# MATH/COMP61-03 Spring 2014 Notes: January 28th

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## 1 Operators on Sets

Just like algebra and boolean algebra, there are operators on sets. Here are some:

**Definition.** *The* union *of two sets $A$ and $B$, written $A \cup B$, is* $\{x : x \in A \text{ or x} \in \text{B}\}$.

**Definition.** *The* intersection *of two sets $A$ and $B$, written $A \cap B$, is* $\{x : x \in A \text{ and x} \in \text{B}\}$.

**Definition.** *Two sets $A$ and $B$ are* disjoint *if $A \cap B = \emptyset$.*

**Definition.** *A collection of $k$ sets $A_1, A_2, \ldots, A_k$ are* pairwise disjoint *if $A_i \cap A_j = \emptyset$ for all $i \neq j$.*

**Definition.** *The* set difference *of two sets $A$ and $B$, written $A - B$ or $A \setminus B$, is* $\{x : x \in A \text{ and x} \notin \text{B}\}$.

**Definition.** *The* symmetric set difference *of two sets $A$ and $B$, written $A \Delta B$, is $(A \cup B) - (A \cap B)$.*

In English, the union of two sets is the set of things in either, the intersection of two sets is the set of things in both, the set difference $A - B$ is the set of things in $A$ and not in $B$, and the symmetric set difference of two sets is the set of things exactly one of the two sets. We finish the lecture with more setception:

**Definition.** *The* power set *of a set $A$ is the set of all subsets of $A$.*

The power set of $\{1, 2\}$ is $\{\{\}, \{1\}, \{2\}, \{1, 2\}\}$. The power set of $\{1, 2, 3\}$ is $\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

**Theorem.** *Let $A$ be a finite set and $A_{\text{power}}$ be the power set of $A$. Then $|A_{\text{power}}| = 2^{|A|}$.*

**Proof.** *Let $A$ be a set with $k$ elements. Then each subset $B \subset A$ can be specified by $k$ choices to include or exclude each element of $A$ in $B$. So by the Multiplication Principle, there are $\prod_{i=1}^{k} 2 = 2^k$ subsets of $A$. Since $|A_{\text{power}}|$ is equal to the number of subsets of $A$, $|A_{\text{power}}| = 2^k = 2^{|A|}$.*

## 2 Proofs about Sets

So far we've seen sets: empty sets ($\{\}$), finite sets ($\{1, 2, 3\}$), infinite sets ($\mathbb{Z}$, $\mathbb{N}$), sets in set-builder notation ($\{n \in \mathbb{Z} : 2 \mid n\}$). We also saw some operators on pairs of sets $A$ an $B$, including the union ($A \cup B$), intersection ($A \cap B$), difference ($A - B$), and symmetric difference ($A \Delta B$). When a set $A$ has an element $x$, it is written $x \in A$, and when all elements of $A$ are also in $B$, we write $A \subseteq B$. We've seen a lot of definitions, and not much done with them. Here we'll see theorems about sets and proofs of these theorems. Let's start with a claim about two specific sets:

**Theorem.** *Let* $A = \{n^2 + n : n \in \mathbb{N}, n \text{ even}\}$. *Let* $B = \{n \in \mathbb{N}, n \text{ even}\}$. *Then* $A \subseteq B$.

This is theorem about one set being a subset of another. To prove such a theorem, show that if $x \in A$, then $x \in B$, since that's how $\subseteq$ is defined.[1]

**Proof.** *Let* $A = \{n^2 + n : n \in \mathbb{Z}, n \text{ even}\}$ *and* $B = \{n \in \mathbb{Z} : n \text{ even}\}$. *Let* $x \in A$. *Then* $x = n^2 + n$ *for some even integer* $n$. *So* $x = (2a)^2 + 2a = 4a^2 + 2a = 2(2a^2 + a)$. *So* $x = 2b$ *for some integer* $b = 2a^2 + a$. *So* $x$ *is an even integer and* $x \in B$. *So* $x \in A \Rightarrow x \in B$. *So* $A \subseteq B$. □

