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Graph Vertex Colorability & the Hardness

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Framework

In General:

- Graph-2-colorability is in N
- Graph-3-colorability is NP-complete (Reduce to $k \geq 4$) [2]
- Graph-3-colorability with $\Delta \leq 4$ is NP-complete [1]

Introduce Planarity:

- Planar graph is 5-colorable (1890), 4-colorable (1977) [3]
- Planar and Δ-free graph is 3-colorable (1959) [4]
- Graph-3-colorability on planar graph is NP-complete [1]
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Introduce Δ-free:

- Graph-3-colorability on Δ-free graph with $\Delta \leq 4$ is NP-complete [5]
- Graph-k-colorability on Δ-free graph with Δ bounded by a function of k, is NP-complete [5]

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DFS, BFS

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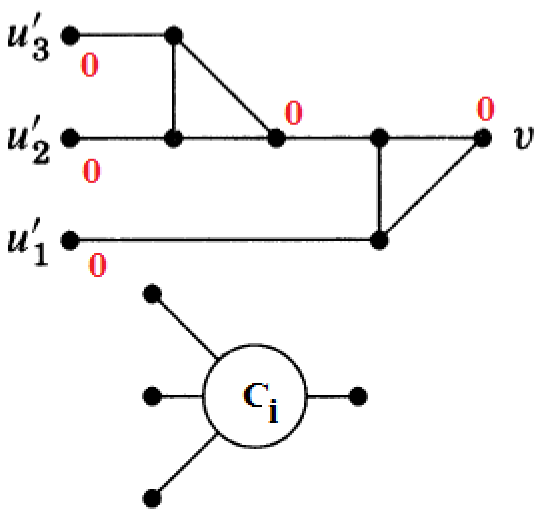
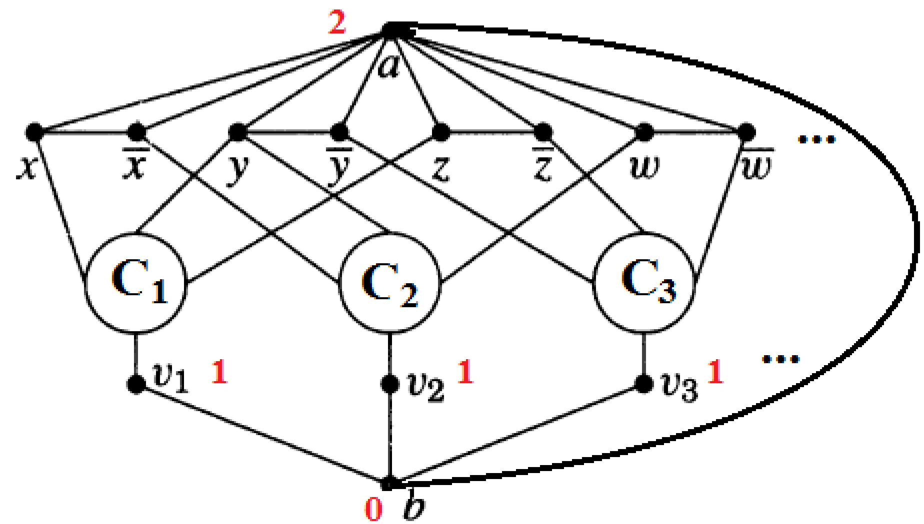
Graph-3-colorability

- In NP: verify a coloring in O(|E|+|V|), BFS or DFS
- Reduction from 3-SAT:
  - ✓ An instance of 3-SAT with variables $U=\{u_j\}$ and clauses $C=\{Ci\}$;
  - ✓ Consider the following transformation to graph G so that there exists a satisfying truth assignment for all $\{C_i\}$ i.f.f. there is a proper 3-coloring {0, 1, 2}:





1) 3-colorable;
2) Output = 0, when all inputs are 0;
   0, 1 or 2, o.w.

