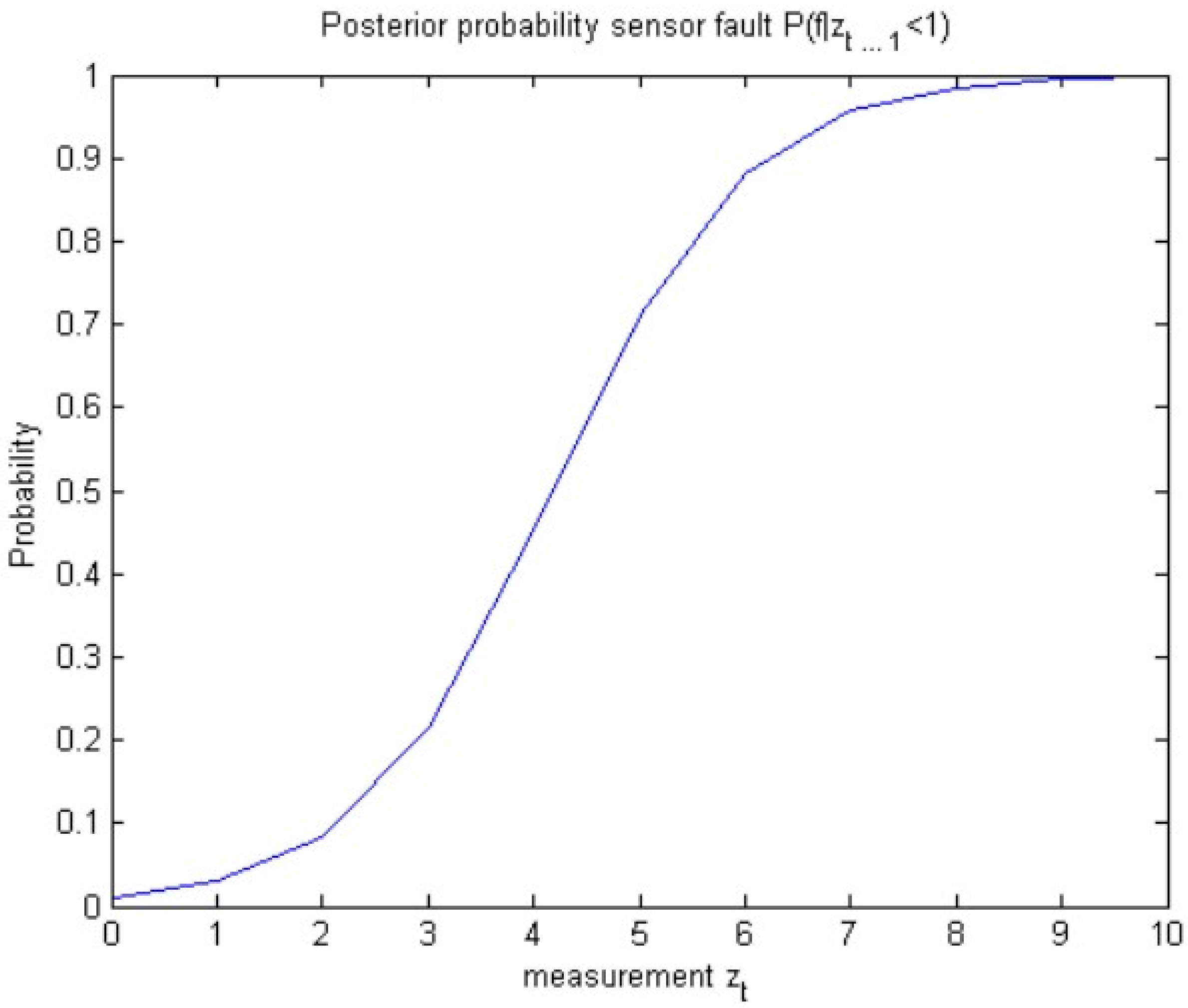
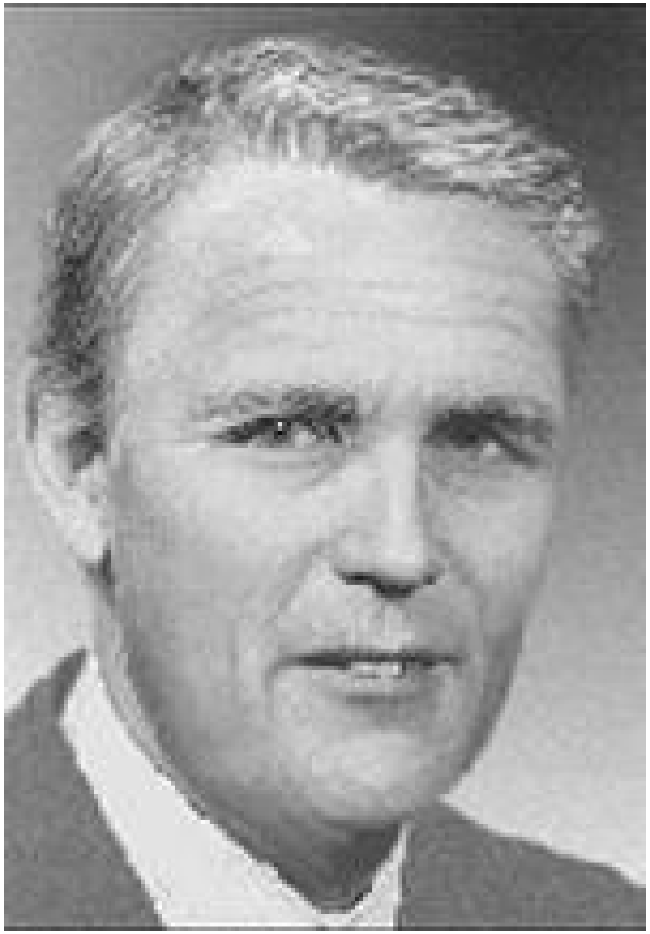
# The Kalman Filter
# (part 1)

# Administrative Stuff


Posterior probability sensor fault P(f|z_t ... 1 <1)
Probability
measurement z_t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7
8
9
10

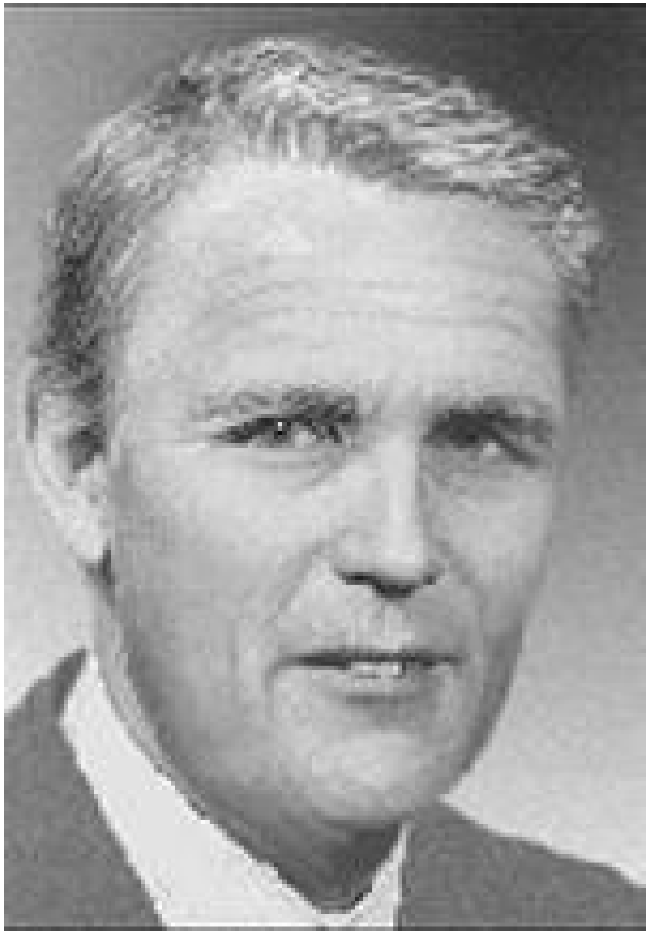
```
Pf(1) = 0.01;

for i = 2:11
    Pz(i) = Pf(i-1) + 0.333 * (1-Pf(i-1))
    Pf(i) = Pf(i-1) / Pz(i);
end
```

# Rudolf Emil Kalman



[http://www.cs.unc.edu/~welch/kalman/kalmanBiblio.html]

# Rudolf Emil Kalman



[http://www.cs.unc.edu/~welch/kalman/kalmanBiblio.html]

# Definition

- A Kalman filter is simply an optimal recursive data processing algorithm

- Under some assumptions the Kalman filter is optimal with respect to virtually any criterion that makes sense.

# Definition

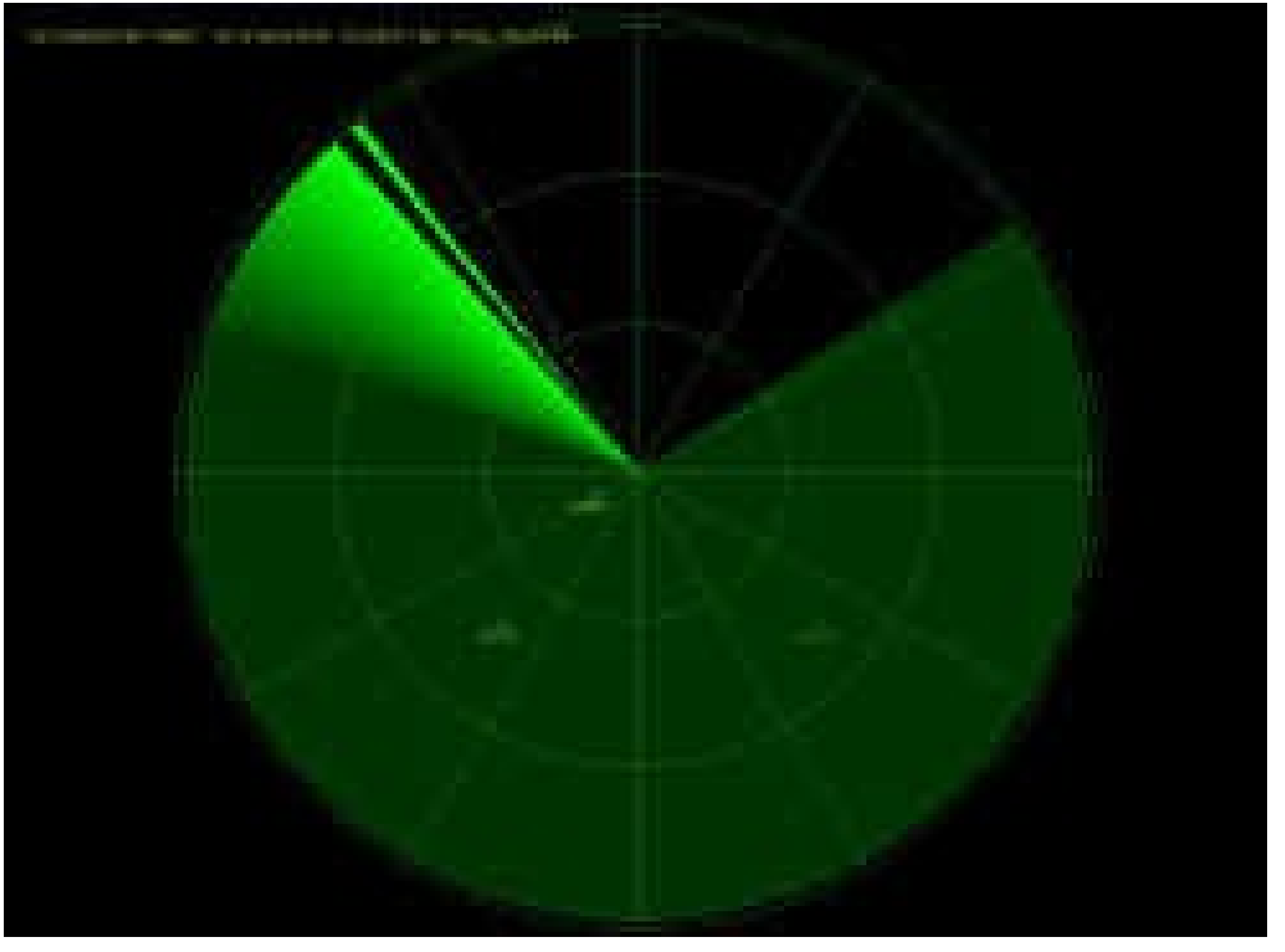
“The Kalman filter incorporates all information that can be provided to it. It processes all available measurements, regardless of their precision, to estimate the current value of the variables of interest.”

