RECURRENCE RELATIONS

A recurrence relation is a sequence of numbers where each term is determined by the previous terms.

Examples: $1, 1, 2, 3, 5, 8, 13, \ldots$ $\quad a_n = a_{n-1} + a_{n-2}$ $\quad a_0 = a_1 = 1$

$3, 8, 63, 3968, 15745023, \ldots$ $\quad a_n = a_{n-1}^2 - 1$ $\quad a_0 = 3$

$1, 2, 4, 8, 16, 32, 64, \ldots$ $\quad a_n = 2a_{n-1}$ $\quad a_0 = 1$

$1, -2, -3, -1, 2, 3, 1, -2, -3, \ldots$ $\quad a_n = a_{n-1} - a_{n-2}$ $\quad a_0 = 1, a_1 = -2$

Just giving the relationship between a_n & $a_{n-1}, a_{n-2}, \ldots$ is not sufficient. Also need initial terms. (How many?)

The order of a recurrence relation $a_n = \ldots$ is the largest k such that a_n depends on a_{n-k}.

$a_n = 2a_{n-1}$ first-order

$a_n = 2a_{n-1} + a_{n-2}$ second-order

$a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$ third-order

We'll just consider first-order and second-order recurrence relations for the most part.

Except for the Tribonacci sequence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

$a_0 = a_1 = 1, a_2 = 2$

FINDING A CLOSED FORM

Thm: The nth term of the recurrence relation $a_n = 2a_{n-1}$ with $a_0 = 1$ is 2^n. That is, $\underbrace{a_n = 2^n.}$

The closed form for this recurrence relation.

Proof: By induction.

BASIS STEP [For $n=0$, $a_n = a_0 = 2^0 = 2^n$.

INDUCTIVE STEP [Assume $\forall k \in \mathbb{N}, 0 \le k < n, a_k = 2^k$.

So $a_n = 2 \cdot a_{n-1} = 2 \cdot 2^{n-1} = 2^{n-1+1} = 2^n$.

Then by induction, $\forall n \in \mathbb{N}, a_n = 2^n$. ▨

Problem: Give a closed form for the recurrence relation $a_n = 2a_{n-1} + 3, a_0 = 1$.

TWO STEPS: 1. Find a closed form you conjecture is valid. SCRATCH WORK

2. Prove by induction that the closed form is valid.

STEP 1:

$a_0 = 1$
$a_1 = 2 \cdot 1 + 3 = 5$
$a_2 = 2 \cdot 5 + 3 = 13$
$a_3 = 2 \cdot 13 + 3 = 29$
$a_4 = 2 \cdot 29 + 3 = 61$
$a_5 = 2 \cdot 61 + 3 = 125$
$a_6 = 2 \cdot 125 + 3 = 253$
$a_7 = 2 \cdot 253 + 3 = 509$

Hmm, does it look like anything closed form? $\sim 2^n$?

$a_0 = 2^2 - 3$
$a_1 = 2^3 - 3$
$a_2 = 2^4 - 3$
$a_3 = 2^5 - 3$
$a_4 = 2^6 - 3$
$a_5 = 2^7 - 3$
$\vdots$

Conjecture: $a_n = 2^{n+2} - 3$

STEP 2: Prove $\forall n \in \mathbb{N},\ a_n = 2^{n+2} - 3$ by induction.

Thm: Let $a_n = 2a_{n-1} + 3,\ a_0 = 1$. Then $\forall n \in \mathbb{N},\ a_n = 2^{n+2} - 3$.

Pf: For $n=0$, $a_n = a_0 = 1 = 4 - 3 = 2^{0+2} - 3 = 2^{n+2} - 3$.

Assume $1 \leq n$ and $a_{n-1} = 2^{(n-1)+2} - 3 = 2^{n+1} - 3$.

So $a_n = 2a_{n-1} + 3 = 2(2^{n+1} - 3) + 3 = 2^{n+2} - 3$.

Then by induction, $\forall n \in \mathbb{N},\ a_n = 2^{n+2} - 3$. ▨

So a closed form for $a_n = 2a_{n-1} + 3, a_0 = 1$ is $a_n = 2^{n+2} - 3$.

Does Step 1 (Find a conjecture) have an algorithmic approach?

The telescope/substitution method:

$$\begin{aligned}
a_n &= 2a_{n-1} + 3 = 2^1 a_{n-1} + 2^0 \cdot 3 \\
&= 2(2a_{n-2} + 3) + 3 = 2^2 a_{n-2} + 2^1 \cdot 3 + 2^0 \cdot 3 \\
&= 2(2(2a_{n-3} + 3) + 3) + 3 = 2^3 a_{n-3} + 2^2 \cdot 3 + 2^1 \cdot 3 + 2^0 \cdot 3 \\
&= 2(2(2(2a_{n-4} + 3) + 3) + 3) + 3 = 2^4 a_{n-4} + 2^3 \cdot 3 + 2^2 \cdot 3 + 2^1 \cdot 3 + 2^0 \cdot 3 \\
&\ \ \vdots
\end{aligned}$$

Conjecture: $a_n = 2^i a_{n-i} + 2^{i-1} \cdot 3 + 2^{i-2} \cdot 3 + \ldots + 2^0 \cdot 3$ for any $i \leq n$.

$$= 2^i a_{n-i} + 3 \sum_{j=1}^{i} 2^{i-j}$$

Let $i = n$

$$= 2^n a_0 + 3 \sum_{j=1}^{n} 2^{n-j} = 2^n + 3 \cdot 2^n \sum_{j=1}^{n} 2^{-j}$$

HW4

$$= 2^n + 3 \cdot 2^n \cdot \left(1 - \frac{1}{2^n}\right) = 2 + 3 \cdot 2^n - 3 = 2^{n+2} - 3.$$

$a_n = 2^{n+2} - 3$.

