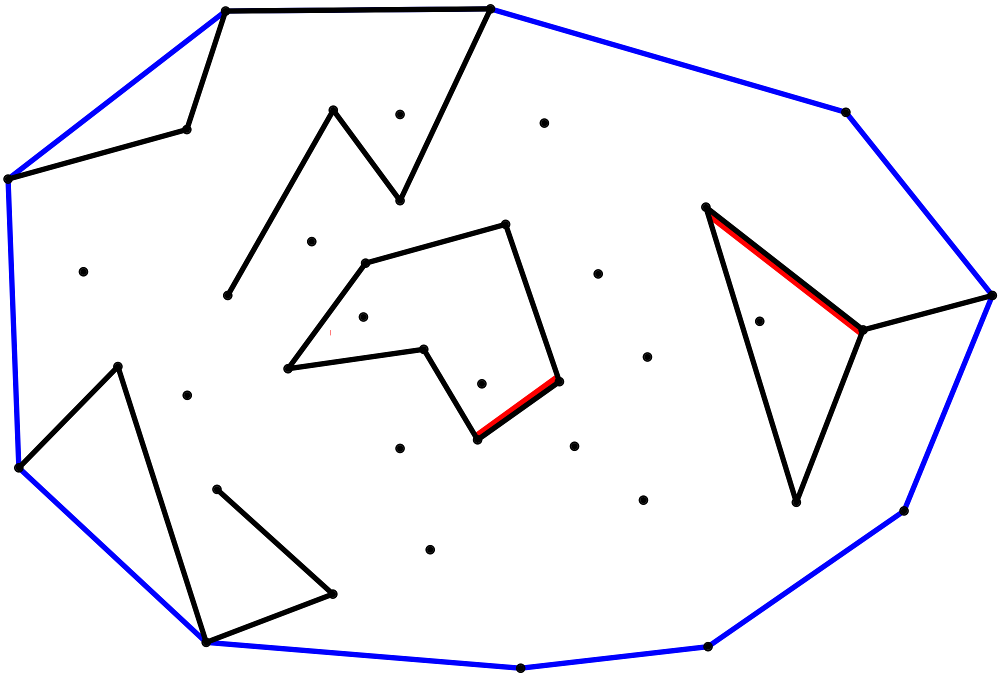

All hull edges are in a single connected component.

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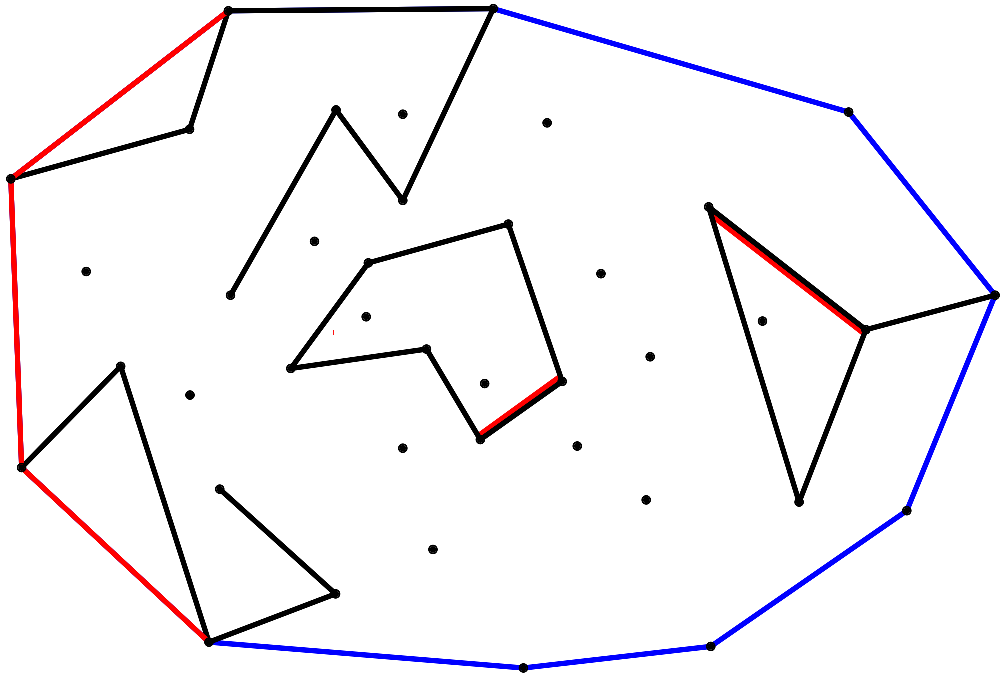

