N CHOOSE K

Defn: For $n, k \in \mathbb{N}$, notation $\binom{n}{k}$, said "n choose k", denotes the number of subsets of size k of a set of size n.

Examples: $\binom{3}{1} = 3$

$\binom{4}{0} = 1$

$\binom{1}{3} = 0$

$\binom{3}{2} = 3$

Thm: $\forall a, b \in \mathbb{N}$ with $a > b$, $\binom{b}{a} = 0$, $\binom{a}{a} = \binom{a}{0} = 1$, $\binom{a}{1} = a$.

$\binom{1}{3} = \binom{5}{100} = \binom{10}{11} = 0$

Can't pick more than available

$\binom{a}{a} = 1$

$\binom{a}{0} = 1$

$\binom{a}{1} = a$

Thm: $\forall n \in \mathbb{N}, n \geq 2, \binom{n}{2} = \frac{n(n-1)}{2}$.

Pf: Consider sequence of choices to select a subset of size 2:

Y/N Y/N Y/N Y/N Y/N Y/N Y/N $\approx$ NYNNYNN 2 Ys, n-2 Ns

Every sequence is a distinct permutation of the string $\overbrace{NN\cdots N}^{n-2}\overbrace{YY}^{2}$.

From last lecture, there are $\frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$ permutations. $\blacksquare$

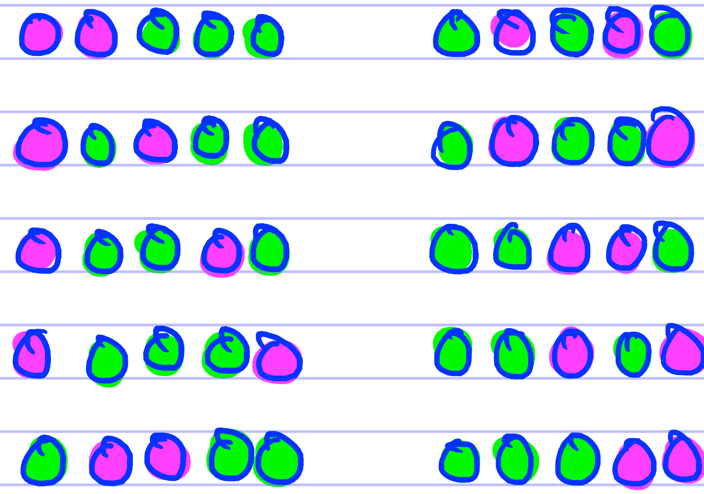
Thm: $\forall n,k \in \mathbb{N}$, $0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$.

Picking k elements to form the subset of size k is equivalent to picking the $n-k$ elements not in the subset.

Example: $\binom{5}{2} = \binom{5}{3}$

= in $k=2$ / not in $k=3$ subset

= in $k=3$ / not in $k=2$ subset



Pf: # subsets of size k of a set of size n is equal to the number of permutations of $\overbrace{NN\cdots N}^{n-k}\underbrace{YY\cdots Y}_{k}$

Namely, $\dfrac{n!}{(n-k)!\,k!}$.

subsets of size $n-k$ of a set of size n is equal to the number of permutations of $\overbrace{NN\cdots N}^{k}\underbrace{YY\cdots Y}_{n-k}$

Namely $\dfrac{n!}{k!\,(n-k)!}$. ▨

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

Thm: $\forall n,k \in \mathbb{N}$, $k \le n$, $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

Thm: Let $k \in \mathbb{N}$ be fixed, let $f(n) = \binom{n}{k}$. Then $f(n)$ is $O(n^k)$.

Pf: $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{1}{k!} \cdot \prod_{i=0}^{k-1}(n-i) \le \frac{1}{k!}\prod_{i=0}^{k-1} n = \frac{n^k}{k!}$$

Let $n_0 = k$, $c = 1$. Then $\forall n \ge n_0$, $\binom{n}{k} \le \dfrac{n^k}{k!} \le c \cdot n^k$. So $\binom{n}{k}$ is $O(n^k)$. ▨

Thm: Let $k \in \mathbb{N}$ be fixed, let $f(n) = \binom{n}{k}$. Then $f(n)$ is $\Omega(n^k)$.

