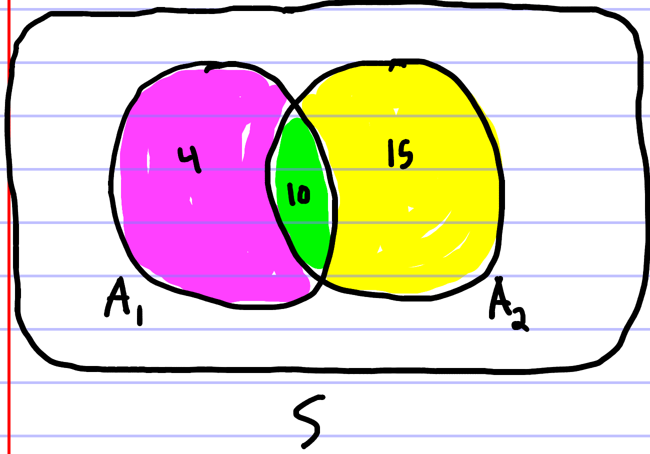
INCLUSION-EXCLUSION

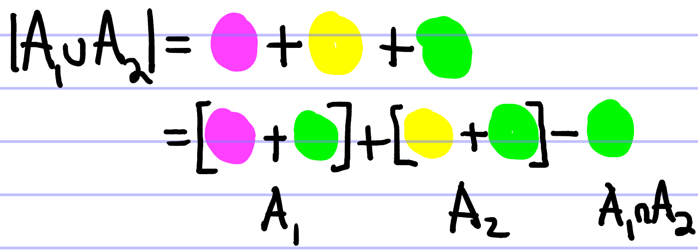
Let $S=\{1,2,\dots,1000\}$. Let $A=\{n\in S: 2\mid n \text{ or } 5\mid n\}\subseteq S$.
What is $|A|$?



Let $A_1=\{n\in S: 2\mid n\}$
$A_2=\{n\in S: 5\mid n\}$

Then $A=A_1\cup A_2$.

So $|A| = \underbrace{|A_1|+|A_2|}_{\text{inclusion}} - \underbrace{|A_1\cap A_2|}_{\text{exclusion}}$

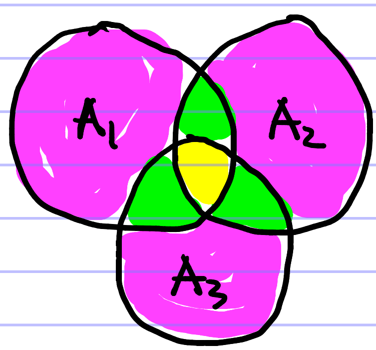


$A_1\cap A_2=\{n\in S: 10\mid n\}$

So $|A_1\cap A_2|=100$.

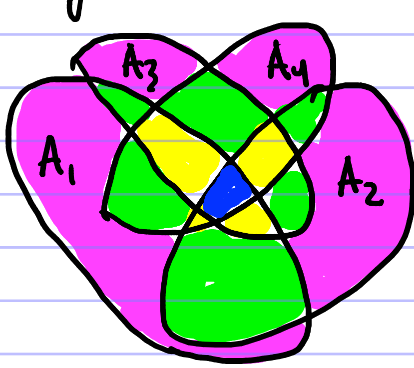
So $|A|=500+200-100=600$

More generally:



$$|A_1\cup A_2\cup A_3| = \overbrace{|A_1|+|A_2|+|A_3|}^{\text{inclusion}}$$
$$\left.-|A_1\cap A_2|-|A_1\cap A_3|-|A_2\cap A_3|\right]\text{exclusion}$$
$$+\underbrace{|A_1\cap A_2\cap A_3|}_{\text{inclusion}}$$