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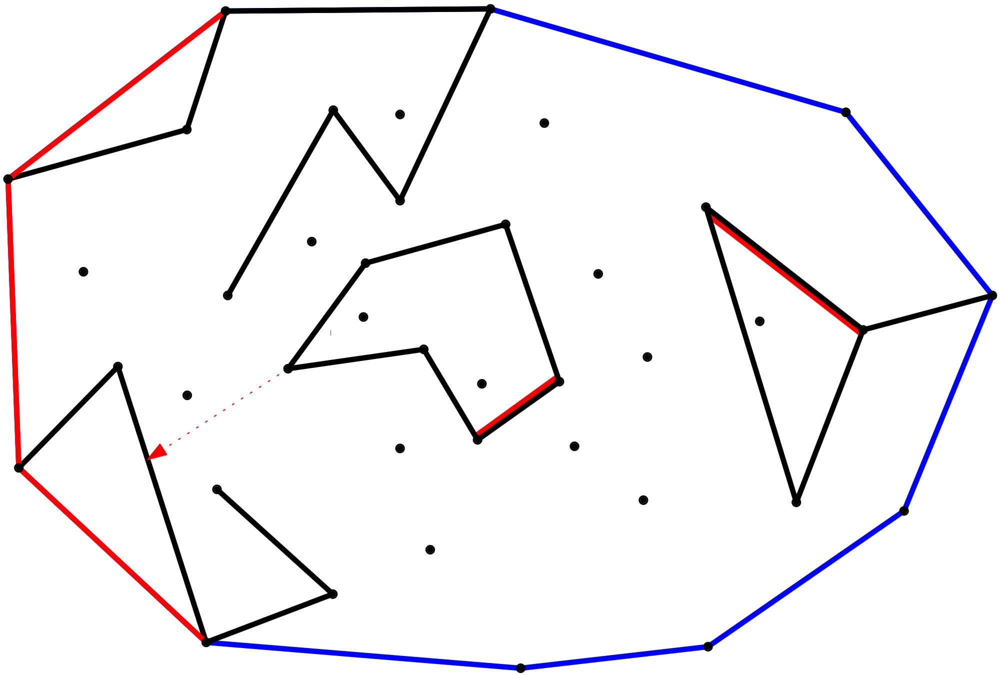
Augmenting the edge connectivity of a PSLG to three

2. Connect all non-singleton components.



One big component and singletons.