**Theorem.** *Let* $A$ *and* $B$ *be two sets. If* $A = B$, *then* $A \subseteq B$ *and* $B \subseteq A$.

**Proof.** *Let* $x \in A$. *Then since* $A = B$, $x \in B$. *So* $x \in A \Rightarrow x \in B$. *So* $A \subseteq B$.

*Now let* $x \in B$. *Then since* $A = B$, $x \in A$. *So* $x \in B \Rightarrow x \in A$. *So* $B \subseteq A$. □

**Theorem.** *Let* $A$ *and* $B$ *be two sets. If* $A \subseteq B$ *and* $B \subseteq A$, *then* $A = B$.

**Proof.** *Let* $x \in A$. *Since* $A \subseteq B$, $x \in B$. *So if* $x \in A$, *then* $x \in B$. *Now let* $x \in B$. *Since* $B \subseteq A$, $x \in A$. *So if* $x \in B$, *then* $x \in A$.

*Then since* $x \in A \Rightarrow x \in B$ *and* $x \in B \Rightarrow x \in A$, $x \in A \Leftrightarrow x \in B$. *So* $\{x : x \in A\} = \{x : x \in B\}$. *So* $A = B$. □

Combining these two theorems gives the following:

**Theorem.** *Let* $A$ *and* $B$ *be two sets.* $A \subseteq B$ *and* $B \subseteq A$ *iff* $A = B$.

This can be written more symbolically as:

**Theorem.** *Let* $A$ *and* $B$ *be two sets.* $A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$.

**Proof.** *Let* $A$ *and* $B$ *be two sets. By the first of the two previous theorems,* $A = B \Rightarrow A \subseteq B \wedge B \subseteq A$. *By the second of the two previous theorems,* $A \subseteq B \wedge B \subseteq A \Rightarrow A = B$. *So* $A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$. □

This theorem is not only interesting, but useful! It gives us an approach to proving two sets are equal: proving that each is a subset of the other. Let's do another proof involving subsets:

**Theorem.** *Let* $A$ *be the prime integers and* $B = \{n \in \mathbb{Z} : 2 \mid n\}$. *Then* $A \cap B = \{2\}$.

[1] This is another instance of unraveling the definition of the conclusion to get us closer to the hypothesis.

This theorem is a statement about two sets being equal. So we'll prove this using the equivalence of $A = B$ and $A \subseteq B \wedge B \subseteq A$.

**Proof.** *Let $n \in A \cap B$. Since $n \in B$, $n$ is an integer and $2 \mid n$. Since $n \in A$, for any integer $a$ with $1 < a < n$, $a \nmid n$. So it must be that $2 \not< n$. So $2 \geq n$. Since $n$ is prime, $1 < n$. So $1 < n \leq 2$ and thus $n = 2$. So $n \in \{2\}$. So $n \in A \cap B \Rightarrow n \in \{2\}$. So $A \subseteq B$.*

*Now let $n \in \{2\}$. So $n$ is prime and so $n \in A$. Also $1 \cdot 2 = 2$, so $n \in B$. So $n \in A$ and $n \in B$ and thus $n \in A \cap B$. So $n \in \{2\} \Rightarrow n \in A \cap B$. So $B \subseteq A$. By the previous theorem, since $A \subseteq B \wedge B \subseteq A$, then $A = B$.* □

# 3 Proofs about the Cardinality of Sets

Recall that the number of elements in a set is the *cardinality* or *size* of the set. In addition to proving theorems sets being contained in one another, we can also prove theorems about the relative sizes of various sets, such as:

**Theorem.** *Let $A$ and $B$ be two finite sets. Then $|A| + |B| - |A \cap B| = |A \cup B|$.*

Notice that I restricted $A$ and $B$ to be finite sets. For this theorem to make sense and be true, I need this restriction.[2] To gain some intuition about why this theorem might be true, we use drawings called *Venn diagrams* that depict sets as regions (usually circular) that contain their elements. Here's a Venn diagram for two sets:

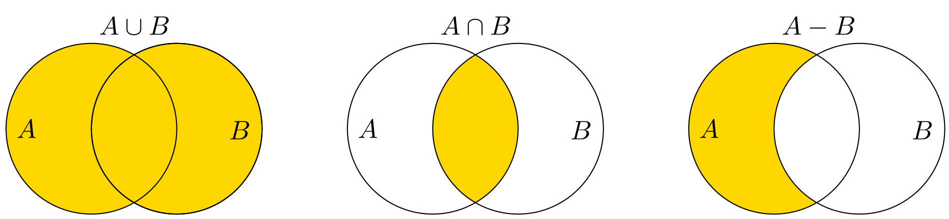


Figure 1: Venn diagram for two sets $A$ and $B$ and their union, intersection, and difference $A - B$.

The "elements" of $A$ and $B$ in a Venn diagram are all the locations (infinitely many) in the regions of $A$ and $B$. This lets us draw Venn diagrams for sets of any size, without worrying about the details. Now consider the previous theorem as a Venn diagram in Figure 2:

In the Venn diagram, the theorem looks to be true since the two disks representing $A$ and $B$ overlap and so have the $A \cap B$ crescent covered twice, whereas $A \cup B$ does not. So removing one copy of the crescent is needed to make the two sides equals. An actual proof looks similar, but formalizes the intuition of the Venn diagram:

**Theorem.** *Let $A$ and $B$ be two finite sets. Then $|A| + |B| - |A \cap B| = |A \cup B|$.*

[2] Arithmetic on $\infty$ is scary.

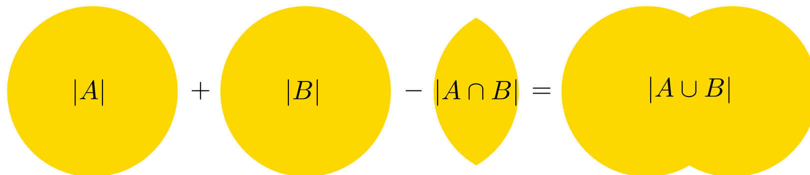


Figure 2: Venn diagram for the claim $|A| + |B| - |A \cap B| = |A \cup B|$.

**Proof.** *Let $A$ and $B$ be two finite sets. The total number of elements that belong to $A$ or $B$ is $|A| + |B|$, except that elements in $A \cap B$ are counted twice, once for being an element of $A$ and once for being an element of $B$. So the number of elements in $A$ and $B$ is $|A| + |B| - |A \cap B|$. Alternatively, the total number of elements that belong to $A$ or $B$ is the number of elements in their union, $|A \cup B|$. So $|A| + |B| - |A \cap B|$.* □

Generally, theorems about two objects having equal size are proved by arguing how the sizes each object both correspond to the size of a common object. If the objects are pieces of sets, drawing the Venn diagram corresponding to the two sides of the equality helps to see why they are equal. For instance, the theorem:

**Theorem.** *Let $A$ and $B$ be two finite sets. Then $|A| + |B| - |A \cap B| = |A - B| + |A \cap B| + |B - A|$.*

can be drawn as a Venn diagram (Figure 3) as seen below, which also suggests that the proof should argue that both sides count the total number of objects in $A \cup B$.

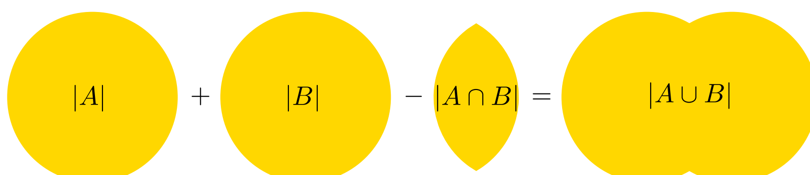


Figure 3: Venn diagram for the claim $|A| + |B| - |A \cap B| = |A \cup B|$.