# The Kalman Filter (part 1)

#### Administrative Stuff



```
Pf(1) = 0.01;
for i = 2:11
Pz(i) = Pf(i-1) + 0.333 * (1-Pf(i-1))Pf(i) = Pf(i-1) / Pz(i);
end
```

#### **Rudolf Emil Kalman**



[http://www.cs.unc.edu/~welch/kalman/kalmanBiblio.html]

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# Definition

• A Kalman filter is simply an optimal recursive data processing algorithm

 Under some assumptions the Kalman filter is optimal with respect to virtually any criterion that makes sense.

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"The Kalman filter incorporates all information that can be provided to it. It processes all available measurements, regardless of their precision, to estimate the current value of the variables of interest."

# Why do we need a filter?

 No mathematical model of a real system is perfect

Real world disturbances

Imperfect Sensors

#### **Application: Radar Tracking**



#### **Application: Lunar Landing**



https://github.com/chrisIgarry/Apollo-11



National Aeronautics and Space Administration

#### Shuttle Docking with Russian Mir Space Station



#### **Application: Missile Tracking**



#### **Application: Sailing**



#### **Application: Robot Navigation**



#### **Application: Other Tracking**



#### **Application: Head Tracking**



#### Face & Hand Tracking



# A Simple Recursive Example

• Problem Statement:

Given the measurement sequence:  $z_1, z_2, ..., z_n$  find the mean

### First Approach

- 1. Make the first measurement  $z_1$ Store  $z_1$  and estimate the mean as  $\mu_1=z_1$
- 2. Make the second measurement  $z_2$ Store  $z_1$  along with  $z_2$  and estimate the mean as  $\mu_2 = (z_1+z_2)/2$

# First Approach (cont'd)

3. Make the third measurement  $z_3$ Store  $z_3$  along with  $z_1$  and  $z_2$  and estimate the mean as

$$\mu_3 = (z_1 + z_2 + z_3)/3$$

# First Approach (cont'd)

n. Make the n-th measurement  $z_n$ Store  $z_n$  along with  $z_1$ ,  $z_2$ ,...,  $z_{n-1}$  and estimate the mean as

$$\mu_n = (z_1 + z_2 + ... + z_n)/n$$

#### Second Approach

1. Make the first measurement  $z_1$ Compute the mean estimate as

$$\mu_1 = Z_1$$

Store  $\mu_1$  and discard  $z_1$ 

# Second Approach (cont'd)

2. Make the second measurement  $z_2$ 

Compute the estimate of the mean as a weighted sum of the previous estimate  $\mu_1$  and the current measurement  $z_{2:}$ 

$$\mu_2 = 1/2 \ \mu_1 + 1/2 \ z_2$$

Store  $\mu_2$  and discard  $z_2$  and  $\mu_1$ 

# Second Approach (cont'd)

3. Make the third measurement  $z_3$ 

Compute the estimate of the mean as a weighted sum of the previous estimate  $\mu_2$  and the current measurement  $z_{3:}$ 

$$\mu_3 = 2/3 \ \mu_2 + 1/3 \ z_3$$

Store  $\mu_3$  and discard  $z_3$  and  $\mu_2$ 

# Second Approach (cont'd)

n. Make the n-th measurement z<sub>n</sub>

Compute the estimate of the mean as a weighted sum of the previous estimate  $\mu_{n-1}$  and the current measurement  $z_{n:}$ 

$$\mu_n = (n-1)/n \mu_{n-1} + 1/n z_n$$

Store  $\mu_n$  and discard  $z_n$  and  $\mu_{n-1}$ 

#### Comparison



#### **Batch Method**

**Recursive Method** 

# Analysis

• The second procedure gives the same result as the first procedure.

• It uses the result for the previous step to help obtain an estimate at the current step.

• The difference is that it does not need to keep the sequence in memory.

Second Approach (rewrite the general formula)  $\mu_n = (n-1)/n \mu_{n-1} + 1/n z_n$ 

$$\mu_n = (n/n) \mu_{n-1} - (1/n) \mu_{n-1} + 1/n z_n$$

Second Approach (rewrite the general formula)  $\mu_n = (n-1)/n \mu_{n-1} + 1/n z_n$  $\mu_n = (n/n) \mu_{n-1} - (1/n) \mu_{n-1} + 1/n z_n$  $1/n (z_n - \mu_{n-1})$ +  $\mu_n = \mu_{n-1}$ Difference Gain Old Between Factor Estimate New Reading and **Old Estimate** 

#### Second Approach (rewrite the general formula)



#### **Gaussian Properties**

#### The Gaussian Function



#### Gaussian pdf



#### Properties

• If 
$$X \sim N(\mu, \sigma^2)$$
 and  $Y = aX + b$ 

# • Then $Y \sim N(a\mu + b, a^2\sigma^2)$

pdf for Y = aX + b $\frac{1}{2}\frac{(y-(a\mu+b))^2}{a^2\sigma^2}$ 2  $f_{Y}(y)$  $\sqrt{2\pi a^2 \sigma^2}^e$
#### Properties