Augmenting the edge connectivity of a PSLG to three

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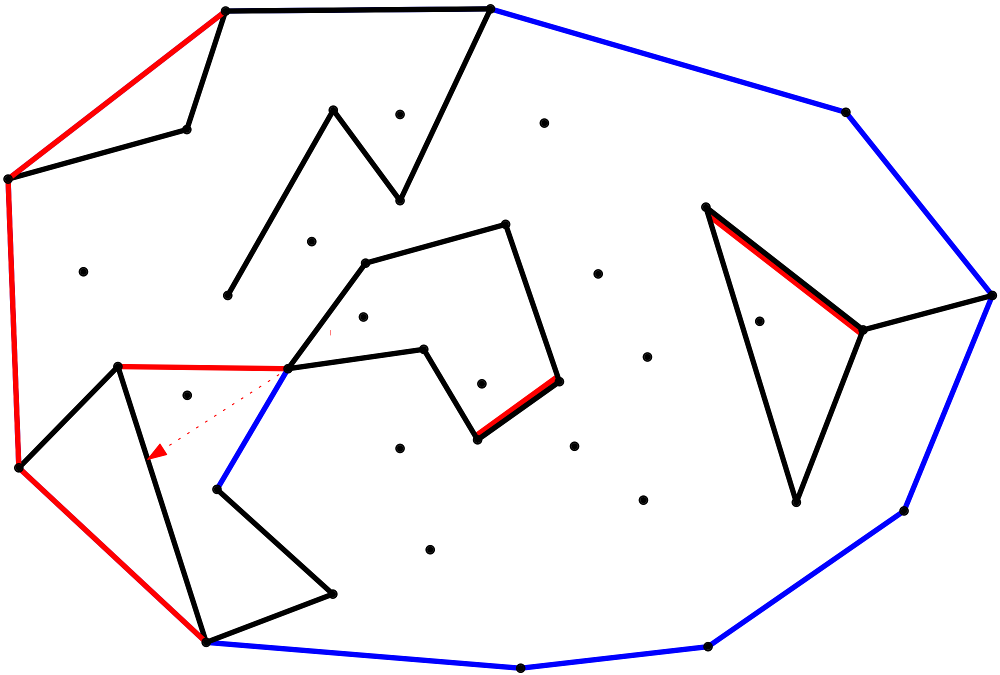
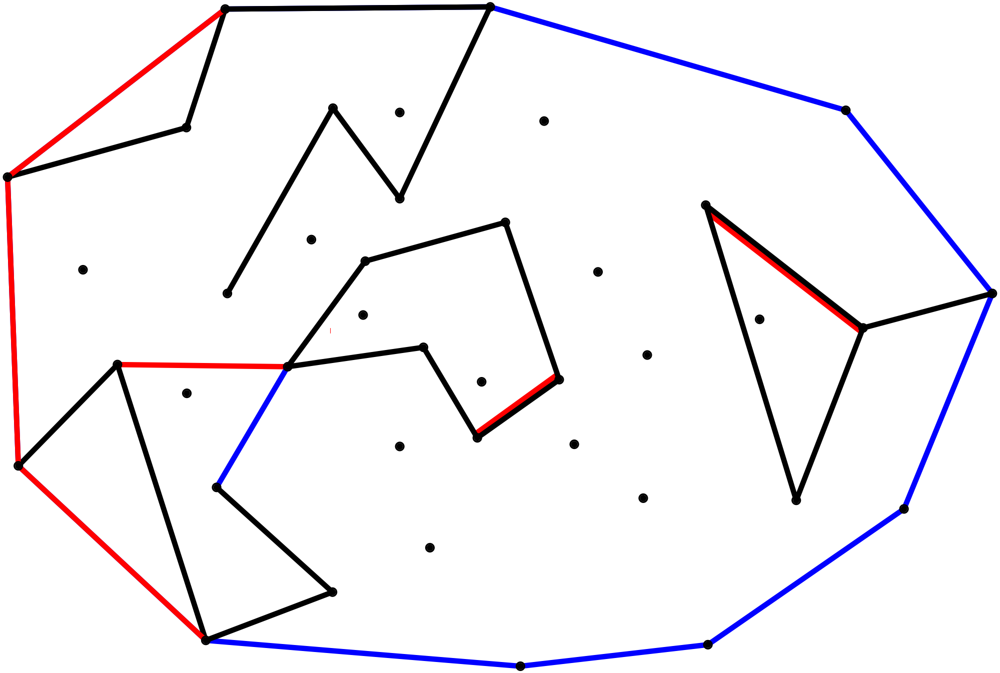


One big component and singletons.

Augmenting the edge connectivity of a PSLG to three

3. Eliminate bridges (Abellanas *et al*, 2008).



A big 2-edge connected component and singletons.