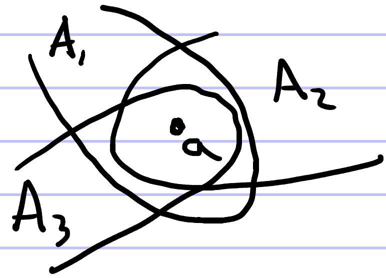
Can go even crazier:



$$|A_1\cup A_2\cup A_3\cup A_4| = \overbrace{|A_1|+|A_2|+|A_3|+|A_4|}^{\text{inclusion}}$$
$$\text{exclusion}\left[\begin{array}{l}-|A_1\cap A_2|-|A_1\cap A_3|-|A_1\cap A_4|\\ -|A_2\cap A_3|-|A_2\cap A_4|-|A_3\cap A_4|\end{array}\right.$$
$$\text{inclusion}\left[\begin{array}{l}+|A_1\cap A_2\cap A_3|+|A_1\cap A_2\cap A_4|\\ +|A_1\cap A_3\cap A_4|+|A_2\cap A_3\cap A_4|\end{array}\right.$$
$$\text{exclusion}\left[\ -|A_1\cap A_2\cap A_3\cap A_4|\right.$$

Why does this work?

Consider an element in the intersection of k of the sets, but not in the in the intersection of any $k+1$ sets.



$a \in A_1 \cap A_2 \cap A_3$ but not in $A_4, A_5, \ldots$

$|A_1| + |A_2| + |A_3| + \ldots$ triple counts $+$ $\binom{3}{1}$

$-|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| - \ldots$ triple counts $-$ $\binom{3}{2}$

$+|A_1 \cap A_2 \cap A_3| + \ldots$ counts once $+$ $\binom{3}{3}$

$- \ldots$ no more counting $\binom{3}{>3}$

The 1-set sum k counts $+$, the 2-set sum $\binom{k}{2}$ counts $-$, the 3-set sum $\binom{k}{3}$ counts $+$, the 4-set sum $\binom{k}{4}$ counts $-$, ... the k-set sum $\binom{k}{k}$ counts $\pm$, all $>k$-set sums 0 count.

So total tally is $\sum_{i=1}^{k} \binom{k}{i} \cdot (-1)^i \cdot -1$. Hope this equals 1. Otherwise inclusion-exclusion is wrong!

Recall $\sum_{i=0}^{k} \binom{k}{i} \cdot (-1)^i = 0$ by last lecture slides.

Then $\sum_{i=1}^{k} \binom{k}{i} \cdot (-1)^i \cdot -1 = -\sum_{i=1}^{k} \binom{k}{i} \cdot (-1)^i$

$= -\sum_{i=0}^{k} \binom{k}{i} \cdot (-1)^i + \binom{k}{0}$

$= -0 + \binom{k}{0} = 1.$

So inclusion-exclusion is correct.

What can we do with this? Solve tough counting problems where over/undercounting cause issues.

Defn: A derangement is a permutation of $S=\{1,2,\dots,n\}$ where $\forall i \in S$, the ith element is not i.

Ex: $S=\{1,2,3,4\}$ 2143 OK 3124 NO 2413 OK 4321 OK 4132 NO

How many derangements of 1,2,3,4 are there?

$4! = 24$ total permutations.

Count non-derangements D. Then derangements $= 4! - D$.

$A_1 = \{$permutations with 1 in first spot$\}$ $|A_i| = 3!$ 1324

$A_2 = \{$permutations with 2 in second spot$\}$ $|A_i \cap A_j| = 2!$ 1243

$A_3 = \{$permutations with 3 in third spot$\}$ $|A_i \cap A_j \cap A_k| = 1!$ 1234

$A_4 = \{$permutations with 4 in fourth spot$\}$ $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0!$ 1234

$$
\begin{aligned}
D = |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\
&\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 - A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= \binom{4}{1} \cdot 3! - \binom{4}{2} \cdot 2! + \binom{4}{3} \cdot 1! - \binom{4}{4} \cdot 0! \\
&= 24 - 12 + 4 - 1 = 15
\end{aligned}
$$

So #derangements is $4! - D = 24 - 15 = 9$.

(inclusive)

How may integers between 1 and 1000 are not divisible by 2, 3, or 5?

