COUNTABILITY

Defn: An infinite set A is countable provided there is a bijection between A and $\mathbb{N}$, i.e. $|A| = |\mathbb{N}|$.

Thm: $\mathbb{Z}$ is countable. HW#5 Problems 12-15

Idea: Map positive integers to evens, negative integers to odds.

Thm: $\mathbb{Q}$ is countable.

Proof Idea: List rational numbers in a way that ensures each appears at least once in the list:

$$\frac{0}{1}\ \frac{1}{1}\ \frac{1}{2}\ \cancel{\frac{2}{2}}\ \frac{2}{1}\ \cancel{\frac{2}{2}}\ \frac{1}{3}\ \frac{2}{3}\ \cancel{\frac{3}{3}}\ \frac{3}{1}\ \frac{3}{2}\ \cancel{\frac{3}{3}}\ \frac{1}{4}\ \cancel{\frac{2}{4}}\ \frac{3}{4}\ \cancel{\frac{4}{4}}\ \frac{4}{1}\ \cancel{\frac{4}{2}}\ \frac{4}{3}\ \cancel{\frac{4}{4}}$$

$$\frac{1}{5}\ \frac{2}{5}\ \frac{3}{5}\ \frac{4}{5}\ \cancel{\frac{5}{5}}\ \frac{5}{1}\ \frac{5}{2}\ \frac{5}{3}\ \frac{5}{4}\ \cancel{\frac{5}{5}}\ \frac{1}{6}\ \cancel{\frac{2}{6}}\ \cancel{\frac{3}{6}}\ \cancel{\frac{4}{6}}\ \frac{5}{6}\ \cancel{\frac{6}{6}}\ \frac{6}{1}\ \cancel{\frac{6}{2}}\ \cancel{\frac{6}{3}}\ \cancel{\frac{6}{4}}\ \frac{6}{5}\ \cancel{\frac{6}{6}}$$

$$\cdots\ \frac{1}{n}\ \frac{2}{n}\ \cdots\ \frac{n-1}{n}\ \cancel{\frac{n}{n}}\ \frac{n}{1}\ \frac{n}{2}\ \cdots\ \frac{n}{n-1}\ \cancel{\frac{n}{n}}\ \cdots$$

Remove any element appering previously, assign $0, 1, 2, \ldots$ to remaining set of rationals. This assignment is a bijection.

Thm: $\mathbb{R}$ is not countable. [Cantor 1891]

Proof idea: Suppose, for the sake of contradiction, $\mathbb{R}$ is countable. Write down sequence of numbers in counting order:

$\mathbb{N}$	$\mathbb{R}$
0	5.91325...
1	3.29134...
2	0.18229...
3	9.00000...
4	4.91163...
5	1.11912...
⋮	⋮

Let $x = 4.17151\ldots$ where digit i is not the ith digit of the real number mapped to i. Then x is not in the list of counted real numbers, a contradiction. So $\mathbb{R}$ is not countable.

PERMUTATIONS

We previously defined a permutation of a set A as: a list containing the elements of A, where each element appears exactly once.

$A = \{1,2,3,4\}$, permutations: $(1,3,2,4)$, $(4,1,3,2)$, $(2,1,3,4)$, ...

This is ok, but hard to describe for large/infinite sets.

Defn: a permutation of a set A is a bijection $f: A \to A$.

$A = \{1,2,3,4\}$, permutations: $\{(1,1),(2,3),(3,2),(4,4)\}$, $\{(1,4),(2,1),(3,3),(4,2)\}$

$A = \mathbb{Z}$, permutations: $\{(n,-n) : n \in \mathbb{Z}\}$, $\{(n,n+3) : n \in \mathbb{Z}\}$, ...

Defn: the identity permutation of a set A is $\{(a,a) : a \in A\}$

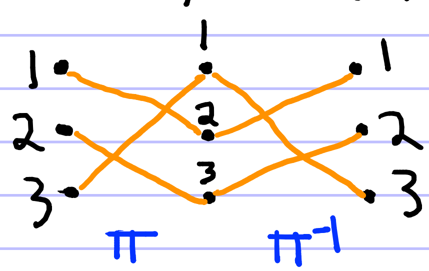
We use π instead of f or R for relations that are permutations.

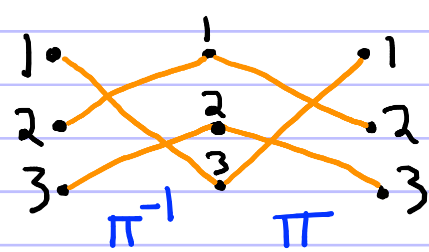
Defn: the inverse of a permutation π is π^{-1}.

$A = \{1,2,3,4\}$, $\pi = \{(1,4),(2,3),(3,1),(4,2)\}$, $\pi^{-1} = \{(4,1),(3,2),(1,3),(2,4)\}$

Thm: Let π be a permutation on a set A.
Then $\forall a \in A$, $\pi^{-1}(\pi(a)) = a$ and $\pi(\pi^{-1}(a))$.

$A = \{1,2,3\}$
$\pi = \{(1,2),(2,3),(3,1)\}$





Defn: Let A, B, C be sets and $f: A \to B$, $g: B \to C$. Then $g(f(x))$ is the composition of f and g, denoted $g \circ f(x)$, and $f \circ g: A \to C$.