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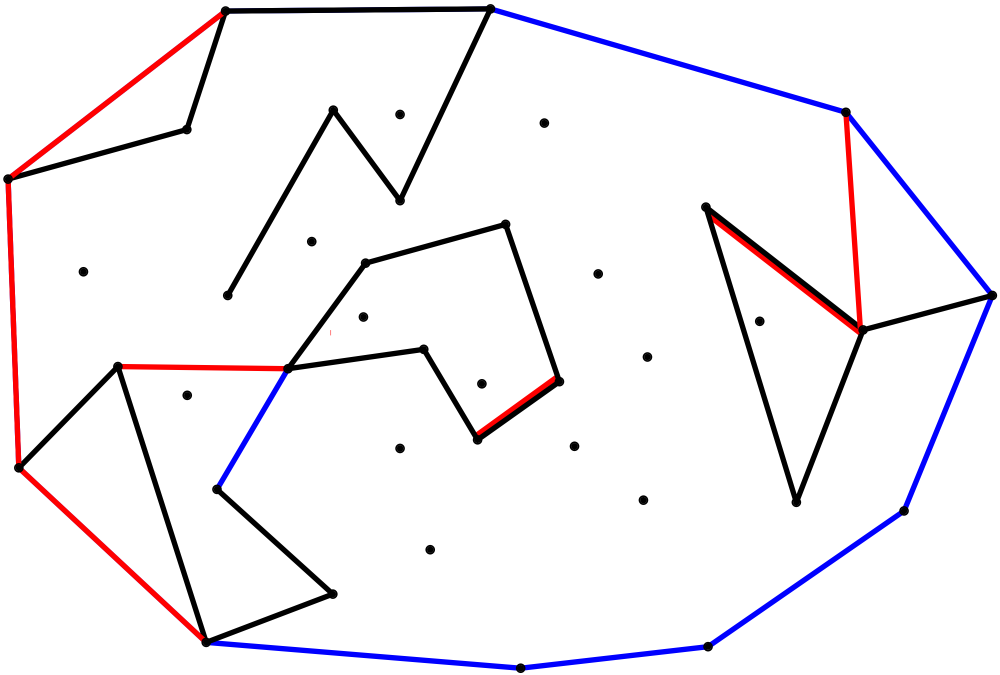

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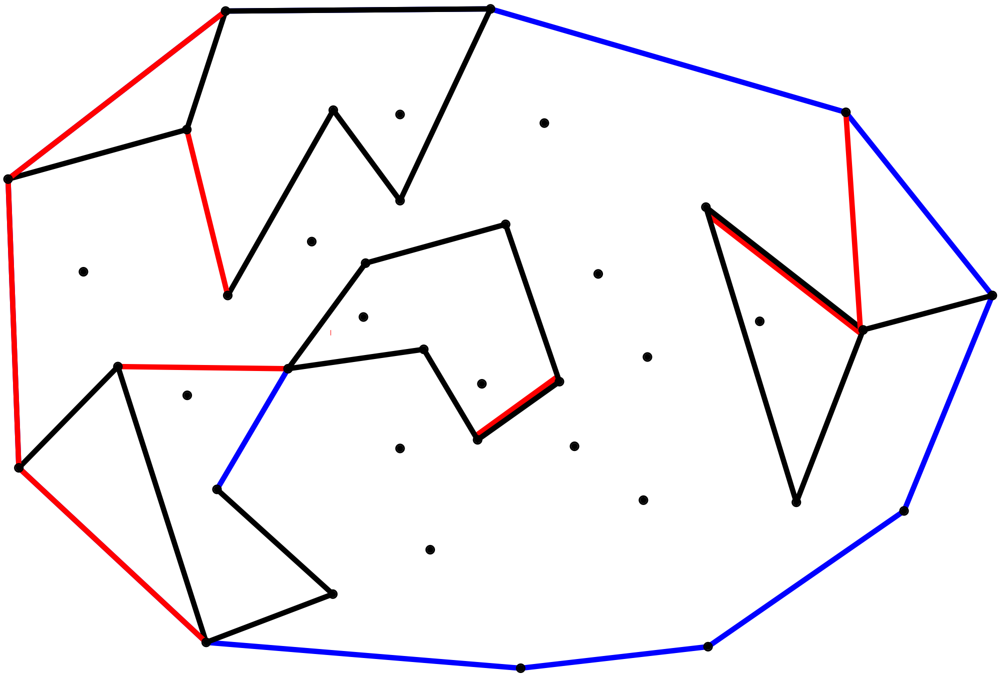