Finally, if  $X_1$  and  $X_2$  are independent (see Section 2.5 below),  $X_1 \sim N(\mu_1, \sigma_1^2)$ , and  $X_2 \sim N(\mu_2, \sigma_2^2)$ , then

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2),$$
 (2.14)

and the density function becomes

$$f_X(x_1 + x_2) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-\frac{1}{2}\frac{(x - (\mu_1 + \mu_2))^2}{(\sigma_1^2 + \sigma_2^2)}}.$$
 (2.15)

### Summation and Subtraction



#### A simple example using diagrams

# Conditional density of position based on measured value of z<sub>1</sub>



# Conditional density of position based on measured value of z<sub>1</sub>



# Conditional density of position based on measured value of z<sub>1</sub>



# Conditional density of position based on measurement of z<sub>2</sub> alone



# Conditional density of position based on measurement of z<sub>2</sub> alone













# Updating the conditional density after the third measurement





#### Questions?

Now let's do the same thing ....but this time we'll use math

# How should we combine the two measurements?



### Calculating the new mean

$$\mu = Scaling Factor 1 \qquad z_1 + Scaling Factor 2 \qquad z_2$$

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$$\mu = Scaling Factor 1 \qquad z_1 + Scaling Factor 2 \qquad z_2$$

$$\mu = \left[\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_1 + \left[\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_2$$

### Calculating the new mean

$$\mu = Scaling Factor 1 \qquad z_1 + Scaling Factor 2 \qquad z_2$$

$$\mu = [\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$
  
Why is this not  $z_1$ ?

#### Calculating the new variance



### Calculating the new variance

$$\sigma^2$$
 = Scaling Factor 1  $\sigma_{z_1}^2$  + Scaling Factor 2  $\sigma_{z_2}^2$ 

#### Remember the Gaussian Properties?



#### Remember the Gaussian Properties?

### • If $X \sim N(\mu, \sigma^2)$ and Y = aX + b

## • Then $Y \sim N(a\mu + b, a^2\sigma^2)$

This is a<sup>2</sup> not a

#### The scaling factors must be squared!



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 $\sigma^2 = \left[\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right]^2 \sigma_{z_1}^2 + \left[\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right]^2 \sigma_{z_2}^2$ 

#### Therefore the new variance is

$$1/\sigma^2 = (1/\sigma_{z_1}^2) + (1/\sigma_{z_2}^2)$$

Try to derive this on your own.

#### Another Way to Express The New Position

$$\begin{aligned} \hat{x}(t_2) &= \left[\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_1 + \left[\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_2 \\ &= \left[z_1 - z_1 + \left[\sigma_{z_2}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_1 + \left[\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_2 \end{aligned}$$

 $= z_1 + [\sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)][z_2 - z_1]$ 

#### Another Way to Express The New Position

#### Another Way to Express The New Position

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)]$$
$$K(t_2) = \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}$$

# The equation for the variance can also be rewritten as

$$\sigma_x^2(t_2) \,=\, \sigma_x^2(t_1) - K(t_2) \sigma_x^2(t_1)$$

### Adding Movement

#### dx/dt = u + w

#### Adding Movement

$$\hat{x}(t_3^-) \,=\, \hat{x}(t_2) + u[t_3 - t_2]$$

### $\sigma_x^2(t_3^-) \,=\, \sigma_x^2(t_2) + \sigma_w^2[t_3 - t_2]$

#### Adding Movement

 $\hat{x}(t_3) = \hat{x}(t_3^-) + K(t_3)[z_3 - \hat{x}(t_3^-)]$ 

 $\sigma_{x}^{2}(t_{3}) = \sigma_{x}^{2}(t_{3}^{-}) - K(t_{3})\sigma_{x}^{2}(t_{3}^{-})$ 

 $K(t_3) = \sigma_x^2(t_3^-) / [\sigma_x^2(t_3^-) + \sigma_z^2]$ 

### Properties of K

• If the measurement noise is large K is small

 $K(t_3) = \sigma_x^2(t_3^-) / [\sigma_x^2(t_3^-) + \sigma_{z_3}^2]$  $\sigma_{z_3}^2 \rightarrow \infty$ ,  $K(t_3) \rightarrow 0$
## The Kalman Filter (part 2)

## **Example Applications**



https://www.youtube.com/watch?v=MxwVwCuBEDA

https://github.com/pabsaura/Prediction-of-Trajectory-with-kalman-filter-and-open-cv

## Demo OpenCV Ball tracker using Kalman Filter

https://www.youtube.com/watch?v=sG-h5ONsj9s

https://www.myzhar.com/blog/tutorials/tutorial-opencv-ball-tracker-using-kalman-filter/

## THE END