1) Assign -> coloring;
2) Coloring -> assign

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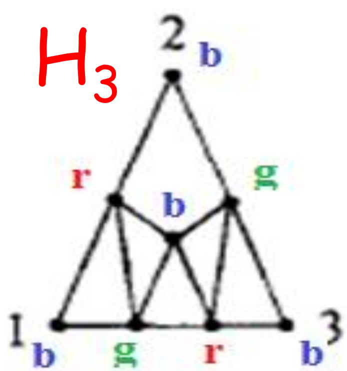
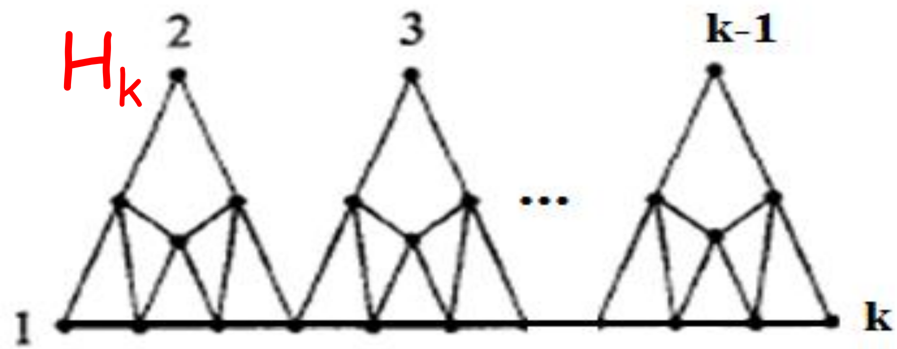
Graph-3-colorability $\Delta \leq 4$ (G_3*)

- In NP: verify a coloring in O(|E|+|V|), BFS or DFS
- Reduction from graph-3-colorability(G_3):
  - ✓ An instance of graph G_3;
  - ✓ Consider the following transformation to graph G_3* so that there exists a proper 3-coloring for G_3 i.f.f. there is a proper 3-coloring for G_3*:
    - ➢ For any u in G_3 with degree k > 4, substitute it with H_k
    - ➢ H_k <- H_{k-1} + H_3

H_3



H_k



1) All the outlet vertices must share the same colors
2) Coloring to G_3 <=> Coloring to G_3*

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Introduce Planarity

- Planar graph is 5-colorable (Heawood, 1890)

Consider 5 components

- Planar graph is 4-colorable (Appel-Haken-Koch, 1977)

Unavoidable configurations

- Planar and Δ-free graph is 3-colorable (Grotzsch, 1959)

Grotzsch's Theorem

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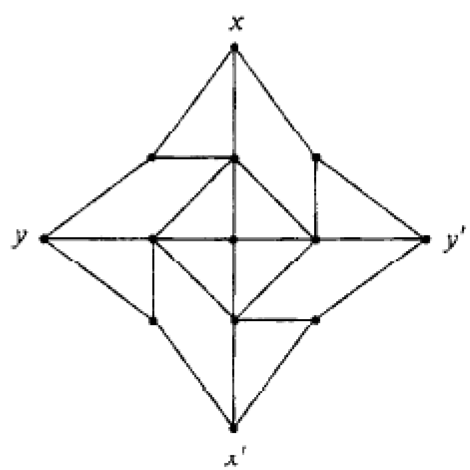
From graph-3-colorability



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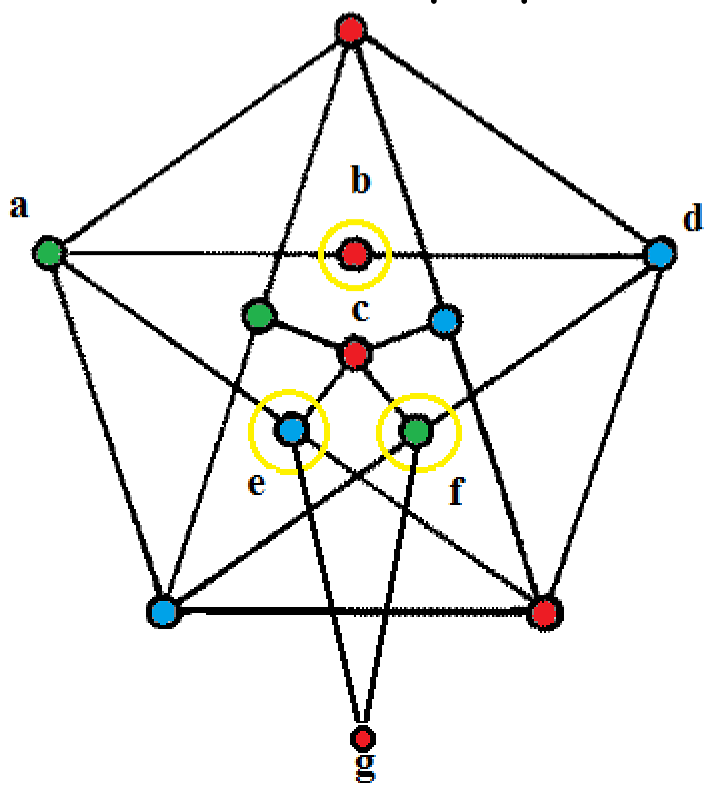
Graph-3-colorability, Δ-free & Δ≤4 (G3*)