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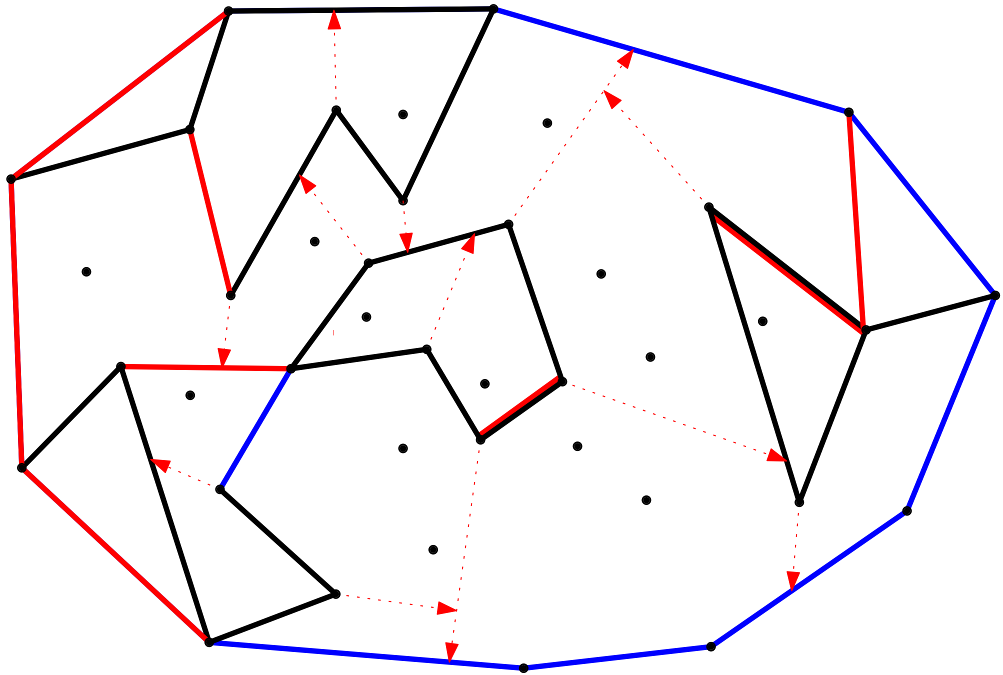
Augmenting the edge connectivity of a PSLG to three

4. Add a new edge at each reflex vertex in the interior of $\mathrm{ch}(G)$.



A big 2-edge connected component, all 2-bridges are along the outer face, and singletons.

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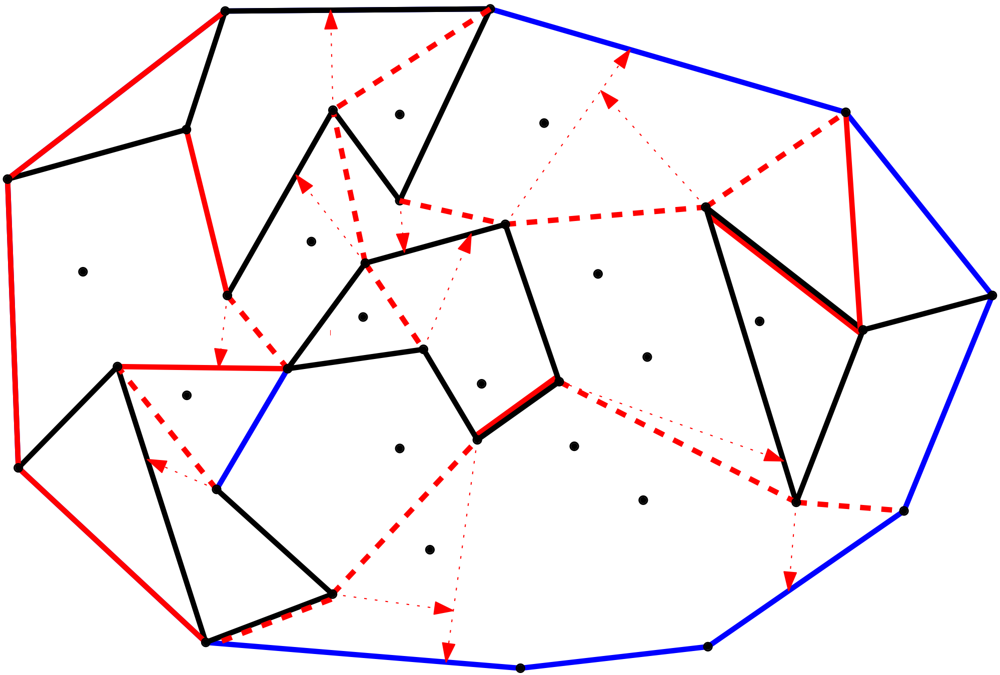