[Maybeck (1979)]

# Why do we need a filter?

- No mathematical model of a real system is perfect
- Real world disturbances
- Imperfect Sensors

# Application: Radar Tracking

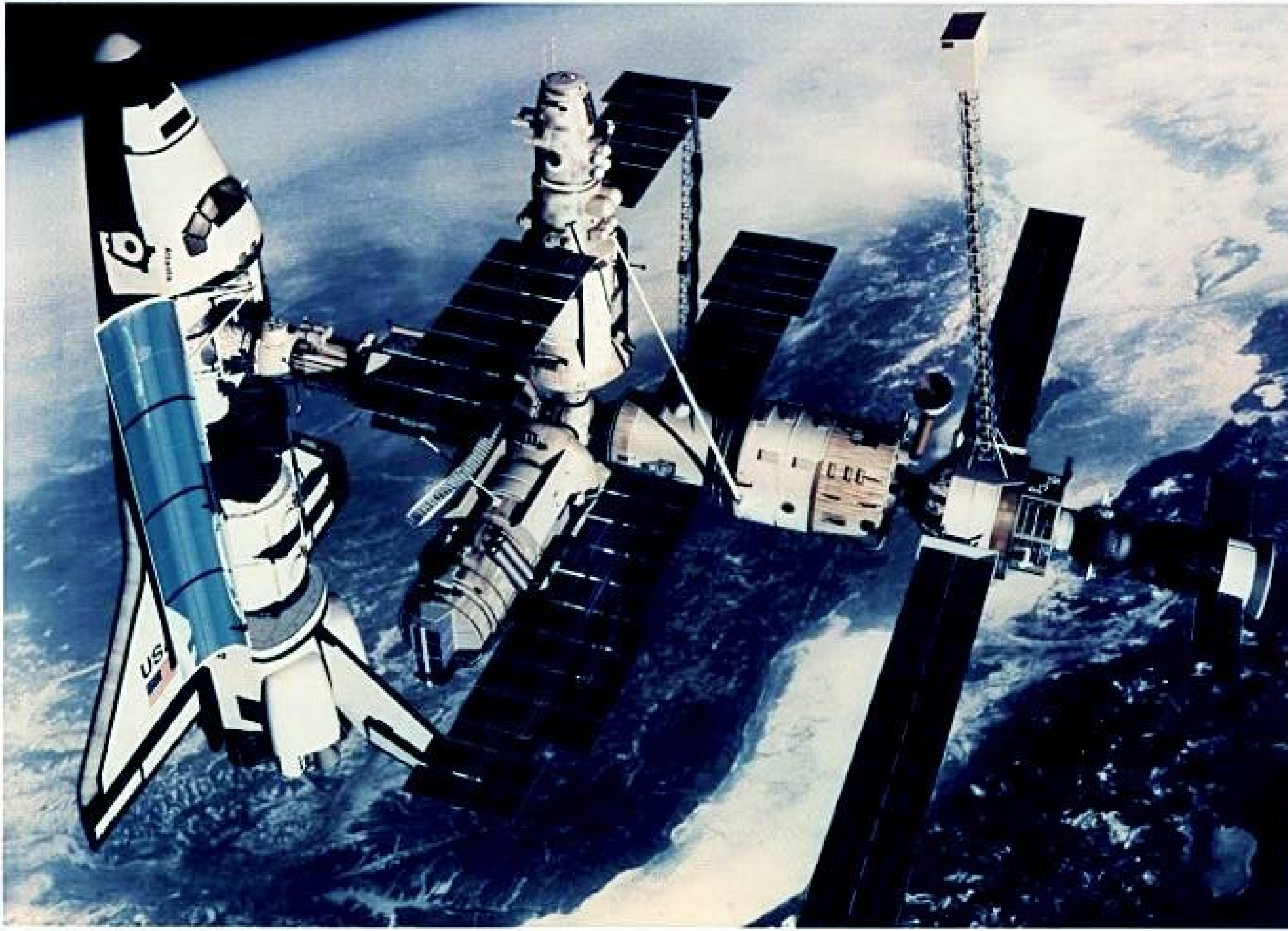


# Application: Lunar Landing

*https://github.com/chrislgarry/Apollo-11*

National Aeronautics and
Space Administration

# Shuttle Docking with Russian *Mir* Space Station



# Application: Missile Tracking

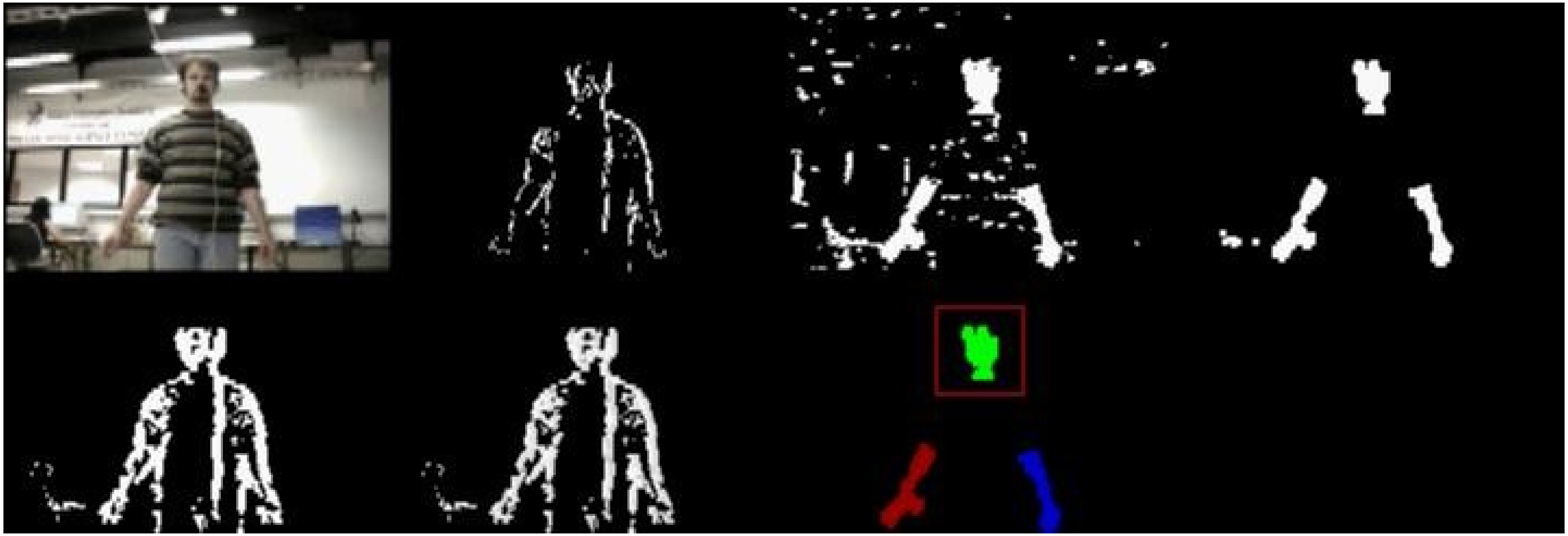
# Application: Sailing

# Application: Robot Navigation

# Application: Other Tracking

# Application: Head Tracking

# Face & Hand Tracking



# A Simple Recursive Example

- Problem Statement:

  Given the measurement sequence:

  $z_1, z_2, \ldots, z_n$ find the mean

[Brown and Hwang (1992)]

# First Approach

1. Make the first measurement $z_1$

   Store $z_1$ and estimate the mean as $\mu_1 = z_1$

2. Make the second measurement $z_2$

   Store $z_1$ along with $z_2$ and estimate the mean as

   $\mu_2 = (z_1 + z_2)/2$

[Brown and Hwang (1992)]

# First Approach (cont'd)

3. Make the third measurement $z_3$
   Store $z_3$ along with $z_1$ and $z_2$ and estimate the mean as

$$\mu_3 = (z_1 + z_2 + z_3)/3$$

[Brown and Hwang (1992)]

# First Approach (cont'd)

n. Make the n-th measurement $z_n$
Store $z_n$ along with $z_1$ , $z_2$ ,…, $z_{n-1}$ and estimate the mean as

$$\mu_n = (z_1 + z_2 + \ldots + z_n)/n$$

[Brown and Hwang (1992)]

# Second Approach

1. Make the first measurement $z_1$

   Compute the mean estimate as

$$\mu_1 = z_1$$

   Store $\mu_1$ and discard $z_1$

[Brown and Hwang (1992)]

# Second Approach (cont'd)

2. Make the second measurement $z_2$

Compute the estimate of the mean as a weighted sum of the previous estimate $\mu_1$ and the current measurement $z_{2:}$

$$\mu_2 = 1/2\ \mu_1 + 1/2\ z_2$$

Store $\mu_2$ and discard $z_2$ and $\mu_1$

[Brown and Hwang (1992)]

# Second Approach (cont'd)

3. Make the third measurement $z_3$

Compute the estimate of the mean as a weighted sum of the previous estimate $\mu_2$ and the current measurement $z_{3:}$

$$\mu_3 = 2/3\ \mu_2 + 1/3\ z_3$$

Store $\mu_3$ and discard $z_3$ and $\mu_2$

[Brown and Hwang (1992)]

# Second Approach (cont'd)

n. Make the n-th measurement $z_n$

Compute the estimate of the mean as a weighted sum of the previous estimate $\mu_{n-1}$ and the current measurement $z_{n:}$

$$\mu_n = (n-1)/n\ \mu_{n-1} + 1/n\ z_n$$

Store $\mu_n$ and discard $z_n$ and $\mu_{n-1}$

[Brown and Hwang (1992)]

# Comparison

**Batch Method**

$$\hat{x}_1 = z_1$$

$$\hat{x}_2 = \frac{z_1 + z_2}{2}$$

$$\hat{x}_3 = \frac{z_1 + z_2 + z_3}{3}$$

$$\hat{x}_n = \frac{z_1 + z_2 + \ldots + z_n}{n}$$

**Recursive Method**

$$\hat{x}_1 = z_1$$

$$\hat{x}_2 = \frac{1}{2}\hat{x}_1 + \frac{1}{2}z_2$$

$$\hat{x}_3 = \frac{2}{3}\hat{x}_2 + \frac{1}{3}z_3$$

$$\hat{x}_n = \frac{n-1}{n}\hat{x}_{n-1} + \frac{1}{n}z_n$$

# Analysis

- The second procedure gives the same result as the first procedure.

