Optimum Transmission Scheme for a MISO Wireless System with Partial Channel Knowledge and Infinite K factor

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***Abstract*— We study the optimum transmission scheme that maximizes ergodic capacity in a $K \to \infty$ regime for 2×1 MISO systems when the channel knowledge at the transmitter is characterized by a known gain imbalance and a known PDF of the phase shift between antennas. Such a channel scenario can arise in a forward link at the base station when there is a single direct path propagation. We show that the optimum transmit solution is beamforming on the mean value of the phase shift with unequal power input to the antennas. When the phase is completely unknown, the solution reduces to a single antenna transmission.**

1. INTRODUCTION

Most of the existing work on multiple antenna coding assumes that the channel is complex Gaussian distributed with zero mean. Capacity achieving transmission characteristics have been studied widely for such channels [4, 5, 6]. In practice however, it is often found that the wireless channel has a *non-zero mean*, i.e., a finite K factor [2, 7]. This motivates the study of transmit schemes for K factor channels.

In this paper we study a limiting case when the K factor is infinity, which corresponds to a direct dominant path propagation. The results however could be applicable to practical channels with high K factors, say 20dB, which occur in practice [7]. The transmit channel knowledge model assumes a perfectly known antenna gain imbalance, including equal-gain, and a random phase shift with a known PDF. This scenario is typical in the forward link at a base station with direct path propagation and large spacing (≈ 10 carrier wavelengths) between the two transmit antennas. We derive the optimum transmission scheme from the ergodic capacity point of view, based on the known channel gain imbalance, PDF of the channel phase shift and the SNR.

In the next section, we will give details about the channel model and assumptions. Section 3 will set up the problem based on ergodic capacity and summarize the main results. The optimum signal phase shift is then established in Section 4. Section 5 presents results for the optimum signal power allocation and covariance magnitude for both cases of imbalance and balance channel gains. Section 6 gives some simulation examples of the results being applied to Ricean phase shift distribution. We close with some concluding remarks in Section 7.

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Some notations used in this paper: E is expectation, $(.)^*$ is complex conjugate and $(.)^\star$ is the optimum value.

2. CHANNEL MODEL

Consider a MISO system with two transmit antennas. Assuming a single direct propagation path and narrowband antenna array [2], the channel can be modeled as a row vector $[h_1 \;\; h_2]$ where

$$h_2 = \alpha e^{j\phi} h_1 \ .$$

Here α is the *channel gain ratio*, ϕ denotes the *phase shift* between the two antennas and h_1 is a fixed complex channel gain. The difference in antenna gains (when $\alpha \neq 1$) is caused by the local scattering from mounting structure near the antennas and is often found in practice.

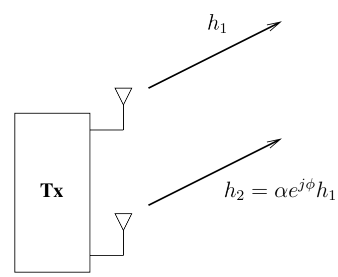

Fig. 1. Single path channel model.

The transmitter can obtain the channel in forward path by estimating the reverse channel in TDD systems or through feedback from the receiver in FDD systems. In both cases, there is likely an error in the estimation due to the time offset/lag between the channel measurement and its use. The antenna gain is likely to be very stable and can be estimated accurately. We therefore assume perfectly known α. The antenna phase shift, however, is highly variable due to the large separation between antennas, leading to errors in the phase estimate. We assume the PDF of the phase shift ϕ is known but not the exact value of the phase. This distribution is circular between 0 and 2π. The precise shape of the phase shift distribution depends on the channel characteristics and the measurement method. A Dirac delta distribution function corresponds to exact phase knowledge, whereas a uniform distribution means no phase information. In fast time varying channels, the phase measurements are more error prone, hence the distribution will tend toward uniform.

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Our analysis requires that the phase shift distribution is *symmetric* around the mean ϕ_0. The exact power allocation at the two transmit antennas does depend on the actual PDF of the phase shift.