Pf: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1}{k!}\prod_{i=0}^{k-1}(n-i) \geq \frac{1}{k!}\prod_{i=0}^{k-1}\frac{n}{2}$ for $n \geq 2k$.

$$\underbrace{n\cdot(n-1)\cdot\ldots\cdot(n-(k-1))}_{k} \geq \underbrace{\frac{n}{2}\cdot\frac{n}{2}\cdot\ldots\cdot\frac{n}{2}}_{k}$$

Also, $\frac{1}{k!}\prod_{i=0}^{k-1}\frac{n}{2} = \frac{1}{2k!}\prod_{i=0}^{k-1} n = \frac{1}{2k!}\cdot n^k = c\cdot n^k$ for $c = \frac{1}{2k!}$.

Let $n_0 = 2k$, $c = \frac{1}{2k!}$. Then $\forall n \geq n_0$, $\binom{n}{k} \geq c\cdot n^k$. So $\binom{n}{k}$ is $\Omega(n^k)$. ▨

Thm: Let $k \in \mathbb{N}$ be fixed, let $f(n) = \binom{n}{k}$. Then $f(n)$ is $\Theta(n^k)$.

Follows from last two theorems & defn of big-Θ.

PASCAL'S KHAYYAM'S HUI'S TARTAGLIA'S TRIANGLE

[Hui, 13th century]

[Pingala, 2nd century BC]

[Al-Karaji, 10th century]

...

row 0: 1

row 1: 1 1

row 2: 1 2 1

row 3: 1 3 3 1

row 4: 1 4 6 4 1

1 5 10 10 5 1

.. 6 15 20 15 6 1

...

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

...

Thm: the nth row of Pascal's Triangle is $\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2} \cdots \binom{n}{n-1}\ \binom{n}{n}$.

row 4 $\quad 1 \quad 4 \quad 6 \quad 4 \quad 1$

row 5 $\quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

$$\binom{4}{1} \quad \binom{4}{2} \;\to\; \binom{5}{2}$$

$$\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$$

$$4 + 6 = 10$$

Is it always true that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$? (for $n \geq k \geq 0$)

$$\frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} + \frac{(n-1)!}{k!\,((n-1)-k)!} \stackrel{?}{=} \frac{n!}{k!\,(n-k)!}$$

$$\frac{k \cdot (n-1)!}{k!\,(n-k)!} + \frac{(n-k)\cdot(n-1)!}{k!\,(n-k)!} \stackrel{?}{=} \frac{n!}{k!\,(n-k)!}$$

$$\frac{(k+n-k)\cdot(n-1)!}{k!\,(n-k)!} \stackrel{?}{=} \frac{n!}{k!\,(n-k)!}$$

$$\frac{n!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!}$$

YES!

Can use this equality + induction to prove theorem.

A combinatorial proof that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$:

Subset of size k either contains first element or not.

If so: $\bullet\ \underbrace{\circ\ \circ \cdots \circ}_{n-1}$ select $k-1$ elements from last $n-1$ elements. $\binom{n-1}{k-1}$

If not: $\circ\ \overbrace{\circ\ \circ \cdots \circ}$ select k elements from last $n-1$ elements. $\binom{n-1}{k}$

So total # subsets, i.e. $\binom{n}{k}$, is $\binom{n-1}{k-1} + \binom{n-1}{k}$. $\blacksquare$

Exercise

$2 \le k \le n-2$

Prove $\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$

$$\begin{array}{ccccccccccccc} & & & 1 & \overset{\times 1}{4} & & \overset{\times 2}{6} & & \overset{\times 1}{4} & 1 & & & \\ & & 1 & 5 & & 10 & & 10 & & 5 & 1 & & \\ & 1 & 6 & & 15 & & 20 & & 15 & & 6 & 1 & \end{array}$$

Hint: break into 4 cases, ● ○ $\underbrace{○ \cdots ○}_{n-2}$

○ ○ $\overbrace{○ ○ \cdots ○}^{n-2}$ $\binom{n-2}{k}$ +

● ○ ○ ○ $\cdots$ ○ $\binom{n-2}{k-1}$ +

○ ● ○ ○ $\cdots$ ○ $\binom{n-2}{k-1}$ +

● ● ○ ○ $\cdots$ ○ $\binom{n-2}{k-2}$

$= \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$

BINOMIAL COEFFICIENTS

Recall that $(x+y)^n = c_0 x^n + c_1 x^{n-1} y + \dots + c_{n-1} x y^{n-1} + c_n y^n$

Ex: $(x+y)^3 = 1\cdot x^3 + 3\cdot x^2 y + 3\cdot x y^2 + 1\cdot y^3$

where $c_0, c_1, \dots, c_n$ are entries of nth row of Pascal's Triangle.

Also, $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ are entries of nth row of Pascal's Triangle.