- It uses the result for the previous step to help obtain an estimate at the current step.

- The difference is that it does not need to keep the sequence in memory.

[Brown and Hwang (1992)]

# Second Approach (rewrite the general formula)

$$\mu_n = (n-1)/n\ \mu_{n-1} + 1/n\ z_n$$

$$\mu_n = (n/n)\ \mu_{n-1} - (1/n)\ \mu_{n-1} + 1/n\ z_n$$

# Second Approach (rewrite the general formula)

$$\mu_n = (n-1)/n\ \mu_{n-1} + 1/n\ z_n$$

$$\mu_n = (n/n)\ \mu_{n-1} - (1/n)\ \mu_{n-1} + 1/n\ z_n$$

$$\mu_n = \underbrace{\mu_{n-1}}_{\text{Old Estimate}} + \underbrace{1/n}_{\text{Gain Factor}}\ \underbrace{(z_n - \mu_{n-1})}_{\text{Difference Between New Reading and Old Estimate}}$$

**Old Estimate**

**Gain Factor**

**Difference Between New Reading and Old Estimate**

# Second Approach
# (rewrite the general formula)

$$
\begin{aligned}
\hat{x}_n &= \left(\frac{n-1}{n}\right)\hat{x}_{n-1} + \left(\frac{1}{n}\right)z_n \\
&= \left(\frac{n}{n}\right)\hat{x}_{n-1} - \left(\frac{1}{n}\right)\hat{x}_{n-1} + \left(\frac{1}{n}\right)z_n \\
&= \hat{x}_{n-1} + \frac{1}{n}\left(z_n - \hat{x}_{n-1}\right)
\end{aligned}
$$

**Old Estimate**

**Gain Factor**

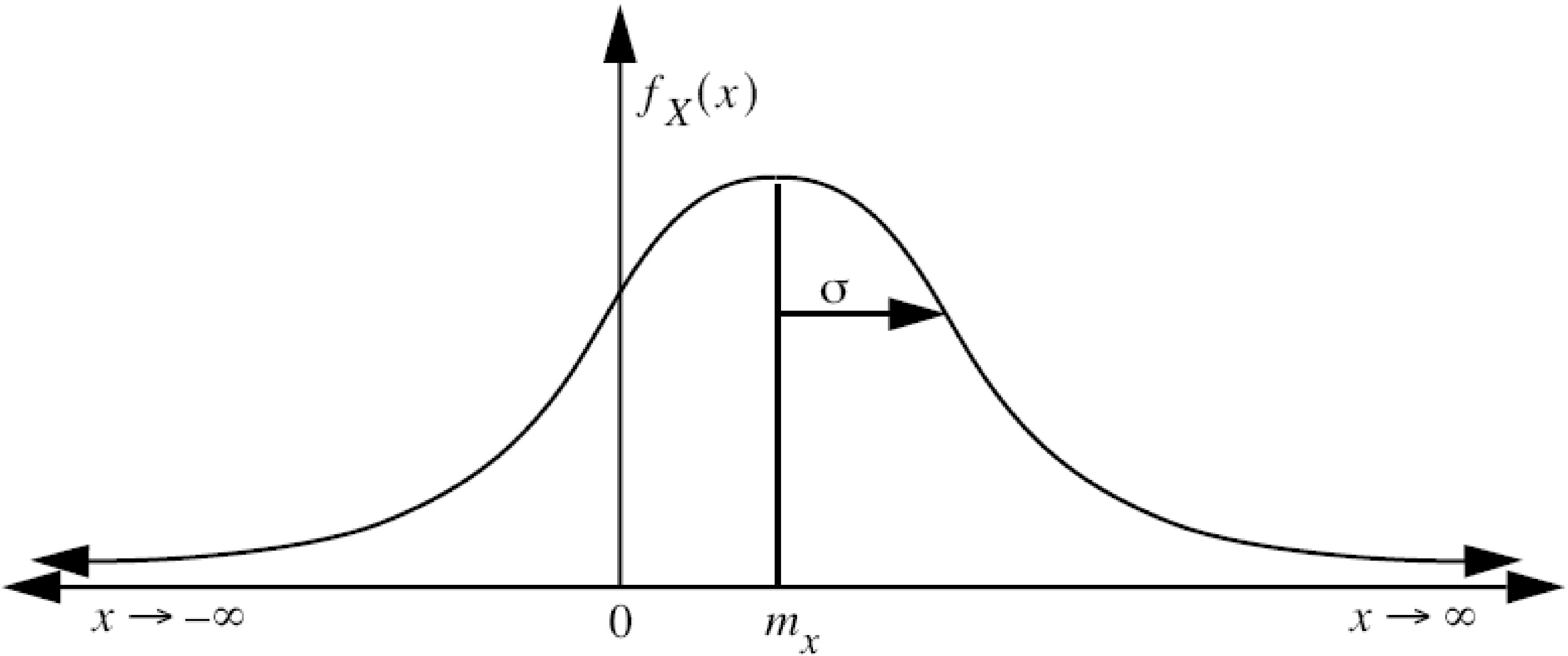
**Difference Between New Reading and Old Estimate**

# Gaussian Properties

# The Gaussian Function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

# Gaussian pdf



# Properties

- If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$
- Then $Y \sim N(a\mu + b, a^2\sigma^2)$

pdf for $Y = aX + b$

$$f_Y(y) = \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{1}{2}\frac{(y-(a\mu+b))^2}{a^2\sigma^2}}$$

# Properties

Finally, if $X_1$ and $X_2$ are independent (see Section 2.5 below), $X_1 \sim N(\mu_1, \sigma_1^2)$, and $X_2 \sim N(\mu_2, \sigma_2^2)$, then

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2), \tag{2.14}$$

and the density function becomes

$$f_X(x_1 + x_2) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-\frac{1}{2}\frac{(x - (\mu_1 + \mu_2))^2}{(\sigma_1^2 + \sigma_2^2)}}. \tag{2.15}$$