3. PROBLEM OVERVIEW

3.1. Problem Setup

We use the ergodic channel capacity under the sum power constraint on transmit antennas as the optimization criterion. It is assumed that the receiver has full knowledge of the channel $[h_1 \quad h_2]$, while the transmitter only knows the PDF $f(\phi)$ of the phase shift and the channel gain ratio α. The ergodic capacity of the channel is achieved by Gaussian input signal $[x_1 \quad x_2]^T$ with zero mean [1] and a *covariance matrix* $\mathbf{R}_{xx}$ which satisfies

$$\begin{array}{lc} \max\limits_{\mathbf{R}_{xx}} & E\log(1+\gamma\mathbf{h}\mathbf{R}_{xx}\mathbf{h}^*) \\ \text{s.t.} & \text{tr}(\mathbf{R}_{xx}) = 1 \ , \end{array} \tag{1}$$

where γ is the total signal to noise ratio with appropriate normalization.

We can absorb h_1 into γ and write the effective channel as

$$\mathbf{h} = [1 \quad \alpha e^{j\phi}] \ ,$$

where $0 \leq \alpha \leq 1$. Taking into account the total transmit power constraint tr$(\mathbf{R}_{xx}) = 1$, the transmit signal covariance matrix $\mathbf{R}_{xx}$ can be expressed as

$$\mathbf{R}_{xx} = \begin{bmatrix} \eta & \frac{1}{2}\rho e^{-j\psi} \\ \frac{1}{2}\rho e^{j\psi} & 1-\eta \end{bmatrix} . \tag{2}$$

Here η is the *fraction of total power* allocated to the first antenna, ψ is the *signal phase shift* and ρ is twice the *magnitude of the covariance* between signals transmitted from the two antennas, $\rho = 2|E[x_1 x_2^*]|$. The three variables η, ψ and ρ define the transmission scheme. The constraints on these variables become

$$\begin{array}{rcccl} 0 & \leq & \eta & \leq & 1 \\ -\pi & \leq & \psi & \leq & \pi \\ 0 & \leq & \rho & \leq & 2\sqrt{\eta(1-\eta)} \ . \end{array} \tag{3}$$

The bounds on η follow immediately from its definition, whereas the bounds on ψ result from a predefined domain of the signal phase shift. The bounds on ρ come from the positive semidefinite property of the covariance matrix $\mathbf{R}_{xx}$.

With these channel and signal models, and the assumption that the phase distribution is symmetric around its mean ϕ_0, the average mutual information can be written as

$$\begin{aligned} \mathcal{I} &= E\log(1+\gamma\mathbf{h}\mathbf{R}_{xx}\mathbf{h}^*) \qquad (4) \\ &= \int_{-\pi}^{\pi} \log\Big[\gamma(1-\alpha^2)\eta + \gamma\alpha\rho\cos(\phi+\psi_0) + \gamma\alpha^2 + 1\Big] f(\phi)d\phi \end{aligned}$$

where $\psi_0 = \phi_0 + \psi$ and $f(\phi)$ is symmetric around zero. We are interested in maximizing (4) by choosing η, ψ and ρ subject to the constraints (3).

3.2. Summary of Results

The optimum transmission scheme is defined by the transmit covariance matrix $\mathbf{R}_{xx}$, which in turn is defined by η, ψ and ρ. These are found based on the known channel parameters at the transmitter, which are the channel phase shift distribution $f(\phi)$, the channel gain ratio α and the SNR γ. The main results can be summarized as

- The optimum signal phase shift $\psi^\star$ is the negative of the estimate channel phase shift ϕ_0 or that plus π, depending on the specific phase shift distribution $f(\phi)$. The $\psi^\star$ value is independent of α and γ. This is derived in Section 4.
- When the channel gain is imbalance ($\alpha < 1$), the optimum transmission scheme is always beamforming. The optimum $\rho^\star$ is a function of $\eta^\star$, and $\eta^\star$ is a function of the phase shift distribution $f(\phi)$, the channel gain imbalance ratio α and the SNR γ. This is derived in Section 5.1.
- When the channel gain is balance ($\alpha = 1$), the optimum solution space includes but is not limited to beamforming. In this case, η vanishes in the average mutual information expression (4). $\rho^\star$ is a function of $f(\phi)$ and γ, then $\eta^\star$ can be chosen arbitrarily within its range subject to the inequality on $\rho^\star$ in (3). This is analyzed in Section 5.2.

4. OPTIMUM $\psi^\star$

The optimum signal phase shift $\psi^\star$ is *independent* of the channel gain ratio α and the SNR γ, and hence, is treated separately in this section.