$S = \{n \in \mathbb{Z} : 1 \le n \le 1000\}$

$A_1 = \{n \in S : 2 | n\}$

$A_2 = \{n \in S : 3 | n\}$

$A_3 = \{n \in S : 5 | n\}$

Count # divisible by 2, 3, or 5. D

Subtract this from 1000 to get # not divisible by 2, 3, or 5

$\neg(2 \nmid n \wedge 3 \nmid n \wedge 5 \nmid n) = (2|n \vee 3|n \vee 5|n)$

$$D = |\{n \in S : 2|n \vee 3|n \vee 5|n\}| = |A_1 \cup A_2 \cup A_3|$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 500 + 333 + 200 - 166 - 100 - 66 + 33$$

$$= 734$$

So $1000 - D = 1000 - 734 = 266$ integers not divisible by 2, 3, or 5.

How may binary strings of length 5 have 000 or 111 as a substring?

$A_1 = \{$binary strings $000XY$ or $111XY\}$ $\quad A_1 \cap A_2 = \{0000X, 1111X\}$

$A_2 = \{$binary strings $X000Y$ or $X111Y\}$ $\quad A_1 \cap A_3 = \{00000, 11111\}$

$A_3 = \{$binary strings $XY000$ or $XY111\}$ $\quad A_2 \cap A_3 = \{X0000, X1111\}$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

$$= 8 + 8 + 8 - 4 - 2 - 4 + 2 = 16$$

(Non-Inclusion-Exclusion)

How may up-right paths from lower-left to upper-right of an $n \times n$ grid.

Write path as UURURUURURRR

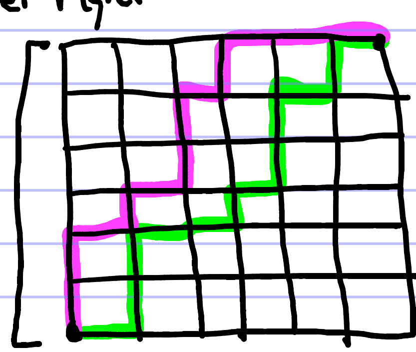
RUURRURUURUR

$\frac{(2n)!}{n! \cdot n!}$

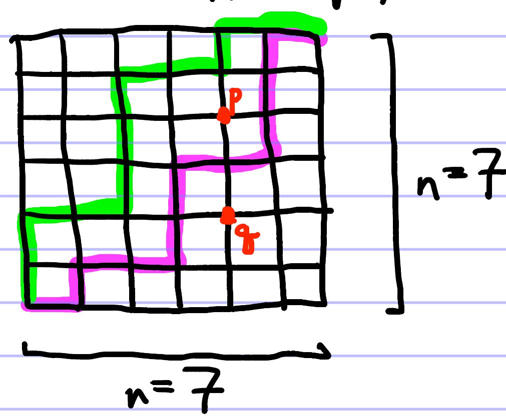
n

n U's, n R's $\Rightarrow \binom{2n}{n}$ paths

$2n$



How many paths from lower-left to upper-right, avoiding two locations?



#ok paths = #paths − #bad paths

#bad paths = $|A_1| + |A_2| - |A_1 \cap A_2|$

$A_1 = \{\text{paths through } p\}$

$A_2 = \{\text{paths through } q\}$

$A_1 = \{\underbrace{URR\cdots U}_{4\text{ U's } 4\text{ R's}},\ \underbrace{URRU}_{2\text{ U's } 2\text{ R's}}\}$ $\quad A_2 = \{\underbrace{URRU\cdots U}_{2\text{ U's } 4\text{ R's}}\ \underbrace{UU\cdots R}_{4\text{ U's } 2\text{ R's}}\}$

$A_1 \cap A_2 = \{\underbrace{URR\cdots U}_{2\text{ Us } 4\text{ Rs}}\ \underbrace{UU}_{2\text{ Us}}\ \underbrace{R\cdots U}_{2\text{ Us } 2\text{ Rs}}\}$

