PROOF BY INDUCTION

**Used:**
- When you want to prove something is true for all natural numbers.
- When a direct proof looks hard.
- When you might want to prove the statement using smallest counterexample. (We will see why)

**Example:** Prove that $3 \mid 4^n - 1$ for all $n \in \mathbb{N}$.

**First,** we prove "$3 \mid 4^0 - 1$ for $n = 0$." **BASIS STEP**

**Second,** we prove "**IF** $3 \mid 4^n - 1$, then $3 \mid 4^{n+1} - 1$" **INDUCTIVE STEP**

**Thm:** $\forall n \in \mathbb{N}, 3 \mid 4^n - 1$.

**Pf:** If $n = 0$, then $4^0 - 1 = 0 - 1 = -1 = 0 \cdot 3$. So $3 \mid 4^0 - 1$. **BASIS STEP**

Now assume $3 \mid 4^n - 1$ and $1 \leq n$. **INDUCTIVE HYPOTHESIS**

So $\exists a \in \mathbb{Z}$ such that $3a = 4^n - 1$.
So $3 \cdot 4a = 4 \cdot (4^n - 1) = 4^{n+1} - 4$.
So $3 \cdot 4a + 3 = 4^{n+1} - 1$ and $3(4a+1) = 4^{n+1} - 1$.
So $3 \mid 4^{n+1} - 1$.

Then by induction, $3 \mid 4^n - 1 \forall n \in \mathbb{N}$. $\Box$

**GENERAL FORM:**

**Thm:** $\forall n \in \mathbb{N}$ (statement about $n$)

**Pf:** If $n = 0$, then (statement about $n$) is true. **BASIS STEP**

Now assume (statement about $n-1$) is true and let **INDUCTIVE HYPOTHESIS**

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So (statement about $n$) is true.
Then by induction, (statement about $n$) is true $\forall n \in \mathbb{N}$. $\Box$
Why does proof by induction work?

\[ n: 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ldots \]

**INDUCTIVE STEP**

**BASIS STEP:** Statement true for \( n=0 \).

**INDUCTIVE STEP:** Statement true for \( n-1 \) \( \Rightarrow \) statement true for \( n \).

So for any specific value of \( n \) (say \( n=100 \)),

- statement true for \( n=0 \) \( \Rightarrow \)
  - statement true for \( n=1 \) \( \Rightarrow \)
    - statement true for \( n=2 \) \( \Rightarrow \)
      - \ldots
        - statement true for \( 99 \) \( \Rightarrow \)
          - statement true for \( n=100 \)

So statement true for \( n=0 \) \( \Rightarrow \) statement true for \( n=100 \)

[Also statement is true for \( n=0 \).]

So statement is true for \( n=100 \). **IMPLIED BY BASIS AND INDUCTIVE STEP**

Thus: \( \forall n \in \mathbb{N}, \sum_{i=0}^{n} 3^i < 3^{n+1} \).

**Pf:** For \( n=0 \), \( \sum_{i=0}^{0} 3^i = 3^0 = 1 < 3 = 3^{n+1} \)

**BASIS STEP**

**INDUCTIVE STEP**

Assume \( \sum_{i=0}^{n-1} 3^i < 3^{n+1} \), \( 1 \leq n \).

Then \( \sum_{i=0}^{n} 3^i = 3^n + \sum_{i=0}^{n-1} 3^i < 3^n + 3^n = 2 \cdot 3^n < 3 \cdot 3^n = 3^{n+1} \).

Then by induction, \( \sum_{i=0}^{n} 3^i < 3^{n+1} \) \( \forall n \in \mathbb{N} \). \( \blacksquare \)
Thm: \( \forall n \in \mathbb{N}, \sum_{i=0}^{n} 3^i < 3^{n+1} \)

Pf: Suppose, by contradiction, that this is not true. Let \( n \in \mathbb{N} \) be the smallest counterexample.