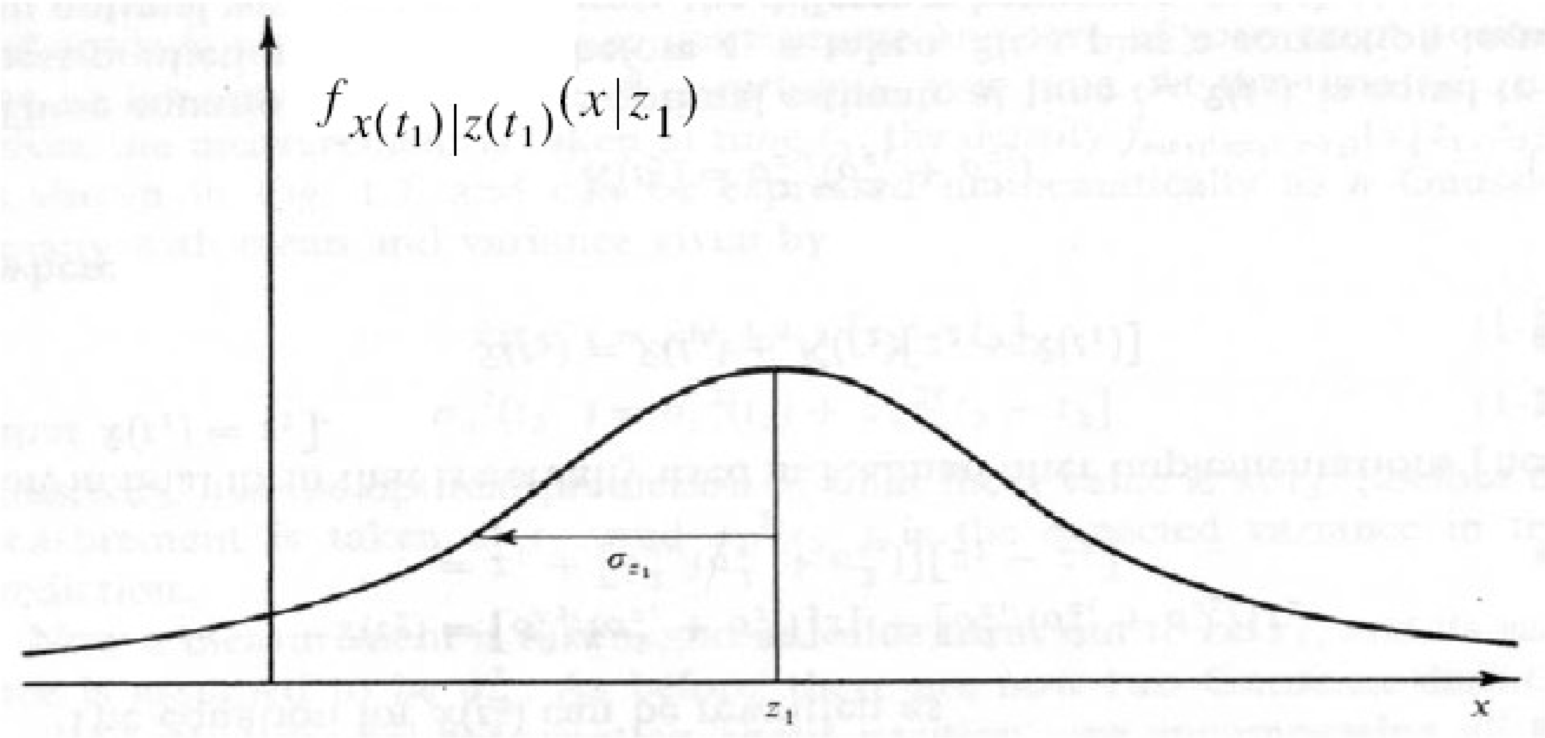
# Summation and Subtraction

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

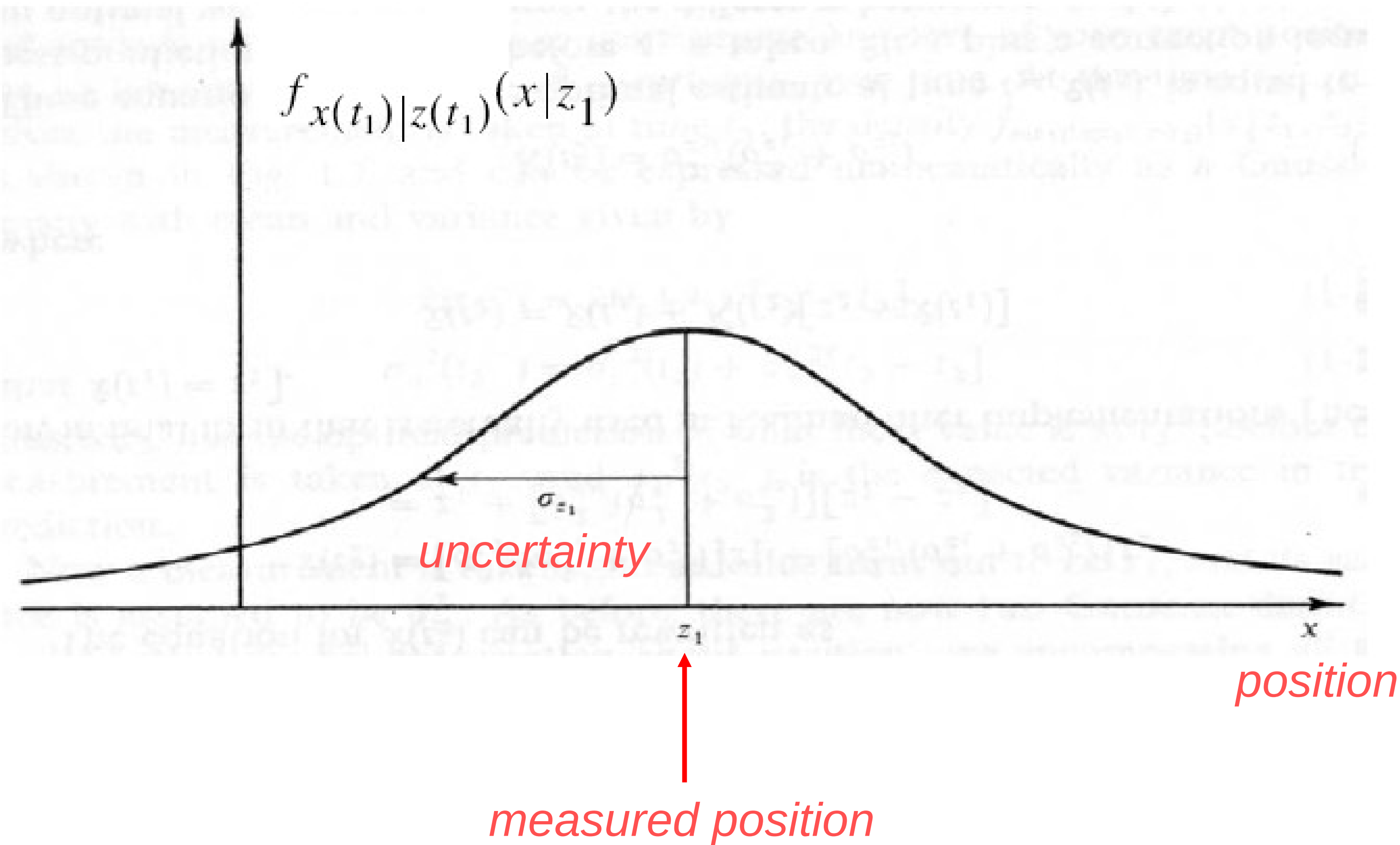
# A simple example using diagrams

# Conditional density of position based on measured value of $z_1$



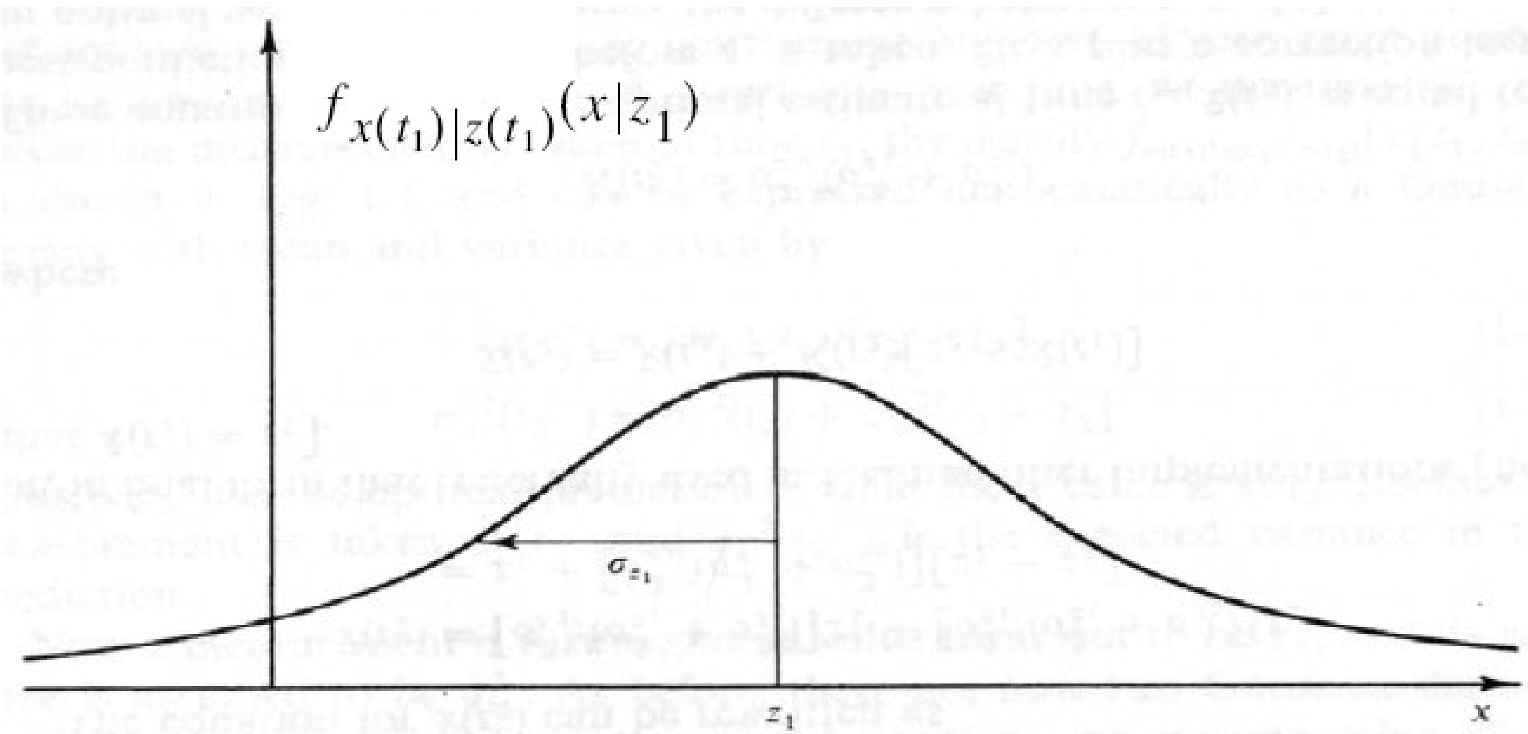

[Maybeck (1979)]

# Conditional density of position based on measured value of $z_1$



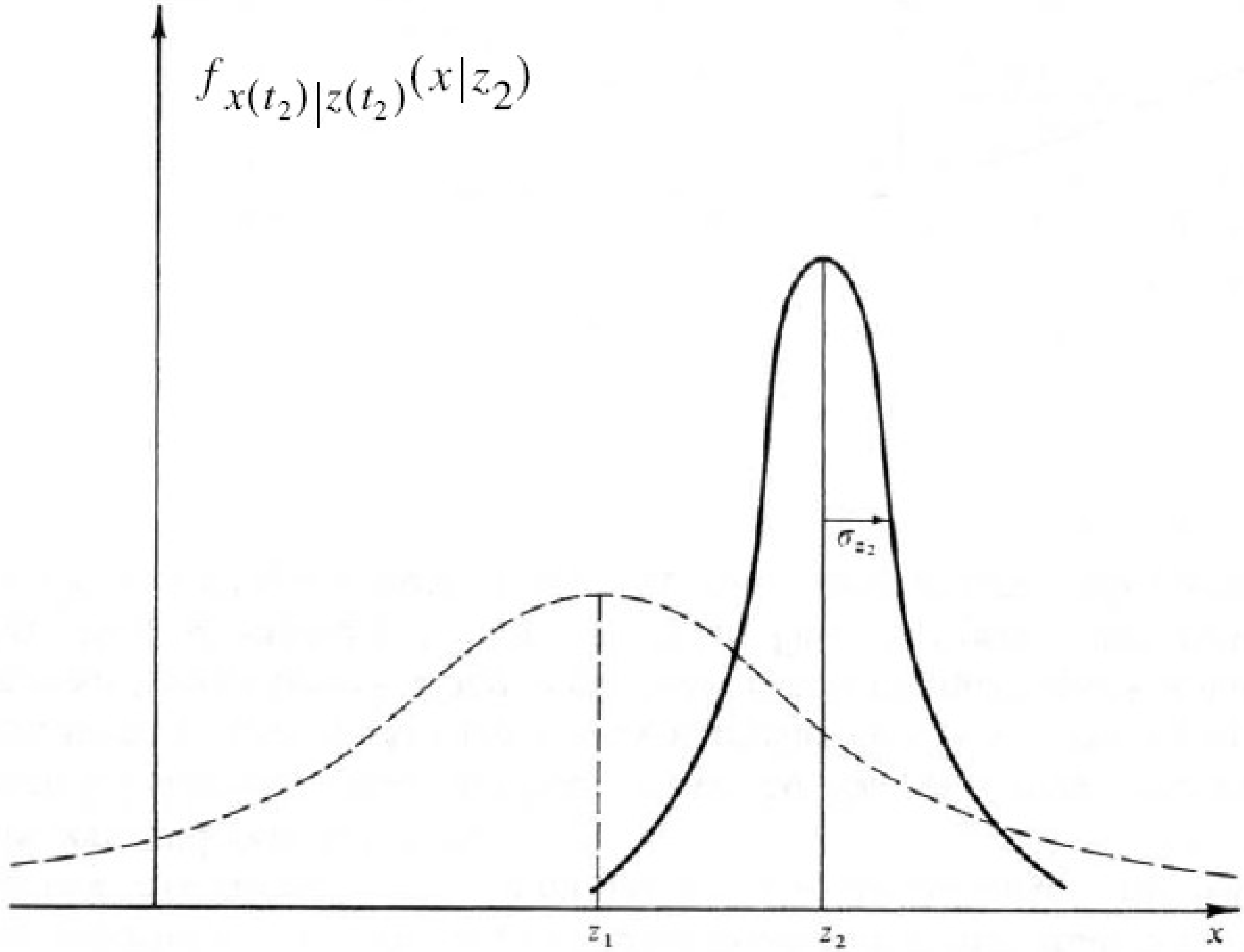

[Maybeck (1979)]