- In NP: verify a coloring in O(|E|+|V|), BFS or DFS
- Reduction from G3 with Δ≤4 (G3):
  - ✓ An instance of graph G3 without 4-clique;
  - ✓ Consider the following auxiliary graph used to transform from G3 to graph G3* so that there exists a 3-coloring on G3 i.f.f. there is a proper 3-coloring on G3*:



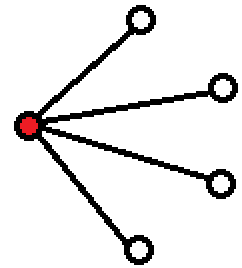
Auxiliary graph H:

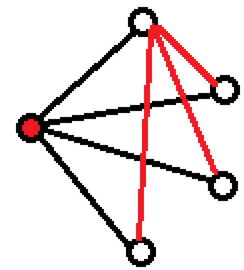
1) Δ-free & 3-colorable;
2) b, e, f are colored 3 different colors;
3) b and c share the same color 1;
4) a and d are colored 2 and 3; therefore e and f are colored 3 and 2.
5) So b and g have same color 1 and degree 2; -- outlets

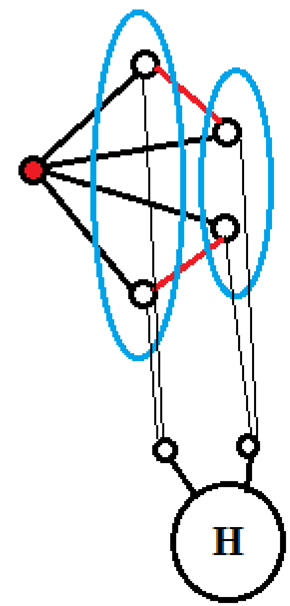
✓ Start with $G_0 = G$:

✓ Repeat for i=1,…,n:

- ➢ Consider each vertex u_j, j=1,2,…,n;
- ➢ If every neighborhood $N(u_j)$ in G_i has no edge or isomorphic to $K_{1,3}$, then $G_i = G_{i-1}$;
- ➢ Else partition $N(u_j)$ into two independent sets with size ≤2 and substitute u_j with $H(u_j)$, connecting the outlets to each sets;







✓ $G^* = G_n$

- ➢ G^* has no triangles and $\Delta \le 4$;
- ➢ A proper 3-coloring to G is also a 3-coloring to G^*
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Graph-k-colorability (Δ?)

- Graph-k-colorability on Δ-free graph with Δ bounded by a function of k, is NP-complete [5]
- A randomized polynomial time algorithm (Karger, Motwani Sudan, 1998 [8]) gives

  $\min\{O(\Delta^{1/3}\log^{1/2}\Delta\log n), O(n^{1/4}\log^{1/2}n)\}$

  approximation on 3-colorable graph; and

  min{O(Δ 1-2/klog1/2 Δlogn), O(n1-3/(k+1)log1/2n)}

  approximation on k-colorable graph.

Approximation Bound

- Non-constant guarantee [10]:
  - For any e>0, it is hard to obtain $\Omega(n^{-e})$ factor approximation unless NP in ZPP(zero-error probabilistic polynomial time).
- Other approximation:
  - 3-color: $O(n^{1/2})$ [7], $\sim O(n^{0.25})$ [8], $O(n^{0.2072})$ [9]

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Thanks!

Questions and Comments?