Theorem 1 *The optimum phase shift $\psi^\star$ between the transmit signals from two antennas is the negative of the estimated channel phase shift ϕ_0 or that plus π, depending on the channel phase shift distribution $f(\phi)$. That is*

$$\psi^\star = -\phi_0 \quad \textit{or} \quad \psi^\star = \pi - \phi_0 \ .$$

Proof. The original problem (1) is a convex optimization one and hence has a unique solution, which leads to a unique solution of $\psi^\star$. Due to symmetry of the phase distribution, from (4), we can rewrite the average mutual information as

$$\begin{aligned} \mathcal{I} = \int_0^{\pi} \log\Big[& g^2 + 2g\gamma\alpha\rho\cos\psi_0\cos\phi + \\ & \gamma^2\alpha^2\rho^2(\cos^2\psi_0 + \cos^2\phi - 1)\Big] f(\phi)d\phi \ , \end{aligned}$$

where $g = \gamma(1-\alpha^2)\eta + \gamma\alpha^2 + 1$. This is a function of $z = \cos\psi_0$, which is an even function of $\psi_0 = \phi_0 + \psi$. If the optimum $z^\star$ is not 1 or -1, then there will be two different values of the optimum phase $\psi^\star$ within the range $[-\pi, \pi]$ that satisfy the original optimization problem. Therefore the optimum value of z must be either 1 or -1, which leads to $\psi^\star = -\phi_0$ or $\psi^\star = \pi - \phi_0$ respectively. This is the result of the symmetry of the channel phase shift distribution. □

The specific value for $\psi^\star$ depends on the phase shift distribution function $f(\phi)$. The choice can be made by evaluating the mutual information $\mathcal{I}$ at the two boundary values $\psi = \phi_0$ and $\psi = \pi - \phi_0$, then pick the value that makes $\mathcal{I}$ larger. Without loss of generality, we assume that the phase shift distribution is such that the optimum signal phase shift is $\psi^\star = -\phi_0$ in the next section.

5. OPTIMUM $\eta^\star$ AND $\rho^\star$

In this section we will derive the optimum set of η and ρ. It turns out that the cases of imbalance channel gain ($\alpha < 1$) and balance channel gain ($\alpha = 1$) have significantly different impact on the optimum $\eta^\star$ and $\rho^\star$. While the solution of imbalance channel gain case can be applied to the balance case too, the latter has a larger solution space. Next we treat these two cases separately.

5.1. Imbalance channel gain

We assume without loss of generality that antenna 1 always has a higher gain than antenna 2, thus α is strictly less than 1. The optimization problem now becomes

$$\begin{array}{ll} \max & \displaystyle\int_0^\pi \log\big[\gamma(1-\alpha^2)\eta + \alpha\rho\gamma\cos\phi + \gamma\alpha^2 + 1\big] f(\phi)d\phi \\ \text{s.t.} & 0 \;\le\; \eta \;\le\; 1 \\ & 0 \;\le\; \rho \;\le\; 2\sqrt{\eta(1-\eta)}\ . \end{array} \tag{5}$$

Optimum signal covariance magnitude $\rho^\star$

Theorem 2 *With $\alpha < 1$, the optimum magnitude of the covariance between the transmit signals from the two antennas is*

$$\rho^\star = 2\sqrt{\eta(1-\eta)}\ . \tag{6}$$

Hence the transmit signals has the form

$$x_2 = \zeta e^{-j\phi_0} x_1\ , \tag{7}$$

with ζ given by

$$\zeta = \sqrt{\frac{1-\eta}{\eta}}\ .$$

In other word, the optimum transmission scheme reduces to simple beamforming with unequal power at each antenna.

Proof. Problem (5) is a convex optimization problem. Form the Lagrangian functional

$$\begin{aligned} \mathcal{L}(\eta,\rho) \;=\; & E\log\big[\gamma(1-\alpha^2)\eta + \alpha\rho\gamma\cos\phi + \gamma\alpha^2 + 1\big] \\ & -\lambda\big[\rho - 2\sqrt{\eta(1-\eta)}\big]\ , \end{aligned}$$

where $\lambda \ge 0$. Then the optimizers $\eta^\star$ and $\rho^\star$ are the solutions of the equations formed by setting the partial derivatives of $\mathcal{L}(\eta,\rho)$ to zero. In particular, setting the partial derivative with respect to η to zero leads to

$$E\Big[\frac{\gamma(1-\alpha^2)}{\gamma(1-\alpha^2)\eta + \alpha\rho\gamma\cos\phi + 1 + \gamma\alpha^2}\Big] = \lambda\frac{2\eta-1}{\sqrt{\eta(1-\eta)}}\ .$$