So \( \sum_{i=0}^{n} 3^i \geq 3^{n+1} \) and \( \sum_{i=0}^{k} 3^i < 3^{k+1} \). Also, \( \sum_{i=0}^{0} 0^i = 0 < 3^0 = 3 \).

So \( \sum_{i=0}^{n-1} 3^i + 3^n < 3^n + 3^n = 3^{n+1} \)

So \( \sum_{i=0}^{n} 3^i \), a contradiction with \( \sum_{i=0}^{n} 3^i \geq 3^{n+1} \).

So \( \forall n \in \mathbb{N}, \sum_{i=0}^{n} 3^i < 3^{n+1} \). \( \square \)

SMALLEST COUNTEREXAMPLE \( \Rightarrow \) INDUCTION

INDUCTION:

IF (statement about \( n=0 \)) and ((statement about \( n-1 \)) \( \Rightarrow \) (statement about \( n \))),

then \( \forall n \in \mathbb{N} \), (statement about \( n \)).

SMALLEST COUNTEREXAMPLE:

IF (statement about \( n=0 \)) and \( \neg \) ((statement about \( n-1 \)) \( \land \) (statement about \( n \))),

then \( \forall n \in \mathbb{N} \), (statement about \( n \)).

SMALLEST COUNTEREXAMPLE \( \Rightarrow \) INDUCTION
by BOOLEAN ALGEBRA.
Thus: \( \forall n \in \mathbb{N}, n \geq 5, 2^n > n^2 \)

Proof: By smallest counterexample.

**Base Step**

\[ 2^5 = 32 > 25 = 5^2, \text{ so smallest counterexample is } n \geq 5. \]

Assume \( 2^n \leq n^2 \) and \( 2^{n-1} > (n-1)^2 \) and \( n \geq 5 \).

So \( 2^{n-1} > (n-1)^2 = n^2 - 2n + 1 \)

\[
2^n > 2n^2 - 4n + 2 = n^2 + (n^2 - 4n + 2) \\
\geq n^2 + (n^2 - 4n - n) \text{ since } n \geq 2 \\
\geq n^2 + (n^2 - 5n) \\
\geq n^2 + (n(n-5)) \\
\geq n^2 + (n) \text{ since } n \geq 5 \\
> n^2 \text{ since } n \geq 1.
\]

So \( 2^n > n^2. \) But also \( 2^n \leq n^2. \) A contradiction.

Removing the \( \cdots \) parts of the proof converts it from *SMALLEST COUNTEREXAMPLE* to *INDUCTION*.

Also possible to covert *INDUCTION* to *SMALLEST COUNTEREXAMPLE*.

Recall that the following theorem can be proven by smallest counterexample.

Thus: For \( n \in \mathbb{N}, F_n \leq 1.7^n \) where \( F_n \) is \( n \)th Fibonacci number.

But had to use that \( F_{n-2} \leq 1.7^{n-2} \), not just that \( F_{n-1} \leq 1.7^{n-1} \)

So first had to prove smallest counterexample is \( n \geq 2. \)
Thm: Let \( F_n \) be the \( n \)th Fibonacci number. Then \( F_n \leq 1.7^n \).

Proof: \( F_0 = 1 \leq 1.7^0, F_1 = 1 \leq 1.7^1 \). \[ \text{Basis} \]

\[ \text{Inductive Hypothesis} \]

Assume \( 2 \leq n \) and \( F_{n-1} = 1.7^{n-1}, F_{n-2} = 1.7^{n-2} \).

So \( F_n = F_{n-1} + F_{n-2} \leq 1.7^{n-1} + 1.7^{n-2} = (1.7 + 1.7) \cdot 1.7^{n-2} \leq 1.7^n \).

Then by induction, \( \forall n \in \mathbb{N}, F_n \leq 1.7^n \). \[ \Box \]

Wut? We can have an inductive hypothesis that is more than "(statement for \( n-1 \)). But the basis step must support it.

Conjecture: \( \forall n \in \mathbb{N}, 2^n = 1 \).