# Conditional density of position based on measured value of $z_1$



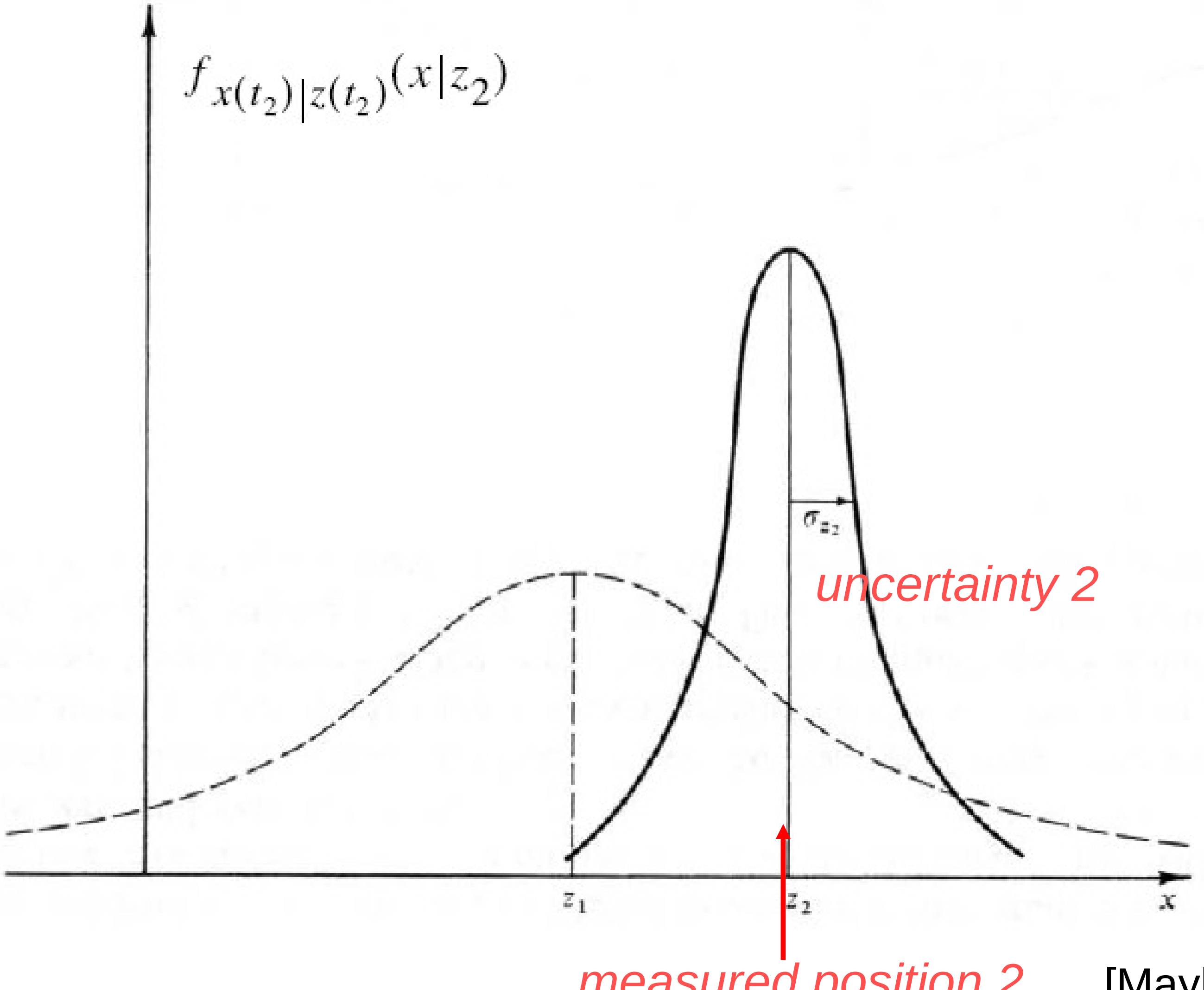

[Maybeck (1979)]

# Conditional density of position based on measurement of $z_2$ alone



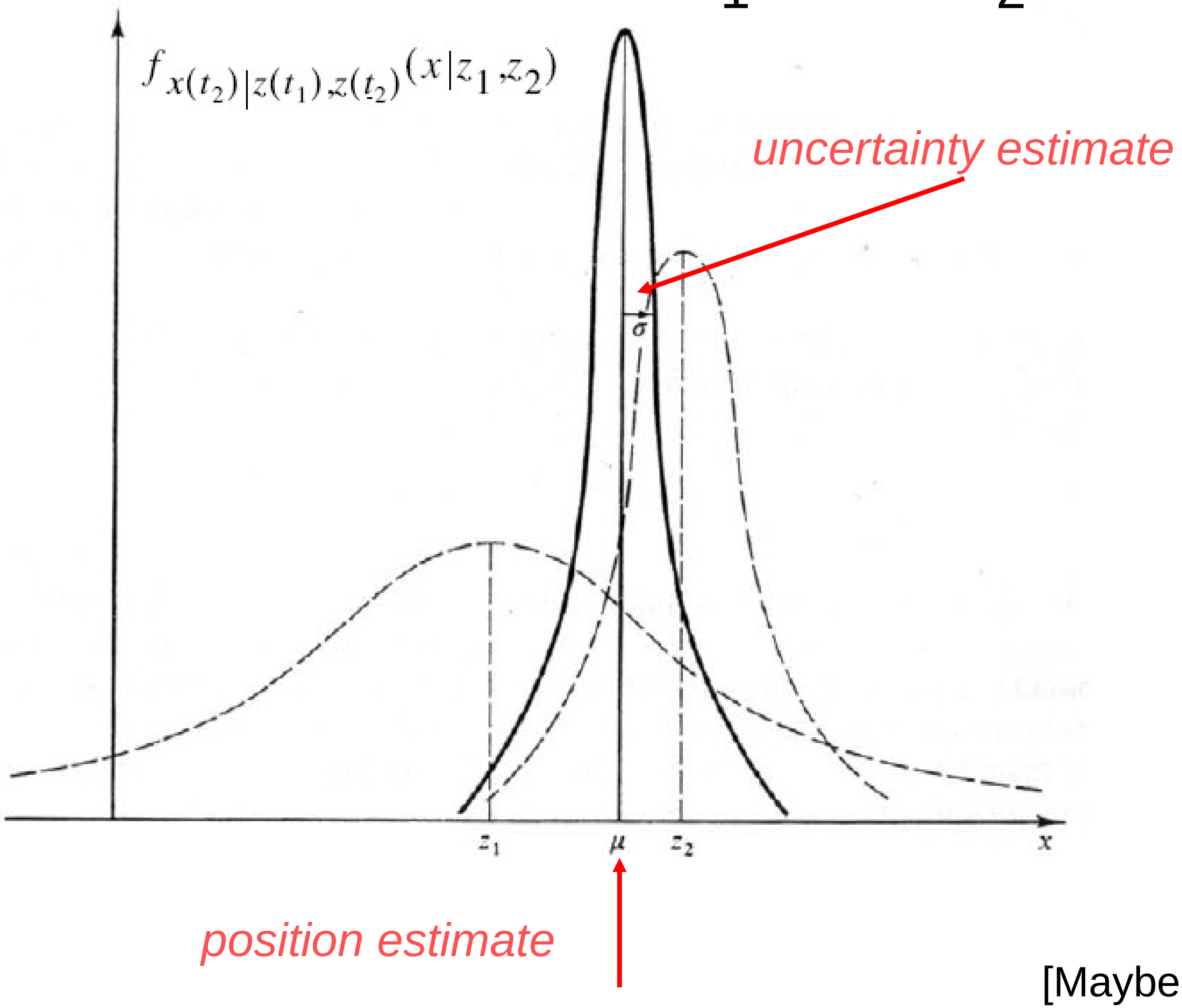

[Maybeck (1979)]

# Conditional density of position based on measurement of $z_2$ alone



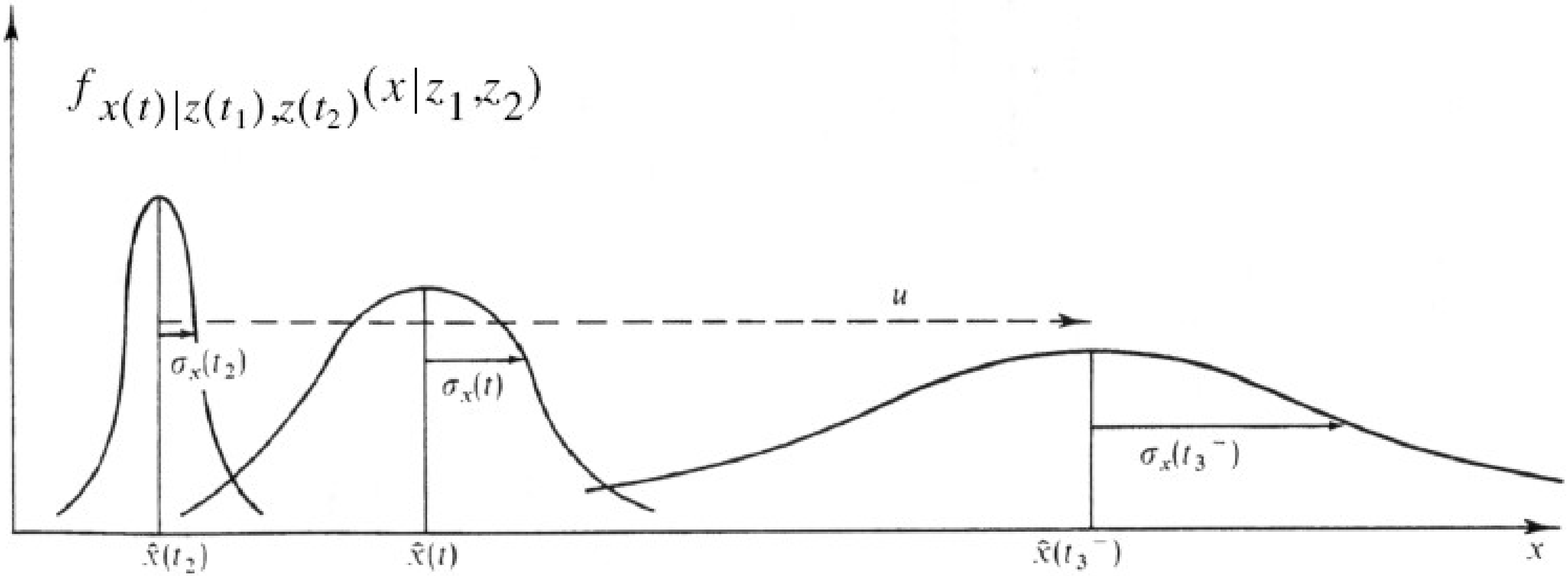

[Maybeck (1979)]

# Conditional density of position based on data $z_1$ and $z_2$



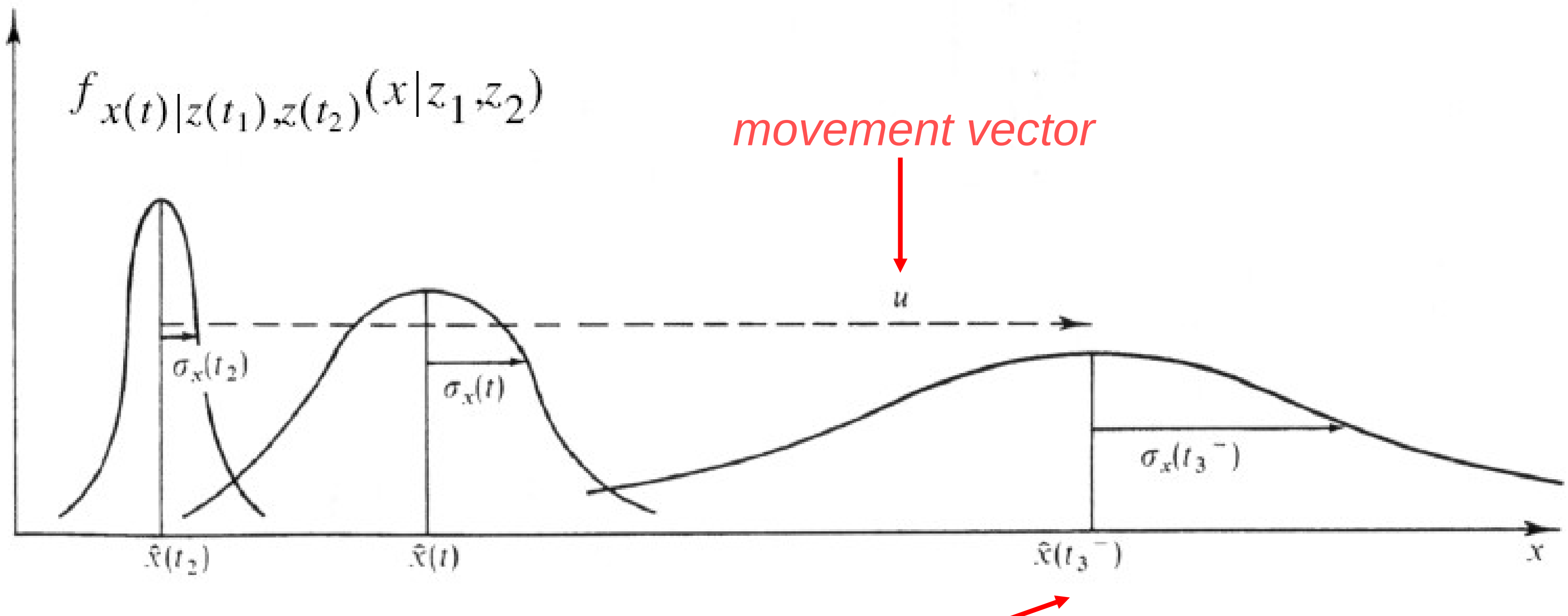

[Maybeck (1979)]