For $\alpha < 1$, the LHS of the above expression is strictly greater than 0 for all distributions of ϕ as the expression under the expectation is always positive. Thus $\lambda^\star > 0$ and $\eta^\star > \frac{1}{2}$. Since $\lambda^\star$ is strictly positive, it means that the upper constraint on ρ is tight (KKT conditions [3]), hence $\rho^\star = 2\sqrt{\eta(1-\eta)}$. This maximum covariance magnitude can be achieved only when the signal sent from one antenna is a scaled version of the signal sent from the other antenna. Applying the phase shift result of Theorem 1, the transmit signals become $x_2 = \zeta e^{-j\phi_0}x_1$. □

Hence the optimum transmit strategy is to do *beamforming* all the time, with the power at each antenna adjusted according to the channel parameters. The optimum covariance matrix $\mathbf{R}_{xx}$ always has rank one in this case.

Optimum power allocation $\eta^\star$

Replacing the optimum $\rho^\star$ into the average mutual information in (5), the problem then becomes finding $0 \le \eta \le 1$ to maximize the following expression

$$E\log\big[(1-\alpha^2)\eta + 2\alpha\sqrt{\eta(1-\eta)}\cos\phi + \alpha^2 + 1/\gamma\big]\ .$$

Since the above expression is concave in η, the optimum $\eta^\star$ is the solution of

$$E\Big[\frac{1-\alpha^2 + \frac{1-2\eta}{\sqrt{\eta(1-\eta)}}\alpha\cos\phi}{(1-\alpha^2)\eta + 2\alpha\sqrt{\eta(1-\eta)}\cos\phi + \alpha^2 + \frac{1}{\gamma}}\Big] = 0\ . \tag{8}$$

The optimum $\eta^\star$ is a function of $f(\phi)$, α, γ.

5.2. Balance channel gain

In this section we treat the case $\alpha = 1$. With the optimum signal phase $\psi^\star = -\phi_0$, the average mutual information becomes

$$\mathcal{I} = 2\int_0^\pi \log\big(\rho\gamma\cos\phi + \gamma + 1\big) f(\phi)d\phi\ .$$

Notice that the signal power allocation η does not appear in this expression as a result of the balance channel gain. Therefore in this case, the covariance magnitude ρ can be found independently of η and the maximization can be taken over $0 \le \rho \le 1$.

Optimum signal covariance magnitude $\rho^\star$

Since the above expression is concave in ρ, the optimum $\rho^\star$ will be the solution of

$$\frac{\partial\mathcal{I}}{\partial\rho} = 2\int_0^\pi \frac{\gamma\cos\phi}{\rho\gamma\cos\phi + \gamma + 1} f(\phi)d\phi = 0\ . \tag{9}$$

The optimum $\rho^\star$ will depend on the specific phase distribution $f(\phi)$ and the SNR γ.

Optimum power allocation $\eta^\star$

Here the optimum $\rho^\star$ and η are only related to each other through the inequality

$$\rho^\star \le 2\sqrt{\eta(1-\eta)}\ . \tag{10}$$

Hence one can choose any power allocation value η that satisfies this relation and design the signal according to the obtained optimum $\mathbf{R}_{xx}$. The rank of this covariance matrix is not restricted to be one as in the imbalance channel gain case. The choice of $\eta^\star$, which influences the rank of $\mathbf{R}_{xx}$, therefore can be divided into two general categories:

- $\mathbf{R}_{xx}$ *rank one - Beamforming:* Here we pick the value of $\eta^\star$ that meets the bound (10) with equality, which gives

$$\eta^\star = \frac{1}{2}(1 \pm \sqrt{1-(\rho^\star)^2}) \ . \tag{11}$$

This is the same solution as the optimum scheme for imbalance channel gain case (6). The optimum signal design is then $x_2 = \zeta e^{-j\phi_0} x_1$, where ζ is given by

$$\zeta = \frac{\rho}{1 \pm \sqrt{1-\rho^2}} \ .$$

- $\mathbf{R}_{xx}$ *full rank:* This can be done by picking a value of η that satisfies the inequality (10) strictly. The signal design problem then becomes finding a coding scheme for the given covariance matrix $\mathbf{R}_{xx}$.