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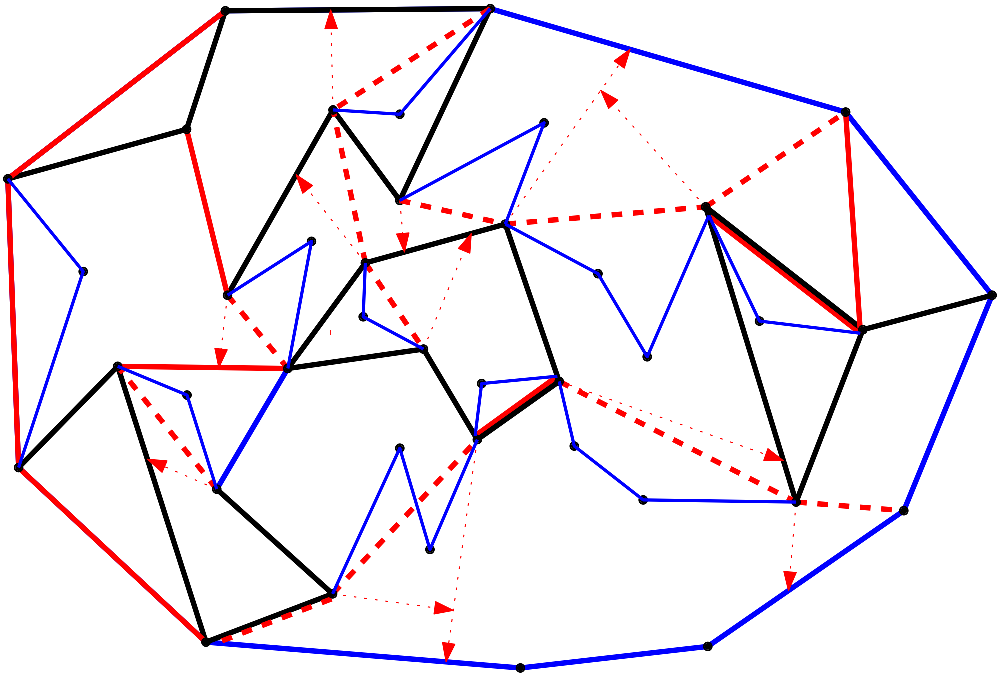
Augmenting the edge connectivity of a PSLG to three

5. Connect singletons lying in the interior of $\mathrm{ch}(G)$.



A 2-edge connected PSLG.