Invalid proof: Let \( n = 0 \). So \( 2^n = 2^0 = 1 \). \[ \text{Basis} \]

\[ \text{Inductive Hypothesis} \]

Assume \( 2^{n-1} = 1, 2^{n-2} = 1 \). \[ \text{Inductive} \]

So \( 2^n = \frac{2^{n-1} \cdot 2^{n-2}}{2^{n-2}} = \frac{1 \cdot 1}{1} = 1. \)

Then by induction, \( \forall n \in \mathbb{N}, 2^n = 1. \)

Missing part of basis step used in inductive step.
**STRONG INDUCTION**

In a proof by induction, the inductive step can use an inductive hypothesis of “∀k∈N, 0≤k≤n (statement about k)” rather than “(statement about n−1)”. Why is this allowed?

If we can assume “(statement about n=5)”, we must also be able to assume “(statement about n=4), ..., (statement about n=0)”. This form of induction with an inductive hypothesis that the statement is true for all smaller values of n is called

**STRONG INDUCTION**

**Thm:** ∀n∈N, 2∤n, n is a product of primes.

**Proof:** 2 is prime, so n≠2 is a product of 1 prime: 2. **[Basis]**

Assume ∀k∈N, 2∤k<n is a product of primes. **[Inductive Hypothesis]**

- If n is prime, then n is a product of 1 prime: n.
- If n is not prime, n is composite, so n=ab with a,b∈N, 1≤a,n.
  - Since a≤n and a,n are positive, b≥2.
  - So a and b are both products of primes.
  - So their product, n, is also a product of primes.

Then by induction, ∀n∈N, 2∤n, n is a product of primes. ⊥
THE RULES OF INDUCTION

1. Two parts: basis step and inductive step.

2. Basis step must prove statement holds for all values of \( n \) used by inductive step that aren't proved by inductive step.

3. Inductive step proves that if the statement is true for some smaller values of \( n \), then it is true for this value of \( n \).

4. Inductive step can use inductive hypothesis that assumes statement is true for all smaller values of \( n \). (Strong induction)

5. Inductive step also can assume \( n \) is larger than values covered by the basis step.

Thm: \( \forall n \in \mathbb{N}, 12 \leq n, \exists a, b \in \mathbb{N} \text{ such that } n = 4a + 5b. \)

Pf: \( 12 = 4 \cdot 3 + 5 \cdot 0. \) [Basis]

\[
\text{Inductive:}
\begin{align*}
\text{Assume } 12 \leq n \text{ and } \exists a, b \in \mathbb{N} \text{ such that } n-1 = 4a + 5b. \quad \text{(I.H.) (Weak)}
\end{align*}
\]

If \( a \geq 0 \), then \( n = 4(a-1) + 5(b+1) \) with \( a-1, b+1 \in \mathbb{N} \).

If \( a = 0 \), then \( b \geq 3 \) and \( n = 4(a+4) + 5(b-3) \) with \( a+4, b-3 \in \mathbb{N} \).

So in either case \( \exists a', b' \in \mathbb{N} \text{ such that } n = 4a' + 5b'. \)

Then by induction, \( \forall n \in \mathbb{N}, 12 \leq n, \exists a, b \in \mathbb{N} \text{ such that } n = 4a + 5b. \) \( \square \)

Pf: \( 12 = 4 \cdot 3 + 5 \cdot 0. \) \( 13 = 4 \cdot 2 + 5 \cdot 1. \) \( 14 = 4 \cdot 1 + 5 \cdot 2. \) \( 15 = 4 \cdot 0 + 5 \cdot 3. \) [Pf:

\[
\text{Inductive:}
\begin{align*}
\text{Assume } 15 \leq n \text{ and } \exists a, b \in \mathbb{N} \text{ such that } n-4 = 4a + 5b. \quad \text{(I.H.) (Strong)}
\end{align*}
\]

So \( n = 4(a+1) + 5b \).