As a special case for both of the above categories, when there is *no phase estimate* (equivalent to uniform phase shift distribution), then the optimum solution is $\rho^\star = 0$, which means sending independent zero-mean Gaussian signals from two antennas with the only constraint that the power adds up to one. Using a single antenna and putting all the transmit power there also achieves the capacity with no randomness, hence single antenna transmission is preferred in this case. That is, $\eta^\star = 1$.

6. SIMULATION EXAMPLES

We use Ricean phase distribution for the channel phase shift in the simulations. This distribution arises from the phase of a constant phasor plus random zero-mean complex Gaussian noise with equal variance on the real and imaginary parts [8, 10]. The *phase estimate quality* can be conveniently described by the Ricean factor β. Assuming an estimated mean ϕ_0 with a given estimate quality β, the phase shift distribution is

$$f_{\mathbf{\Phi}}(\phi) = \frac{1}{2\pi} e^{-\beta^2} \Big\{ 1 + \tag{12}$$
$$\sqrt{\pi}\beta \cos(\phi - \phi_0) e^{\beta^2 \cos^2(\phi-\phi_0)} \left[1 + \mathrm{erf}(\beta \cos(\phi - \phi_0))\right] \Big\} \ .$$

If $\beta = 0$, the phase distribution is uniform, corresponding to no phase estimate. When $\beta \to \infty$, the distribution converges to the Dirac delta function, which means that the estimate is exact.

6.1. Imbalance channel gain

We solve equation (8) numerically to find $\eta^\star$. It turns out that the SNR γ has a very little effect on $\eta^\star$, which can be seen from this equation as $1/\gamma$ can be approximated off for reasonably large values of γ. Simulation results show that we get practically the same value for $\eta^\star$ for all $\gamma \geq -20$dB. Figure 2 shows the plot of the optimum power allocation $\eta^\star$ as a function of the channel gain imbalance ratio α and the phase estimate quality β, at SNR $\gamma = 10$dB.

When $\eta = 1$, it means that only one antenna is used. This is the case when no phase estimate exists ($\beta = 0$), where one antenna transmission on the stronger channel is optimum regardless of the actual α value ($\alpha > 1$ here). As the phase estimate quality increases, the scheme approaches transmit maximum ratio combining (MRC) beamforming, which is optimum when

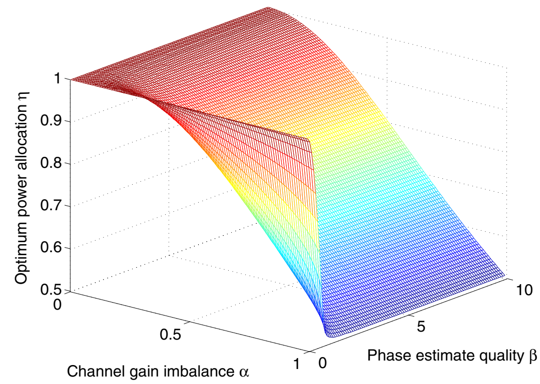

Fig. 2. Optimum $\eta^\star$ in imbalance channel gain case at SNR=10dB.

the channel is known perfectly at the transmitter. The MRC beamforming power allocation is a function of α and is given as $\eta_{\text{MRC}} = (1+\alpha^2)^{-1}$.

6.2. Balance channel gain

Solving equation (9) with the Ricean phase distribution again by numerical means, we obtain the plot for the optimum $\rho^\star$ in Figure 3. The value of $\rho = 1$ means beamforming where signal sent from one antenna is a scaled version of signal sent from the other, whereas $\rho = 0$ means independent signals from the two antennas.

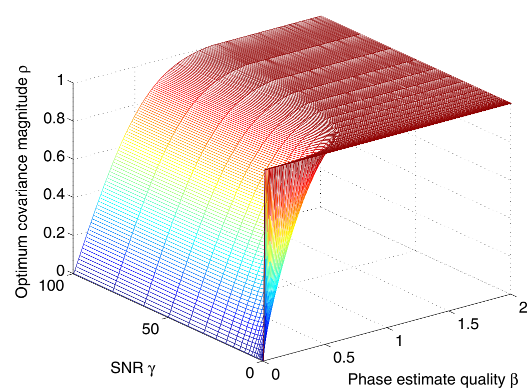

Fig. 3. Optimum $\rho^\star$ in balance channel gain case.