Augmenting the edge connectivity of a PSLG to three

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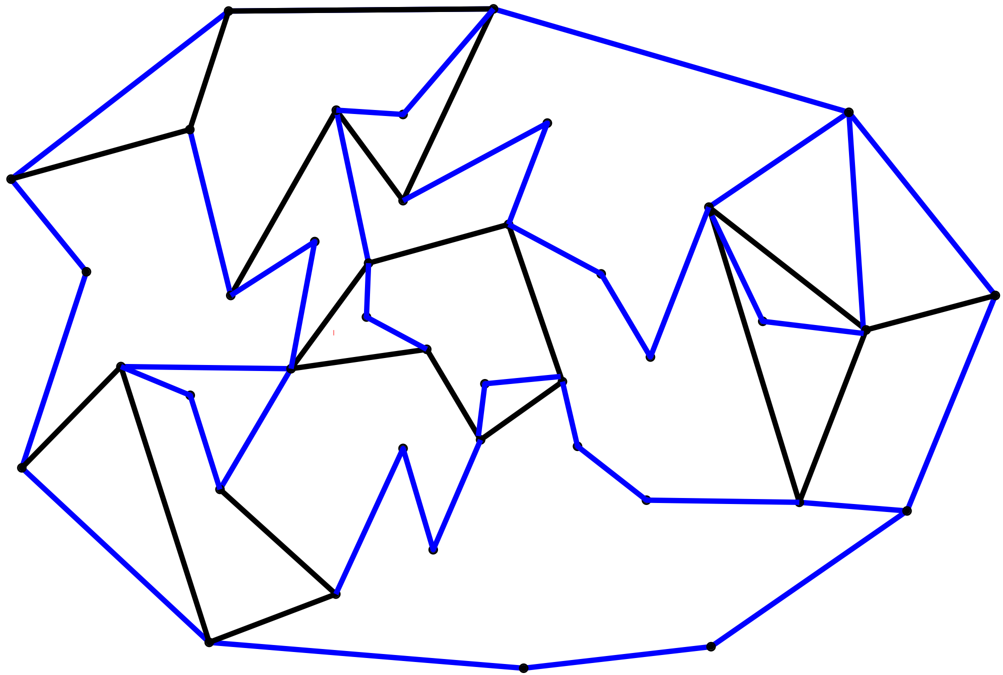
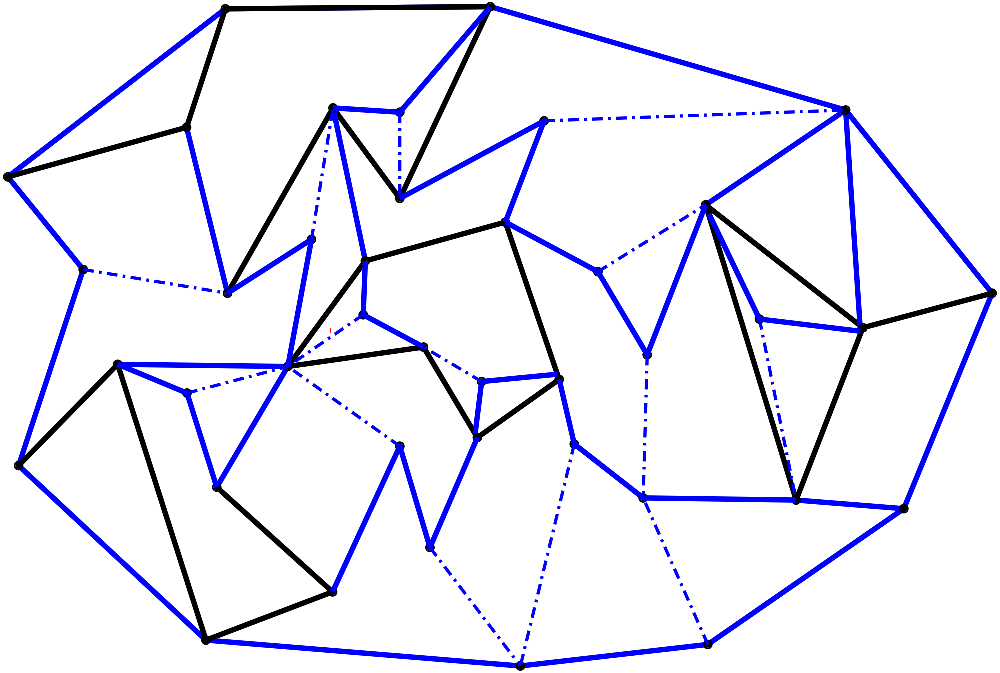


A 2-edge connected PSLG.

Augmenting the edge connectivity of a PSLG to three

6. Eliminate 2-bridges (Tòth and Valtr, 2009)



A 3-edge connected PSLG.

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Theorem: If a PSLG with $n \geq 4$ vertices is 3-edge augmentable, then it has an embedding preserving augmentation to a 3-edge connected PSLG with at most $2n-2$ new edges.

0. Create a deformable edge for each compact face.
1. Add all convex hull edges
   [all hull edges are in one connected component].
2. Connect all non-singleton components
   [one big component and singletons].
3. Eliminate bridges
   [a big 2-edge connected component and singletons].
4. Add a new edge at each reflex vertex in the interior of $\mathrm{ch}(G)$
   [a big 2-edge connected component, all 2-bridges are along the outer face, and singletons].
5. Connect singletons lying in the interior of $\mathrm{ch}(G)$
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Augmenting the vertex connectivity of a PSLG to three?

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Conjecture. If a PSLG with $n \geq 4$ vertices is 3-vertex augmentable, then it has an embedding preserving augmentation to a 3-vertex connected PSLG with at most $2n-2$ new edges.