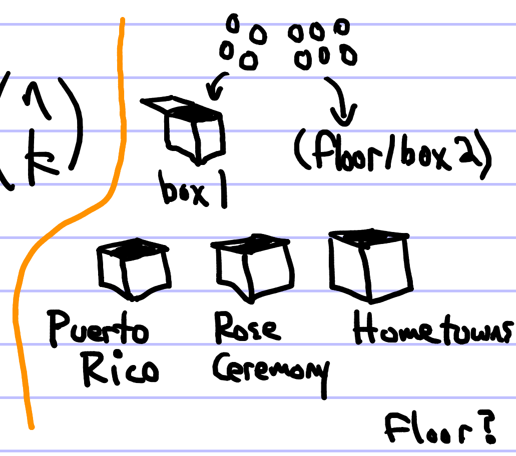
$|A_1| = \binom{8}{4}\cdot\binom{4}{2} \quad |A_2| = \binom{6}{2}\cdot\binom{6}{4} \quad |A_1 \cap A_2| = \binom{6}{2}\cdot\binom{4}{2}$

#ok paths $= \binom{12}{6} - \left[\binom{8}{4}\binom{4}{2} + \binom{6}{2}\binom{6}{4} - \binom{6}{2}\cdot\binom{4}{2}\right]$

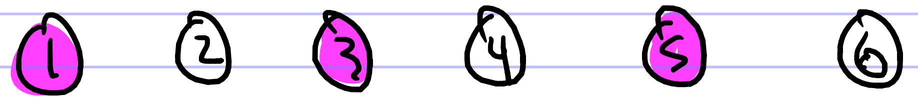
We have seen a variety of counting problem flavors:

- Permutations of distinct things $\quad n!$
- Permutations of identical things $\quad \dfrac{n!}{\prod_{c \in \Sigma} f(c)!}$
- Putting distinct things into boxes $\quad \binom{n}{k}$
- Putting identical things into boxes?

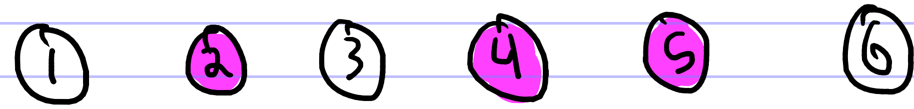
[Section 18: Counting Multisets]



$\binom{n}{k}$ means picking k distinct things from n distinct things

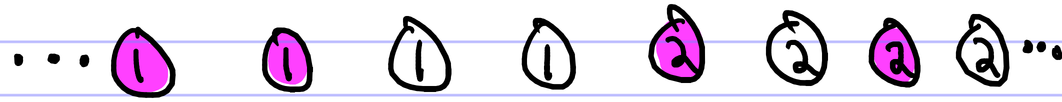
 $\{1,3,5\}$

is different than

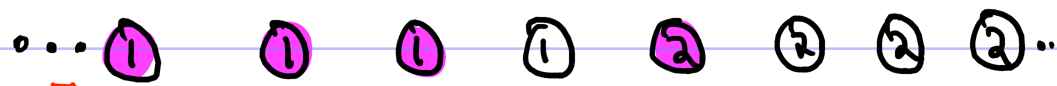
 $\{2,4,5\}$

What about counting the number of ways to pick a subset of 4 things from a ~~set~~ multiset of 2 types of identical things?

unorded, but repeats

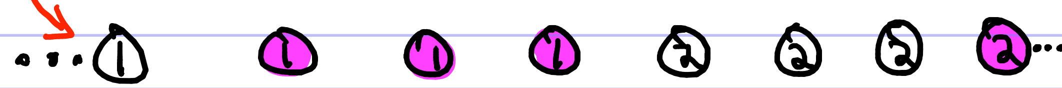
 $\langle 1,1,2,2\rangle$

is different than

 $\langle 1,1,1,2\rangle$

"different" identical things

is the same as

 $\langle 1,1,1,2\rangle$

How many such sets? Each determined by # of 1s & 2s, with #1s + #2s = 4.

5 such sets: 0x1, 4x2 1x1, 3x2 2x1, 2x2 3x1, 1x2 4x1, 0x2

Decision is just "how many of type 1,
how many of type 2,
⋮
how many of type n, so that total number chosen is k"

Ex: previous example has $n=2, k=4$, and there were 5 ways.