- The telescope/substitution method is for Step 1 (conjecture).
- It does not constitute a proof of a closed form!
- Also not necessarily effective... making the jump from examples to general form of the sum has a "guessing" component to it.

EXERCISE

Find a closed form for $a_n = 3a_{n-1} + 2$ using telescoping method.

$$
\begin{aligned}
a_n &= 3a_{n-1}+2 = 3^1 a_{n-1} + 3^0\cdot 2\\
&= 3(3a_{n-2}+2)+2 = 3^2 a_{n-2} + 3^1\cdot 2 + 3^0 \cdot 2\\
&= 3(3(3a_{n-3}+2)+2)+2 = 3^3 a_{n-3} + 3^2\cdot 2 + 3^1\cdot 2 + 3^0 + 2\\
&= 3(3(3(3a_{n-4}+2)+2)+2)+2 = 3^4 a_{n-4} + 3^3\cdot 2 + 3^2\cdot 2 + 3^1\cdot 2 + 3^0\cdot 2\\
&\vdots\\
&= 3^i a_{n-i} + 3^{i-1}\cdot 2 + 3^{i-2}\cdot 2 + \ldots + 3^1\cdot 2 + 3^0\cdot 2\\
&= 3^i a_{n-i} + \sum_{j=0}^{i-1} 3^j \cdot 2 \overset{\text{Let } i=n}{=} 3^n a_0 + 2\sum_{j=0}^{n-1} 3^j = 3^n a_0 + 2\cdot \frac{1-3^n}{1-3}\\
&= 3^n a_0 + 3^n - 1 = 3^n(a_0+1) - 1
\end{aligned}
$$

Problem: Find a closed form for the recurrence $a_n = a_{n-1} + n$, $a_0 = 0$.

$$
\begin{aligned}
a_n &= a_{n-1} + n\\
&= (a_{n-2} + n-1) + n\\
&= ((a_{n-3} + n-2) + n-1) + n\\
&= (((a_{n-4} + n-3) + n-2) + n-1) + n\\
&\vdots\\
&= a_{n-i} + \sum_{j=0}^{i-1} n-j \overset{\text{Let } i=n}{=} a_0 + \sum_{j=0}^{n-1} n-j = \sum_{j=0}^{n-1} n - \sum_{j=0}^{n-1} j = n\cdot n - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}
\end{aligned}
$$

AN APPLICATION OF RECURRENCE CLOSED FORMS

An algorithm implemented in code:

```
1. int sum(int i)
2. {
3.     if (i <= 0)
4.         return 0;
5.     else
6.         return 3+ sum(i-1);
7. }
```

recursive algorithm calls itself!

What does sum(i) return?

If $i \le 0$, sum(i) returns 0. (Lines 3-4) X Ignore

If $i > 0$, sum(i) returns $3 +$ sum$(i-1)$. (Lines 5-6)

Let s_i be what sum(i) returns. Then $s_i = 3 + s_{i-1}$, $s_0 = 0$.

So
$$\begin{aligned} s_i &= 3 + s_{i-1} \\ &= 3 + (3 + s_{i-2}) = 2 \cdot 3 + s_{i-2} \\ &= 3 + (3 + (3 + s_{i-3})) = 3 \cdot 3 + s_{i-3} \\ &\vdots \\ &= 3 \cdot j + s_{i-j} \text{ for any } j \le i \\ &= 3 \cdot i + s_0 = 3 \cdot i \end{aligned}$$

$a_n = O(1) + a_{n/2}$

$a_n = \Theta(\log n)$

Conjecture: If $i \ge 0$, sum(i) returns $3i$.

Recurrence relations can describe what recursive code does.

what it returns

What time it takes

GENERAL FORMS OF CLOSED FORMS

Proving a closed form for each new reccurrence relation is so tedious. Replace numbers with variables to get a general form.

Problem: Find a closed form for $a_n = s a_{n-1}$, where s is a real number.

STEP 1:
$$\begin{aligned} a_n &= s a_{n-1} \\ &= s(s a_{n-2}) = s^2 a_{n-2} \\ &= s(s(s a_{n-3})) = s^3 a_{n-3} \\ &= s(s(s(s a_{n-4}))) = s^4 a_{n-4} \\ &\vdots \\ &= s^i a_{n-i} \overset{\text{Let } i=n}{=} s^n a_0 \end{aligned}$$

STEP 2: Thm: $\forall n \in \mathbb{N}$ and real number s, the recurrence relation $a_n = s a_{n-1}$ has the closed form $a_n = s^n a_0$.