# Propagation of the conditional density




[Maybeck (1979)]

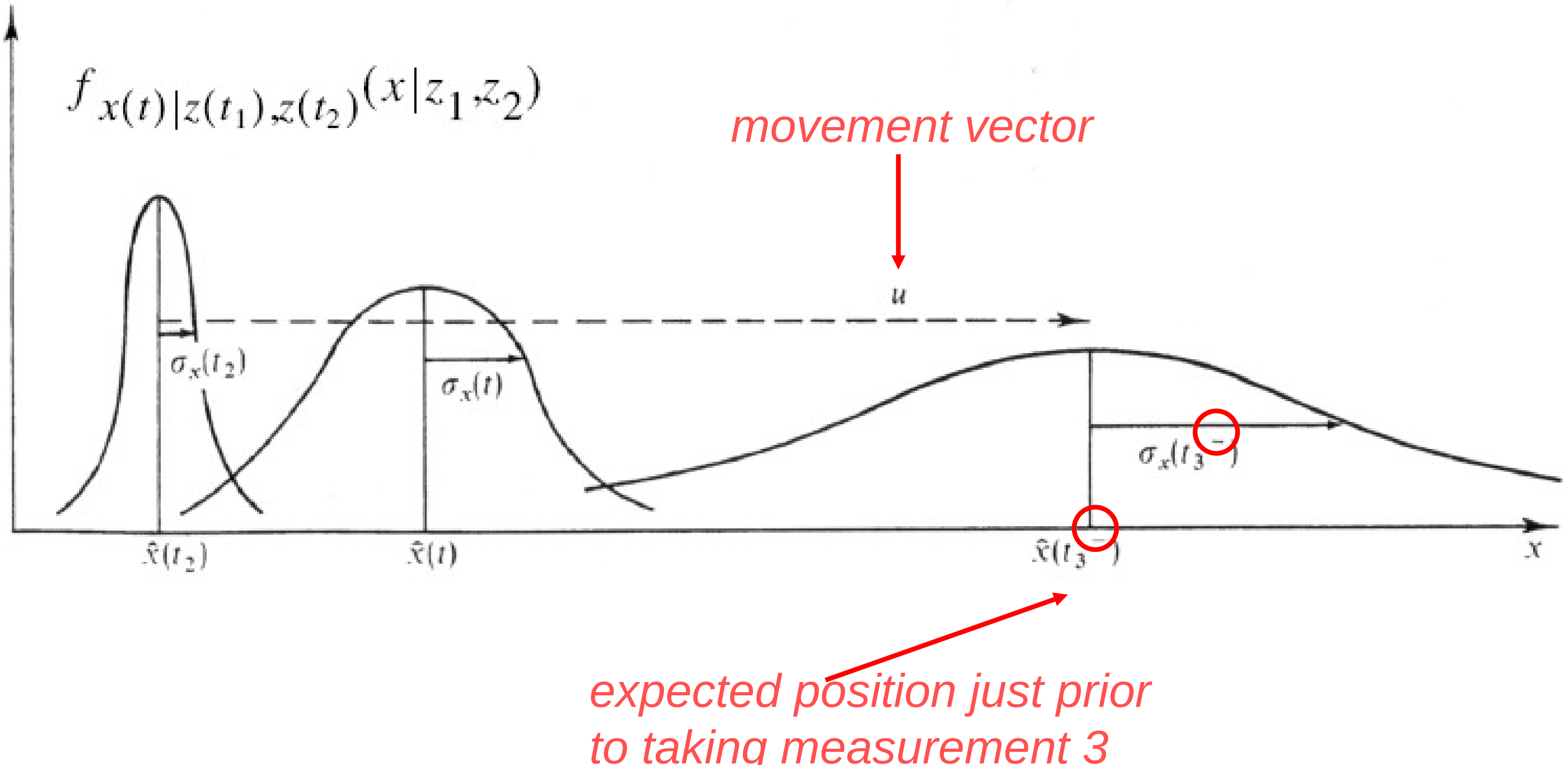
# Propagation of the conditional density




*expected position just prior to taking measurement 3*

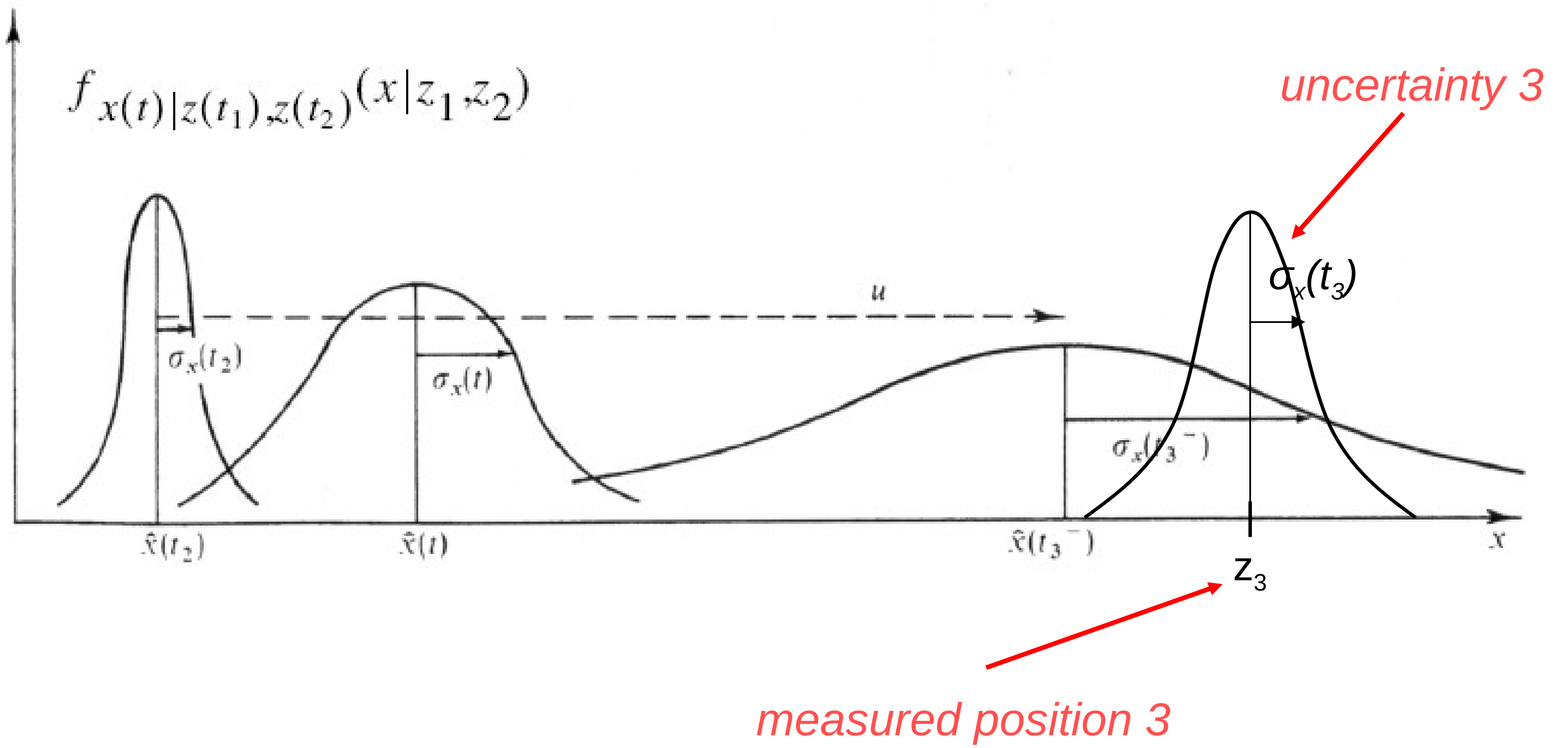
[Maybeck (1979)]