Thm: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. Binomial Theorem

Exercise

Compute $(x+1)^4$. $\qquad x^4 + 4x^3 + 6x^2 + 4x + 1$

$\binom{4}{0} x^4 \cdot \cancel{1^0} + \binom{4}{1} x^3 \cdot \cancel{1} + \binom{4}{2} x^2 \cancel{1^2} + \binom{4}{3} x^1 \cancel{1^3} + \binom{4}{4} \cancel{x^0} \cdot 1^4$

Thm: The sum of the numbers in the nth row of Pascal's Triangle is 2^n.

row 4: $1+4+6+4+1=16=2^4$

row 5: $1+5+10+10+5+1=32=2^5$

Pf: The sum of the numbers in the nth row is $\sum_{k=0}^{n}\binom{n}{k}$

That is, $\sum_{k=0}^{n}$(# subsets of size of a set S of size n)

So $\sum_{k=0}^{n}\binom{n}{k}=\sum_{k=0}^{n}\left|\{A : A\subseteq S, |A|=k\}\right|$

$= |\text{power set of } S| = 2^{|S|} = 2^n$. $\blacksquare$

How many ways are there to elect a committee of 6 people, and 2 chairs of committee, from a group of 10 people?

Two decisions: 1. who is on committee? $\binom{10}{6}$

2. who are chairs? $\binom{6}{2}$

Total ways: $\binom{10}{6}\cdot\binom{6}{2}=\frac{10!}{4!6!}\cdot\frac{6!}{4!2!}=\frac{10!}{4!4!2!}=3150$

Two decisions: 1. who are chairs? $\binom{10}{2}$

2. who are non-chair members? $\binom{8}{4}$

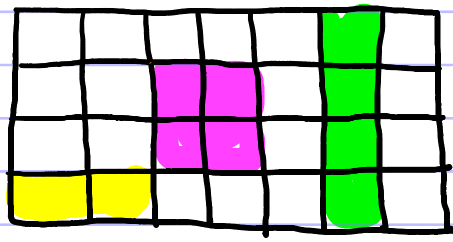
Total ways: $\binom{10}{2}\cdot\binom{8}{4}=\frac{10!}{2!8!}\cdot\frac{8!}{4!4!}=\frac{10!}{2!\cdot4!\cdot4!}=3150$

Two decisions: 1. who is not on committee? $\binom{10}{4}$

2. who is not a chair? $\binom{6}{4}$

Total ways: $\binom{10}{4}\cdot\binom{6}{4}=\frac{10!}{4!6!}\cdot\frac{6!}{4!2!}=\frac{10!}{4!4!2!}$

How many rectangles does the following drawing contain?



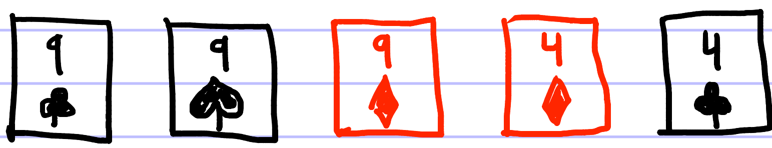
Pick left and right edges, pick bottom & top edges.

$$\binom{8}{2} \times \binom{5}{2}$$

How many full house hands exist?

52 cards to choose from 4 suits, 13 values in each suit. Deck

3 cards w/one value, 2 cards w/a second, different value. Hand



1. Pick 2 values. $\binom{13}{1} \cdot \binom{12}{1}$ why?
2. Pick 3 suits for first value. $\binom{4}{3}$
3. Pick 2 suits for second value. $\binom{4}{2}$

$$\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2} = 13 \cdot 12 \cdot 4 \cdot 6 = 3744$$

Chance of drawing a full house? $\dfrac{3744}{\binom{52}{5}} \approx 0.144\%$

Thm: $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\dots\pm\binom{n}{n}=0$ for $n\in\mathbb{N}, n>0$.

Ex: $\binom{3}{0}-\binom{3}{1}+\binom{3}{2}-\binom{3}{3}=1-3+3-1=0$

$\binom{4}{0}-\binom{4}{1}+\binom{4}{2}-\binom{4}{3}+\binom{4}{4}=1-4+6-4+1=0$

Pf: $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\dots\pm\binom{n}{n}=\sum_{k=0}^{n}\binom{n}{k}\cdot(-1)^k\cdot 1^{n-k}$

$=(1+-1)^n$ by binomial thm

$=0^n=0$ since $n>0$ $\blacksquare$