Defn: a <u>multiset</u> is an unordered collection with repeats.
A multiset is denoted $\langle x_1, x_2, \ldots, x_k \rangle$.

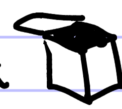
Ex: $\langle 1,2,3\rangle \neq \langle 1,1,2,3\rangle = \langle 3,2,1,1\rangle \neq \langle 2,3,1\rangle$

Defn: $\left(\!\binom{n}{k}\!\right)$, said "n multichoose k" is the number of multisets of size k of elements in a set of size n.

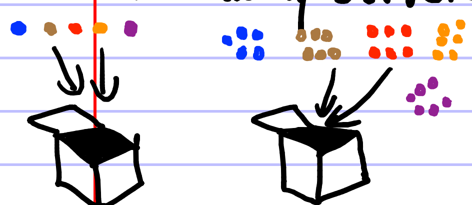
Ex: Let $n=2$ and set of size n be $\{1,2\}$. Let $k=3$.

$\langle 1,1,1\rangle, \langle 1,1,2\rangle, \langle 1,2,2\rangle, \langle 2,2,2\rangle$ are all multisets of size $k=3$ consisting of the elements of a set $\{1,2\}$ of size 2.

So $\left(\!\binom{2}{3}\!\right) = 4$.

Suppose you have a box  that holds 4 things.

How many different ways:



\#1 \#2

\#1 Can I fill the box from a set of 6 distinct things? $\binom{6}{4}$

\#2 Can I fill the box from a set of 3 distinct types of things? $\left(\!\binom{3}{4}\!\right)$

$\binom{n+k-1}{n-1}$ $\underline{\ }\ \underline{*}\ \underline{\ }\ \underline{*}\ \underline{\ }\ \underline{\ }\ \underline{*}\ \underline{\ }$

Thm: $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ For $n,k \in \mathbb{N}$, $n>0$. $k>n$ is ok!

"Stars & bars" Picking a size-k multiset is equivalent to picking partitions (bars) to divide the multiset elements into groups of each type:

$**|**|*||*$ $\qquad$ $*|*|***|*|$

$**|**|*||*$ $\qquad$ $*|*|***|*|$

Key:
n-1 bars, k stars
pick n-1 bars/k stars

$2\times\bullet, 2\times\bullet, 1\times\bullet, 0\times\bullet, 1\times\bullet$ $\qquad$ $1\times\bullet, 1\times\bullet, 3\times\bullet, 1\times\bullet, 0\times\bullet$

$\langle \bullet,\bullet,\bullet,\bullet,\bullet,\bullet\rangle$ $n=5, k=6$ $\qquad$ $\langle \bullet,\bullet,\bullet,\bullet,\bullet,\bullet\rangle$

For 2 types of identical things ($n=2$) and box that holds 4 things ($k=4$) we get:

$$\left(\!\!\binom{2}{4}\!\!\right) = \binom{2+4-1}{4} = \binom{5}{4} = 5$$

Stars and bars: |**** *|*** **|** ***|* ****|

Multisets: $\langle 2,2,2,2 \rangle$ $\langle 1,2,2,2 \rangle$ $\langle 1,1,2,2 \rangle$ $\langle 1,1,1,2 \rangle$ $\langle 1,1,1,1 \rangle$

assuming types are $\{1,2\}$

Exercise

How many ways are there to choose 10 things from 4 identical types? (Ex: types are 1,2,3,4 and one way is $\langle 1,1,2,2,2,2,3,3,3,4 \rangle$)

$$\left(\!\!\binom{4}{10}\!\!\right) = \binom{4+10-1}{10} = \binom{4+10-1}{4-1} = \frac{(4+10-1)!}{(4-1)!\,10!} = \frac{13!}{3!\,10!}$$

Exercise

What is $\left(\!\!\binom{1}{3}\!\!\right)$? Don't use the formula.

What is $\left(\!\!\binom{1}{k}\!\!\right)$? 1, since there is only one way to select k things of 1 type (k stars, 0 bars