# Propagation of the conditional density

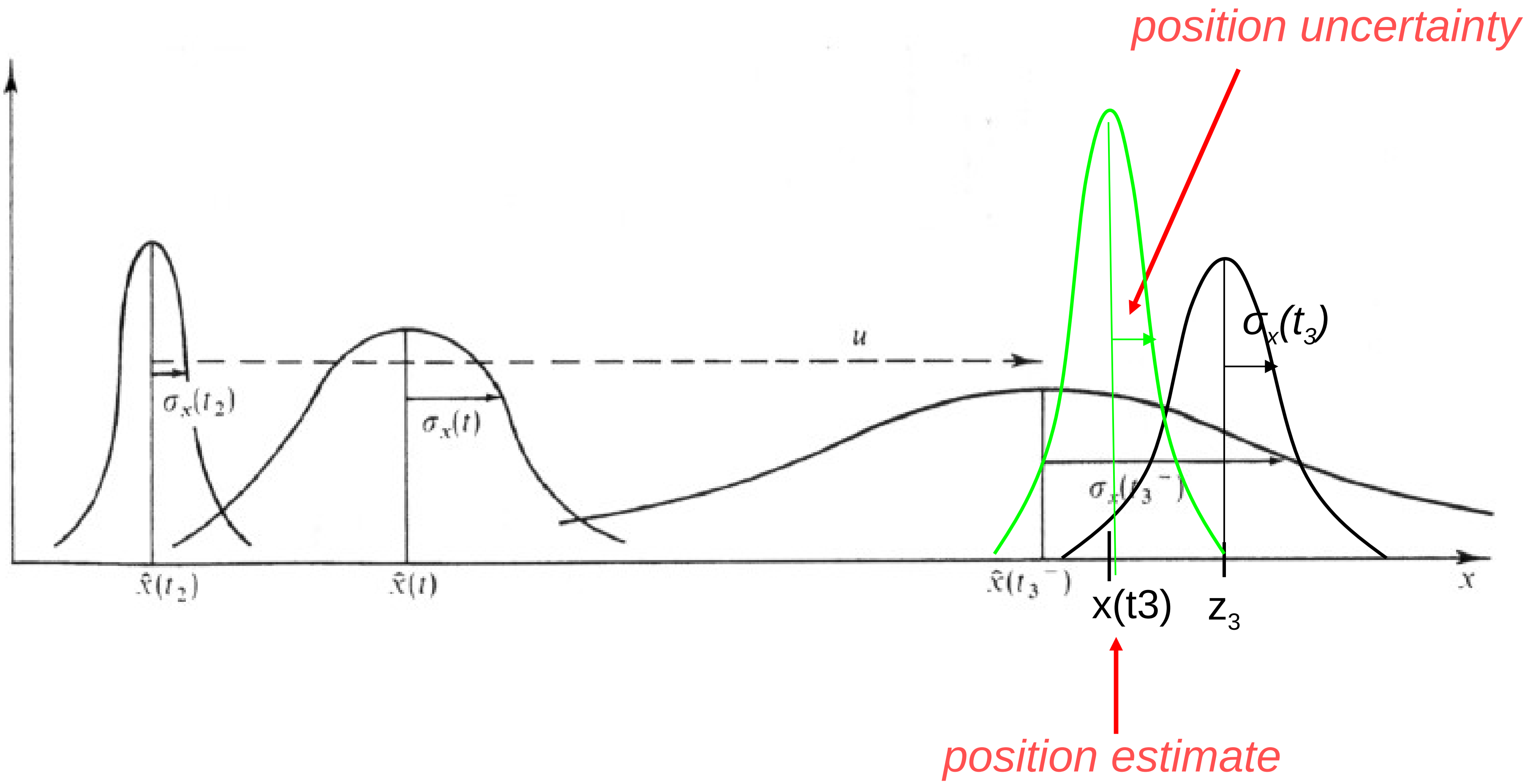


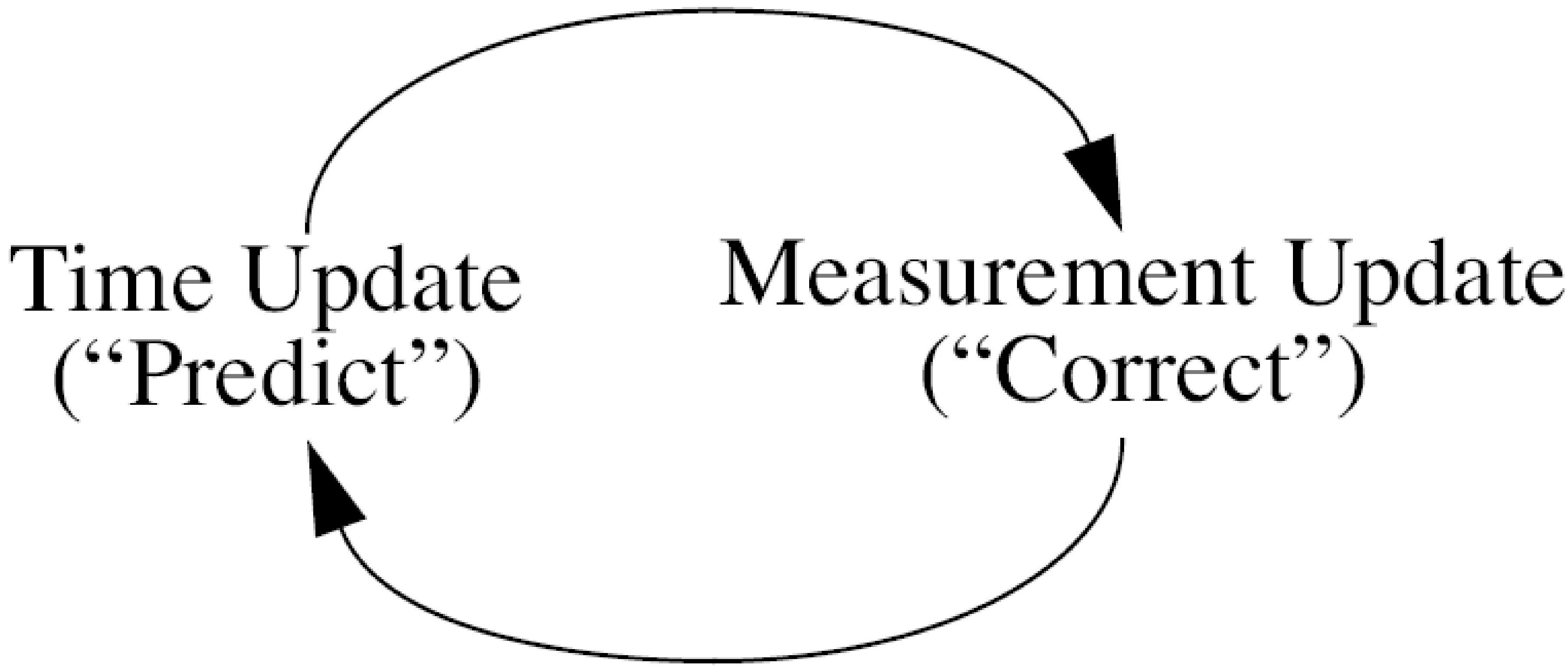

[Maybeck (1979)]

# Propagation of the conditional density



# Updating the conditional density after the third measurement

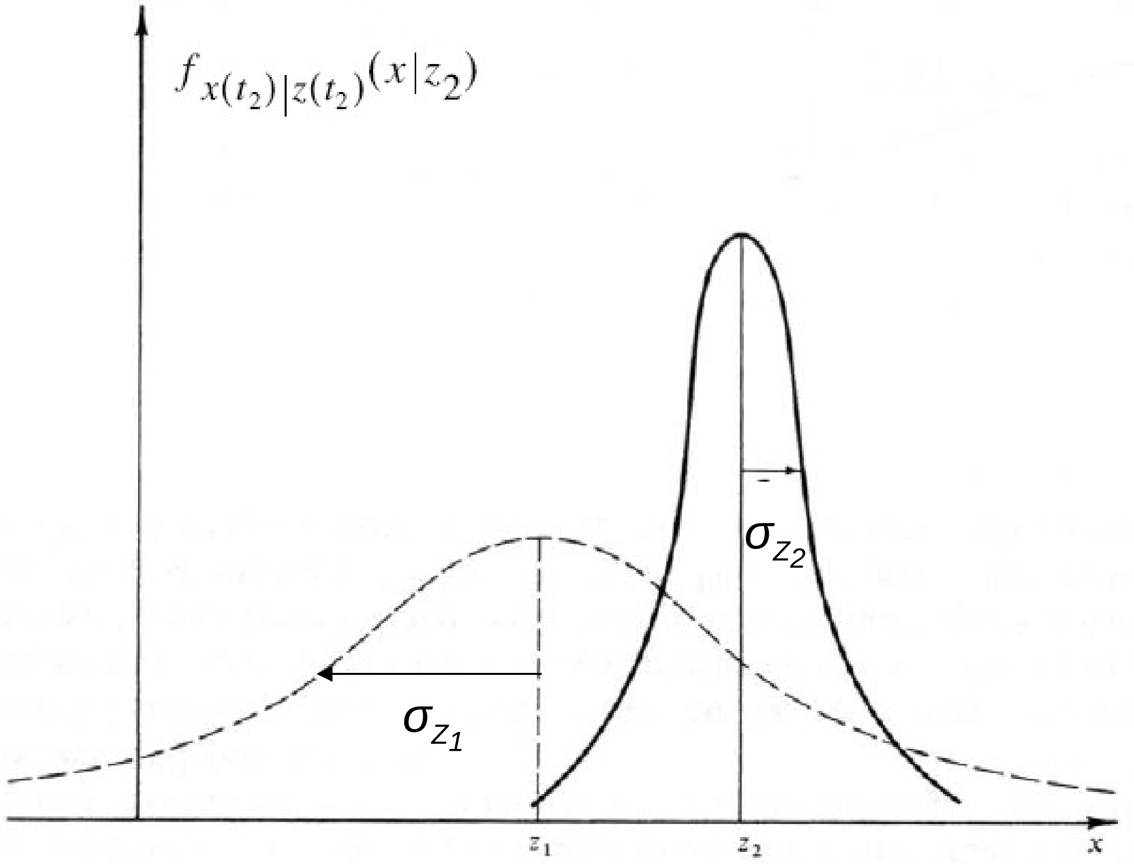



Time Update
("Predict")
Measurement Update
("Correct")

# Questions?

Now let’s do the same thing
…but this time we’ll use math

# How should we combine the two measurements?




[Maybeck (1979)]

# Calculating the new mean

$$\mu = \boxed{\textit{Scaling Factor 1}}\, z_1 + \boxed{\textit{Scaling Factor 2}}\, z_2$$

# Calculating the new mean

$$\mu = [\text{Scaling Factor 1}]z_1 + [\text{Scaling Factor 2}]z_2$$

$$\mu = [\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$

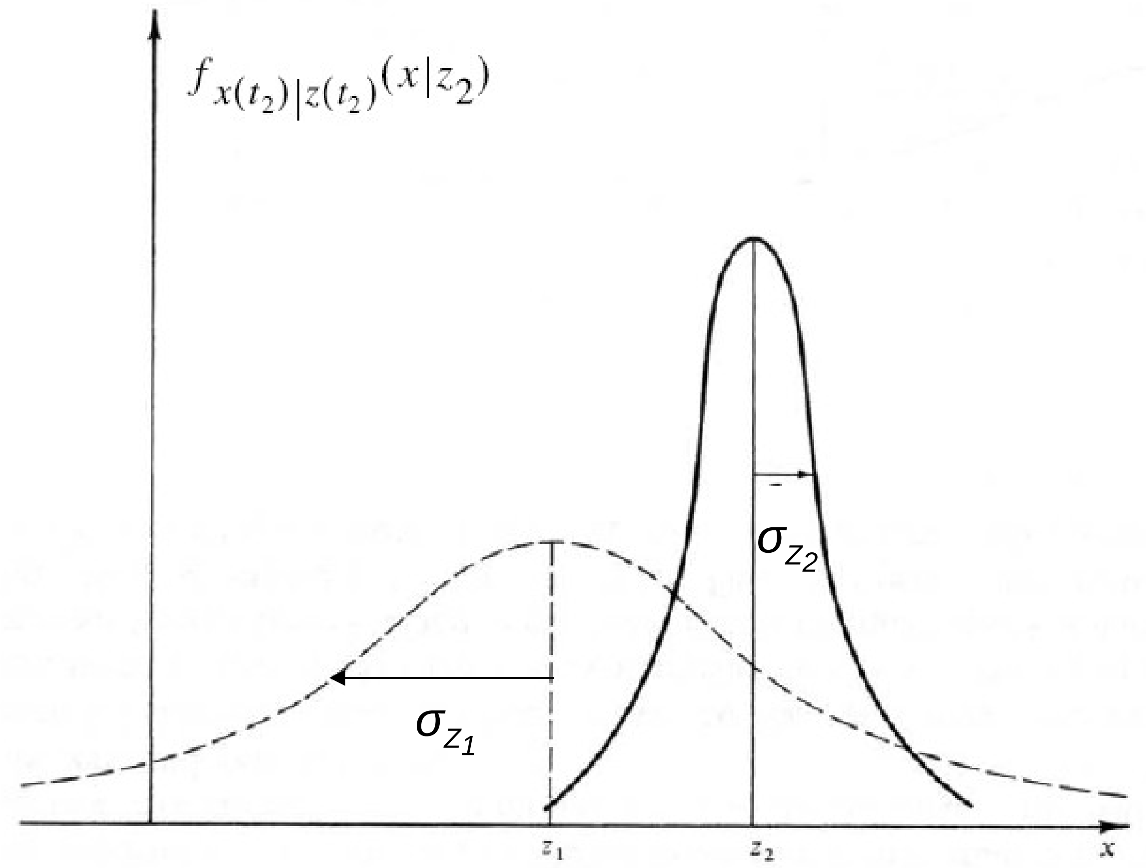
# Calculating the new mean

$$\mu = [\text{Scaling Factor 1}] z_1 + [\text{Scaling Factor 2}] z_2$$

$$\mu = [\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)] z_1 + [\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)] z_2$$

*Why is this not $z_1$?*

# Calculating the new variance




[Maybeck (1979)]

# Calculating the new variance

$$\sigma^2 = \boxed{\textit{Scaling Factor 1}}\ \sigma_{z_1}^2 + \boxed{\textit{Scaling Factor 2}}\ \sigma_{z_2}^2$$

# Remember the Gaussian Properties?

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

# Remember the Gaussian Properties?

- If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$
- Then $Y \sim N(a\mu + b, a^2\sigma^2)$

