Fast methods for oscillatory hydraulic tomography with multiple frequency data

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Oscillatory hydraulic tomography

Figure: Experimental setup of pumping tests

- Collect pressure (head) measurements from pumping tests
- Recover aquifer properties such as conductivity, storage etc.
- To better locate natural resources, treat pollution and manage underground sites.
Time-dependent groundwater flow equations.

Pumping well - source oscillating at fixed frequency and amplitude

At long time, head can be decomposed into phasor and a complex exponential.

\[ h(x, t) = \Re(\Phi(x) \exp(i\omega t)) \]
Governing equations - Forward problem

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\[ h(x, t) = \Re(\Phi(x) \exp(i\omega t)) \]

The phasor satisfies

\[-\nabla \cdot (K(x)\nabla \Phi(x)) + i\omega S_s(x)\Phi(x) = Q_0\delta(x - x_s), \quad x \in \Omega \]

\[ \Phi(x) = 0, \quad x \in \partial\Omega_D \]

\[ \nabla \Phi(x) \cdot n = 0, \quad x \in \partial\Omega_N \]
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Making non-dimensional and discretizing, leads to a shifted system.

\[ (\text{laplacian} + i\omega \text{complex shift storage}) \Phi(\sigma) = b \]

Solve as many systems as number of frequencies. Computationally expensive!
Denoising the signal

Location of measurement and pumping wells

Location 1

Location 2

Pumping well

Location of measurement and pumping wells

Location 1

Location 2

Pumping well

Hydraulic head (m)

Time (s)

Hydraulic head (m)

Time (s)
Consider the measurement equation

\[ y = h(s) + v \quad v \sim \mathcal{N}(0, R) \]

where,

- \( y \) := measurements.
- \( s \) := model parameters
- \( h(s) \) := measurement op.

Further, assume Gaussian prior

\[ s \sim \mathcal{N}(X\beta, Q) \]
Quasi-linear geostatistical approach

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Maximum a posteriori estimate

\[ \arg\min_{s, \beta} \frac{1}{2} \| y - h(s) \|_{R^{-1}} + \frac{1}{2} \| s - X\beta \|_{Q^{-1}} \]

subject to

likelihood prior
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\[ \text{likelihood} \quad \text{prior} \]
Fast solution for multiple frequencies

The forward problem

\[(K + iω_j M)x_j = b \quad j = 1, \ldots, n_f\]
Fast solution for multiple frequencies

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- Developed and analysed Krylov subspace solver for shifted systems.

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The cost is nearly independent of number of frequencies.

Build a smaller dimension approximation space, search for optimal solutions.

200 systems, 90k unknowns.

Figure: Comparison of time taken using a direct solver vs. using an iterative solver.
Fast solution for multiple frequencies

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Figure: Comparison of time taken using a direct solver vs. using an iterative solver.
Adjoint approach to compute Jacobian

Measurements collected at a given location $x_i$ and time $t$

$$\Re \left\{ e^{i\omega t} \Phi(x_i) \right\} = A \cos \omega t + B \sin \omega t$$

At each measurement well, measure two pieces of information.
Adjoint approach to compute Jacobian

Measurements collected at a given location $x_i$ and time $t$

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At each measurement well, measure two pieces of information.

Steps involved in computing sensitivity

- Solve forward problem for each source (and for each frequency).
- Solve adjoint problem for each measurement location (and for each frequency).
- Compute inner products.
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At each measurement well, measure two pieces of information.

Steps involved in computing sensitivity

- Solve forward problem for each source (and for each frequency).
- Solve adjoint problem for each measurement location (and for each frequency).
- Compute inner products.

The adjoint field $\Psi_i$ satisfies

$$- \nabla \cdot (K \nabla \Psi_i) + i \omega S_s \Psi_i = - \delta(\mathbf{x} - \mathbf{x}_i), \quad \mathbf{x} \in \Omega$$

$$\Psi_i = 0, \quad \mathbf{x} \in \partial \Omega_D$$

$$\mathbf{n} \cdot \nabla \Psi_i(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial \Omega_N$$

The adjoint calculation can be sped up using this fast solver.
Visualization of sensitivity fields

Measurements collected at a given location $x_i$ and time $t$

$$\Re \left\{ e^{i\omega t} \Phi(x_i) \right\} = A \cos \omega t + B \sin \omega t$$

**Figure:** Sensitivity at frequency $\frac{2\pi}{30}$ [s$^{-1}$]. Measurement and source locations are 50 m apart
Visualization of sensitivity fields

Measurements collected at a given location $x_i$ and time $t$

$$\Re \left\{ e^{i\omega t} \Phi(x_i) \right\} = A \cos \omega t + B \sin \omega t$$

Figure: Sensitivity at frequency $\frac{2\pi}{150}$ [s$^{-1}$]. Measurement and source locations are 50 m apart.
Time to compute Jacobian

![Graphs showing time taken for different components in the Jacobian.](image)

**Figure:** Comparison of time taken for different components in the Jacobian. **Forward** refers to solving the forward problem for multiple frequencies. **Adjoint** refers to solving the adjoint field for multiple frequency at each measurement location. **Inner prod.** refer to forming the inner product to form the rows of the Jacobian.
Results for the inversion for log transmissivity

Figure: Comparison of inversion results for log transmissivity with single and multiple frequencies. Frequency range was $\omega \in \left[\frac{2\pi}{150}, \frac{2\pi}{30}\right]$. The volumetric flow rate is 0.62 lit/sec.

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<th>Parameters</th>
<th>Values</th>
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<tr>
<td>L (m)</td>
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<tr>
<td>S (m$^{-1}$)</td>
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<tr>
<td>$\mu(T)$ (m/s)</td>
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<td>var(log $T$) ((m/s)$^2$)</td>
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<tr>
<th>$N_f$</th>
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<tr>
<td>20</td>
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Wrap up

Contributions

- Developed and tested flexible Krylov solvers for shifted systems.
- Applied it to oscillatory hydraulic tomography and observed significant speedups.
  - 640 measurements, 10k unknowns, 200k state variables.

Future work

- Joint inversion for conductivity and storage.
- More realistic conductivity fields.

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