PF: Let $n=0$. $a_0 = s^0 a_0 = a_0$.

Assume $1 \leq n$ and $a_{n-1} = s^{n-1} a_0$.
So $a_n = s a_{n-1} = s(s^{n-1} a_0) = s^n a_0$.
Then by induction, $\forall n \in \mathbb{N}$ and real numbers s, the recurrence relation $a_n = s a_{n-1}$ has closed form $a_n = s^n a_0$ ▨

EXERCISE

Find a closed form for the recurrence $a_n = (a_{n-1})^k$ with $k \in \mathbb{N}$.

$$\begin{aligned} a_n &= (a_{n-1})^k \\ &= ((a_{n-2})^k)^k = (a_{n-2})^{k^2} \\ &= (((a_{n-3})^k)^k)^k = (a_{n-3})^{k^3} \end{aligned}$$

$$a_n = (a_{n-i})^{k^i} = a_0^{k^n}$$

SECOND-ORDER RECURRENCE RELATIONS

Problem: Find a closed form for $a_n = 3a_{n-1} + 4a_{n-2}$, $a_0 = 3$, $a_1 = 2$.

Telescoping/substitution will be ugly and painful.

Theorem: Let $a_n = s_1 a_{n-1} + s_2 a_{n-2}$. Consider the roots r_1, r_2 of $x^2 - s_1 x - s_2 = 0$.

(Case 1) If $r_1 \neq r_2$, $a_n = c_1 r_1^n + c_2 r_2^n$ for some real numbers c_1, c_2.

(Case 2) If $r_1 = r_2$, $a_n = c_1 r_1^n + c_2 n r_2^n$ for some real numbers c_1, c_2.

Thm: $\forall n \in \mathbb{N}$, $a_n = 4^n + 2(-1)^n$

PF: Since $a_n = 3a_{n-1} + 4a_{n-2}$, $s_1 = 3$, $s_2 = 4$.

The roots of $x^2 - 3x - 4 = (x-4)(x+1) = 0$ are $r_1 = 4$, $r_2 = -1$.

Since $r_1 \neq r_2$, Case 1 applies. So $a_n = c_1 4^n + c_2 (-1)^n$ for some c_1, c_2.

$$3 = a_0 = c_1 \cdot 4^0 + c_2 \cdot (-1)^0 = c_1 + c_2$$
$$2 = a_1 = c_1 \cdot 4^1 + c_2 \cdot (-1)^1 = 4c_1 - c_2$$

$$\xrightarrow{+} 5 = 5c_1 \rightarrow 1 = c_1 \rightarrow 3 - 1 = c_2$$
$$2 = c_2$$

So $a_n = 4^n + 2(-1)^n = 4^n + 2(-1)^n$. ∎

φ "The Golden Ratio"

Thm: Let F_n be the nth Fibonacci number. Then $F_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$.

Pf: Since $F_n = F_{n-1} + F_{n-2}$, $s_1 = s_2 = 1$. Let r_1, r_2 be roots of $x^2 - x - 1 = 0$.

Then $r_1, r_2 = \dfrac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot -1}}{2 \cdot 1} = \dfrac{1 \pm \sqrt{5}}{2}$ by quadratic Formula. So Case 1 applies.

So $F_n = c_1 r_1^n + c_2 r_2^n$. $0 = F_0 = c_1 + c_2$. $1 = F_1 = c_1 r_1 + c_2 r_2$. So $1 = c_1 r_1 - c_1 r_2 = c_1(r_1 - r_2)$.

So $c_1 = \dfrac{1}{r_1 - r_2} = \dfrac{1}{\sqrt{5}}$, $c_2 = -\dfrac{1}{\sqrt{5}}$.

So $F_n = \dfrac{r_1^n - r_2^n}{\sqrt{5}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$ ∎

EXERCISE

Prove a closed form for the recurrence $a_n = a_{n-1} + 6a_{n-2}$, $a_0 = a_1 = 4$, using the previous theorem.

$s_1 = 1$, $s_2 = 6$. $x^2 - x - 6 = (x-3)(x+2)$, so $r_1 = 3$, $r_2 = -2$ are roots of $x^2 - s_1 x - s_2 = 0$

Then by (Case 1) of previous theorem, $a_n = c_1 3^n + c_2(-2)^n$.

$4 = a_0 = c_1 3^0 + c_2(-2)^0 = c_1 + c_2$ → $8 = 2c_1 + 2c_2$

$4 = a_1 = c_1 \cdot 3^1 + c_2(-2)^1 = 3c_1 - 2c_2$ → $4 = 3c_1 - 2c_2$

$12 = 5c_1$ so $c_1 = \frac{12}{5}$. So $c_2 = 4 - \frac{12}{5} = \frac{8}{5}$.

So $a_n = \frac{12}{5} \cdot 3^n + \frac{8}{5} \cdot (-2)^n$. ▨

CHALLENGE

Find a closed form for the recurrence relation

$a_n = \sum_{i=0}^{n-1} a_i$, $a_0 = 1$.

:)