*This is $a^2$ not a*

# The scaling factors must be squared!

$$\sigma^2 = \boxed{\textit{Scaling Factor 1}}\ \sigma_{z_1}^2 \quad \boxed{\textit{Scaling Factor 2}}\ \sigma_{z_2}^2$$

$$[\sigma_{z_2}^2/(\sigma_{z_1}^2+\sigma_{z_2}^2)]^2 \qquad [\sigma_{z_1}^2/(\sigma_{z_1}^2+\sigma_{z_2}^2)]^2$$

# The scaling factors must be squared!

$$\sigma^2 = \boxed{\textit{Scaling Factor 1}}\ \sigma_{z_1}^2 + \boxed{\textit{Scaling Factor 2}}\ \sigma_{z_2}^2$$

$$\left[\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right]^2 \qquad \left[\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right]^2$$

$$\sigma^2 = \left[\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right]^2 \sigma_{z_1}^2 + \left[\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right]^2 \sigma_{z_2}^2$$

# Therefore the new variance is

$$1/\sigma^2 = (1/\sigma_{z_1}^2) + (1/\sigma_{z_2}^2)$$

*Try to derive this on your own.*

# Another Way to Express The New Position

$$\hat{x}(t_2) = [\sigma^2_{z_2}/(\sigma^2_{z_1}+\sigma^2_{z_2})]z_1 + [\sigma^2_{z_1}/(\sigma^2_{z_1}+\sigma^2_{z_2})]z_2$$

$$= \boxed{z_1 - z_1} + [\sigma^2_{z_2}/(\sigma^2_{z_1}+\sigma^2_{z_2})]z_1 + [\sigma^2_{z_1}/(\sigma^2_{z_1}+\sigma^2_{z_2})]z_2$$

$$= z_1 + [\sigma^2_{z_1}/(\sigma^2_{z_1}+\sigma^2_{z_2})][z_2 - z_1]$$

[Maybeck (1979)]

# Another Way to Express The New Position

$$\hat{x}(t_2) = [\sigma_{z_2}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$

$$= z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)][z_2 - z_1]$$

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2) [z_2 - \hat{x}(t_1)]$$

[Maybeck (1979)]

# Another Way to Express The New Position

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)]$$

$$K(t_2) = \sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)$$

[Maybeck (1979)]

# The equation for the variance can also be rewritten as

$$\sigma_x^2(t_2) = \sigma_x^2(t_1) - K(t_2)\sigma_x^2(t_1)$$

[Maybeck (1979)]

# Adding Movement

$$dx/dt = u + w$$

[Maybeck (1979)]

# Adding Movement

$$\hat{x}(t_3^-) = \hat{x}(t_2) + u[t_3 - t_2]$$

$$\sigma_x^2(t_3^-) = \sigma_x^2(t_2) + \sigma_w^2[t_3 - t_2]$$

[Maybeck (1979)]

# Adding Movement

$$\hat{x}(t_3) = \hat{x}(t_3^-) + K(t_3)[z_3 - \hat{x}(t_3^-)]$$

$$\sigma_x^2(t_3) = \sigma_x^2(t_3^-) - K(t_3)\sigma_x^2(t_3^-)$$

$$K(t_3) = \sigma_x^2(t_3^-)/[\sigma_x^2(t_3^-) + \sigma_{z_3}^2]$$

[Maybeck (1979)]

# Properties of K

- If the measurement noise is large K is small

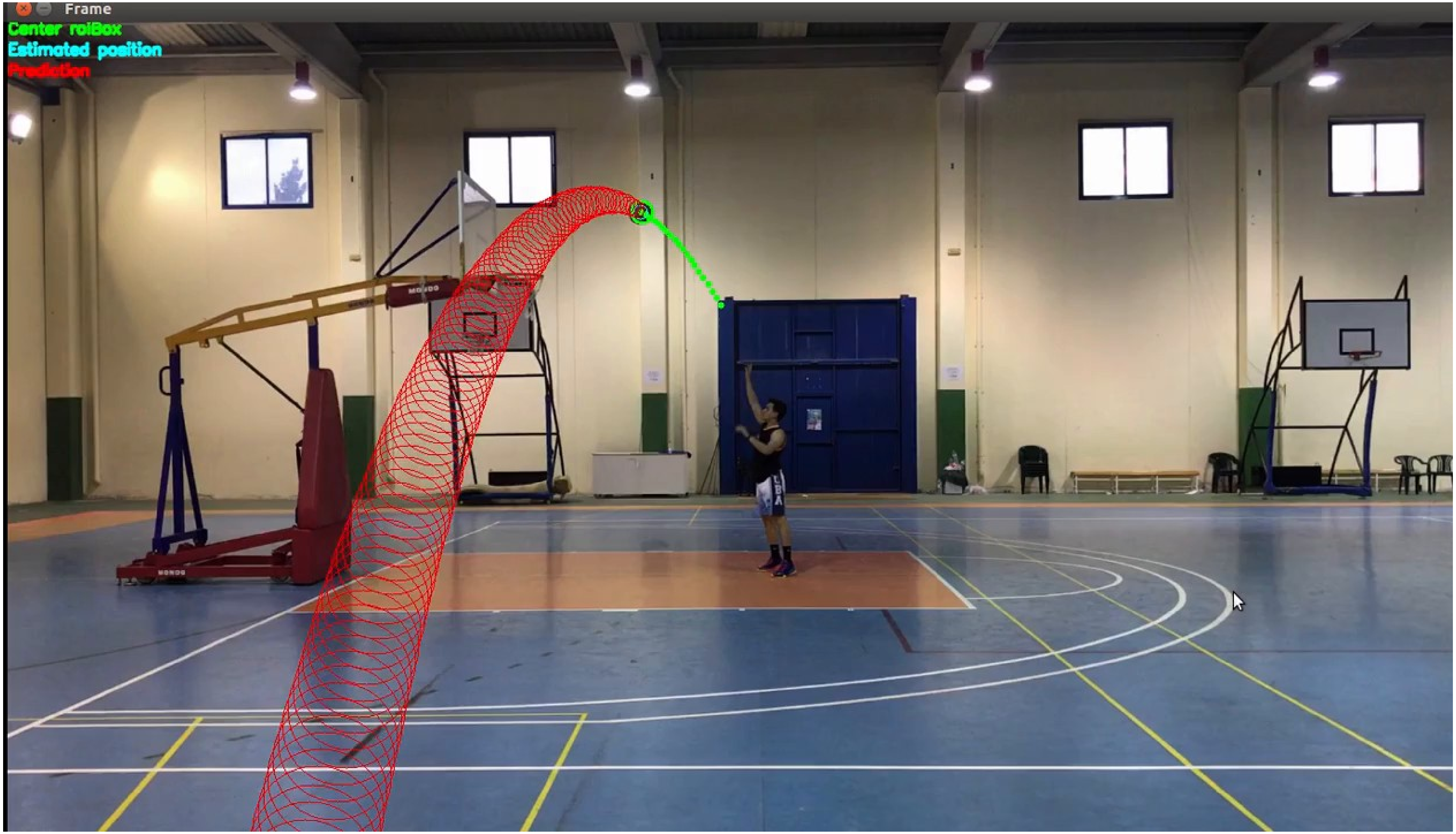
$$K(t_3) = \sigma_x^2(t_3^-)/[\sigma_x^2(t_3^-) + \sigma_{z_3}^2]$$

$$\sigma_{z_3}^2 \rightarrow \infty \ , \ K(t_3) \rightarrow 0$$

[Maybeck (1979)]

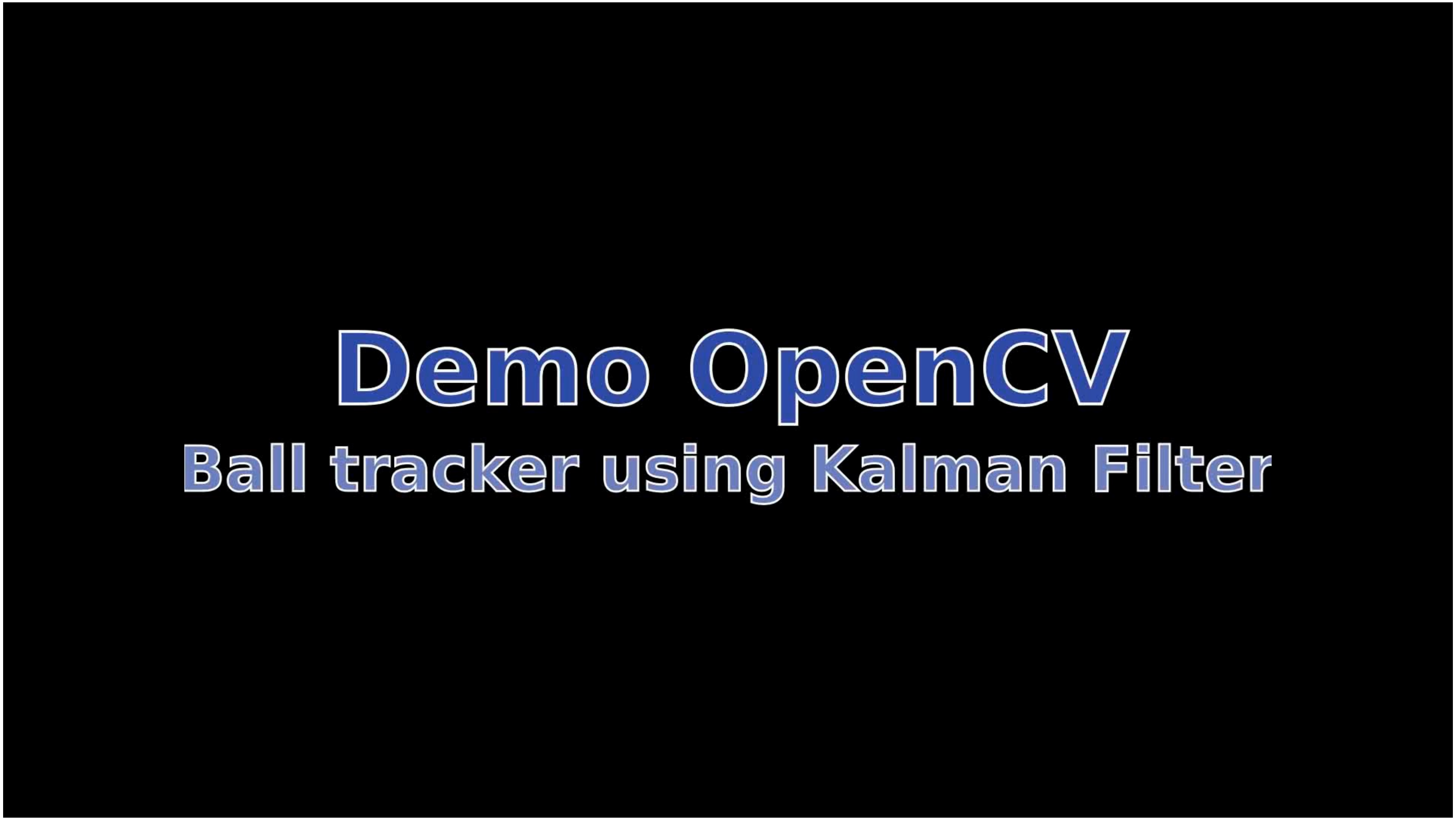
# The Kalman Filter
# (part 2)

# Example Applications




https://www.youtube.com/watch?v=MxwVwCuBEDA

https://github.com/pabsaura/Prediction-of-Trajectory-with-kalman-filter-and-open-cv




https://www.youtube.com/watch?v=sG-h5ONsj9s

https://www.myzhar.com/blog/tutorials/tutorial-opencv-ball-tracker-using-kalman-filter/

THE END