In case of a beamforming solution ($\mathbf{R}_{xx}$ rank one), the power split between the two antennas (11) is regulated according to the phase estimate quality β and the SNR γ. A plot of the optimum power allocation $\eta^\star$ versus β at different values of γ is shown in Figure 4. Since the roles of the two antennas here are symmetric, we only show values for $\eta^\star$ up to 0.5. Notice that in this case, $\eta^\star$ depends significantly on the SNR, in contrast to the imbalance channel gain case.

If the phase estimate quality β is *above* a certain threshold, which is a function of the SNR γ, then the integral on the LHS of (9) is always non-negative for $0 \leq \rho \leq 1$, which leads to *only beamforming* being optimum (where $\rho^\star = 1$ and $\eta^\star = \frac{1}{2}$ in this particular case). This threshold is plotted in Figure 5.

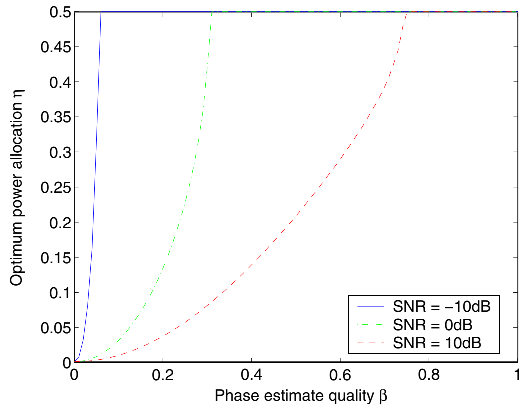

Fig. 4. Portion of the total power allocated to the first antenna in beamforming with balance channel gain.

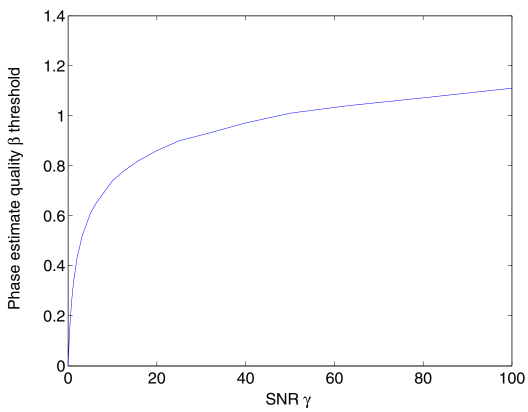

Fig. 5. Phase estimate quality threshold above which only beamforming is optimum for balance channel gain.

7. CONCLUSION

We have studied 2×1 MISO channels with partial transmit channel knowledge when $K \rightarrow \infty$. The optimum signaling scheme is shown to be beamforming on the estimated channel phase with power adjusted according to the known parameters. This is also an indication that beamforming is or close to being optimum in practical MISO channels with high K factors, where the mean amplitude of the channel gain is known but the channel phase is unknown and random.

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REFERENCES

[1] T.M.Cover and J.A.Thomas, *"Elements of Information Theory"*, John Wiley & Sons, Inc., 1991.

[2] A.Paulraj, R.Nabar and D.Gore, *"Introduction to Space-Time Wireless Communications"*, Cambridge University Press, May 2003.

[3] S.Boyd and L.Vandenberghe, *"Convex Optimization"*, Cambridge University Press 2003. Also avail. at URL http://www.stanford.edu/ boyd/cvxbook.html.

[4] I.E.Telatar, *"Capacity of Multi-antenna Gaussian Channels"*, AT&T-Bell Lab. Int. Tech. Mem., June 1995.

[5] E.Visotsky and U.Madhow, *"Space-Time Transmit Precoding with Imperfect Feedback"*, IEEE Trans. on Info. Theory, pp. 2632-2639, Vol. 47, No. 6, Sep. 2001.

[6] S. Jafar, A. Goldsmith, *"Transmitter Optimization and Optimality of Beamforming for Multiple Antenna Systems"*, to appear in Tran. Wireless Comm.

[7] D.S.Baum, et. al., *"Measurement and Characterization of Broadband MIMO Fixed Wireless Channels at 2.5GHz"*, Int. Conf. Per. Wireless Comm., pp.203-206, Dec 2000.

[8] S.O.Rice, *"Statistical Properties of a Sine Wave plus Random Noise"*, Bell Sys. Tech. J. 27, pp. 109-157, Jan 1948.

[9] W.C.Jakes, ed., *"Microwave Mobile Communications"*, IEEE Press, 1993.

[10] P.Z.Peebles,JR., *"Probability, Random Variables, and Random Signal Principles"*, McGrawHill, 1993.