So \( \exists a', b' \in \mathbb{N} \text{ such that } n = 4a' + 5b'. \)

Then by induction, \( \forall n \in \mathbb{N}, 12 \leq n, \exists a, b \in \mathbb{N} \text{ such that } n = 4a + 5b. \) \( \square \)
EXERCISE

Prove by induction that \( \forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).

Let \( n=0 \). Then \( \sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2} \). [**B**A**S**I**S**

Assume \( \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} + 1 \leq n \). [I.H.]

Inductive

So \( \sum_{i=1}^{n} i = \frac{(n-1)n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} \).

So \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).

Then by induction, \( \forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \). \( \square \)

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These are triangulated simple polygons with ears.

Define: An ear of a triangulated polygon is a triangle with two boundary edges and one triangulation edge.
Two-Ear Theorem: Every triangulated simple polygon with \( \geq 4 \) sides has at least two ears. \( \text{t.s.p.} \)

**Proof:** By induction.

**Basis Step**
- Every t.s.p. with 4 sides has 1 diagonal and two triangles. So both triangles have 2 boundary edges and 1 triangulation edge.

![Basis Step Diagram]

Now consider a t.s.p. \( P \) with \( n \geq 4 \) sides. Pick an edge \( e \) of the triangulation, and split the t.s.p. \( P \) into two t.s.p.s with fewer sides, \( A \) & \( B \).

![Inductive Step Diagram]

Assume that every t.s.p. with fewer than \( n \) and at least 4 sides has two ears. \( A \) and \( B \) are either triangles or t.s.p.s with between 4 and \( n-1 \) sides.

**Inductive Hypothesis**

**Inductive Step**
- If \( A \) is a triangle, then \( A \) is an ear in \( P \).
- Otherwise \( A \) has two ears, and one does not have \( e \) as a side. This ear is an ear in \( P \).
- So \( A \) contains an ear of \( P \). By symmetry, so does \( B \).
- So \( P \) contains at least two ears.

Then by induction, every t.s.p. with \( \geq 4 \) sides has at least two ears.
**EXERCISE**

Prove by induction if $n \in \mathbb{N}$ and $n$ is odd, then $8 \mid n^2 - 1$.

Let $n=1$. Then $n^2 - 1 = 1^2 - 1 = 0$. Since $8 \mid 0$, $8 \mid n^2 - 1$. \textbf{Basis}

Assume $n \geq 3$ and $n$ is odd. Also, assume $8 \mid (n-2)^2 - 1$ since $1 \leq n-2 \leq n-2$ odd.

Since $n \geq 3$ and $n$ is odd, $n = 2a+1$ for some $a \in \mathbb{Z}$, $a \geq 1$.

So $n - 2 = 2a - 1$.

So $8 \mid (2a-1)^2 - 1$.

So $8 \mid 4a^2 - 4a - 1$.

So $8 \mid 4a^2 - 4a - 1$.

So $8 \mid 4a^2 - 4a - 1$.

Then by induction, $\forall n \in \mathbb{N}$, $n$ odd, $8 \mid n^2 - 1$. \(\Box\)

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**CHALLENGE**

Prove by induction and by smallest counterexample that $\forall n \in \mathbb{N}, 1 \leq n, 133 \mid 11^{n+1} + 12^{2n-1}$.

\textbf{Basis} [Let $n=1$. Then $11^{n+1} + 12^{2n-1} = 11^2 + 12 = 133$. So $133 \mid 11^{n+1} + 12^{2n-1}$.

Assume $2 \leq n$ and $133 \mid 11^n + 12^{2n-3}$, \textbf{I.H.}

So $133 \mid 11^{n+1} + 12^{2n-1}$.

So $133 \mid 11^{n+1} + 12^{2n-3}$.

So $133 \mid 11^{n+1} + 12^{2n-3}$.

Then by induction, $\forall n \in \mathbb{N}, 1 \leq n, 133 \mid 11^{n+1} + 12^{2n-1}$, \(\Box\).