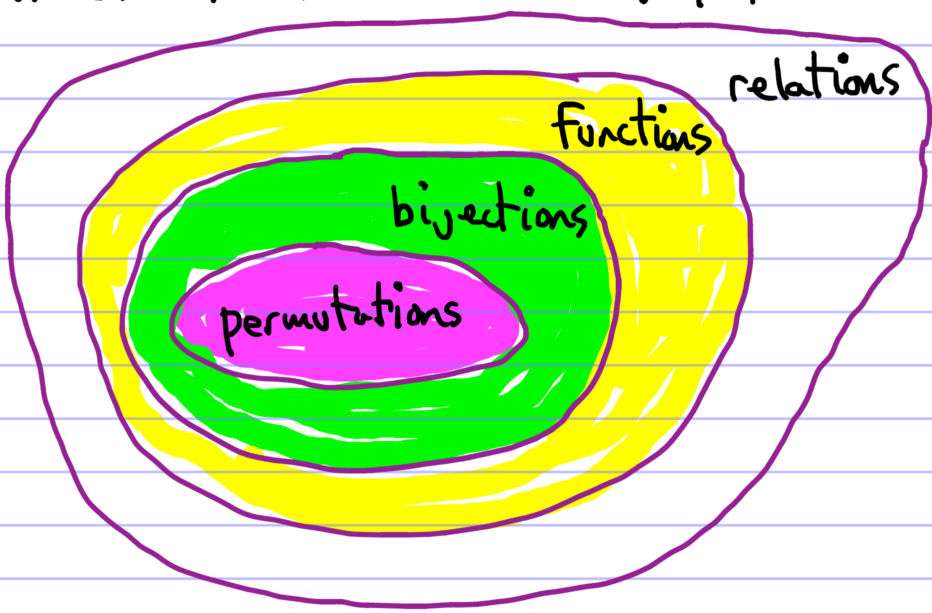
$A = \{1,2,3\}$, $f = \{(1,2),(3,1),(2,3)\}$, $g = \{(3,1),(2,2),(1,3)\}$, $g \circ f = \{(1,2),(3,3),(2,1)\}$

Thm: for any permutation π on a set A, $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi$ is the identity permutation.

permutations are bijections from a set to itself.

bijections are one-to-one and onto functions.

functions are relations with $(a,b),(a,c) \in R \Rightarrow b = c$.



Exercise:

Let $A = \{1,2\}$, $B = \{3,4\}$. Give a relation that is not a function, a function that is not a bijection, and a bijection that is not a relation.

COUNTING PERMUTATIONS

How many permutations of a set $A=\{a_1,a_2,\ldots,a_n\}$?

$$\pi=\{(a_1,\underset{\substack{\uparrow\\ n\text{ options}}}{\_\_}),(a_2,\underset{\substack{\uparrow\\ n-1\text{ options}}}{\_\_}),(a_3,\underset{\substack{\uparrow\\ n-2\text{ options}}}{\_\_}),\ldots,(a_{n-1},\underset{\substack{\uparrow\\ 2\text{ options}}}{\_\_}),(a_n,\underset{\substack{\uparrow\\ 1\text{ option}}}{\_\_})\}$$

$= n!$ sequences of decisions

How may permutations of a string "word"?

We know there are 4! If we write them as permutations:

$\{(w,d),(o,r),(r,o),(d,w)\}$, $\{(w,d),(o,w),(r,o),(d,r)\}$, . . .

"drow" "dwor"

Can also use positions (useful if duplicate letters):

$\{(1,4),(2,3),(3,2),(4,1)\}$, $\{(1,4),(2,1),(3,2),(4,3)\}$, . . .

$4321 \approx$ "drow" $4123 \approx$ "dwor"

How many permutations of "hello"? (12345)

$\{(1,5),(2,3),(3,4),(4,2),(5,1)\}$, $\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$53421 \approx$ "olleh" $54321 \approx$ "olleh"

Not 5! Swapping locations of two "l"'s gives same string.

$\{(1,3),(2,1),(3,2),(4,5),(5,4)\}$, $\{(1,4),(2,1),(3,2),(4,5),(5,3)\}$

$31254 \approx$ "lheol" $41253 \approx$ "lheol"

These are the only two permutations that give the same string.

So $\dfrac{5!}{2}=60$ permutations of the string "hello".

The repeated characters cause overcounting.
Need to divide by the number of different permutations that give the same string.

How many permutations of "alaska"?

Mark "a"'s for convenience: "alaska"
 1 2 3 4 5 6

Then $\underbrace{\text{"aaalsk"}}_{\pi_1}$ = "aaalsk" = "aaalsk" = "aaalsk" = "aaalsk" = $\underbrace{\text{"aaalsk"}}_{\pi_2}$

Written as permutations:

$\pi_1 = \{(1,1),(2,3),(3,6),(4,2),(5,4),(6,5)\}$ $\pi_2 = \{(1,3),(2,6),(3,1),(4,2),(5,4),(6,5)\}$

136245 = "aaalsk" $\qquad$ 361245 = "aaalsk"

The values 1,3,6 in second location in ordered pairs can be permuted arbitrarily. So $3! = 6$ permutations give the same string.

So $\dfrac{6!}{3!} = 120$ permutations of the string "alaska".

What about "mississippi"? $\qquad$ m**i**ss**i**ss**i**pp**i** $\quad$ 4 "i"s
Three groups of the same character: $\quad$ mi**ss**i**ss**ippi $\quad$ 4 "s"s
$\qquad$ mississi**pp**i $\quad$ 2 "p"s

So $\dfrac{11!}{4! \cdot 4! \cdot 2!} = 34560$ permutations of "mississippi".

Thm: Let "$c_1 c_2 \cdots c_n$" be a string with $\Sigma = \{c_1, c_2, \ldots, c_n\}$.
Let $f(c)$ be the number of times a character $c \in \Sigma$ occurs in the string. Then there are $\dfrac{n!}{\prod_{c\in\Sigma}[f(c)!]}$ permutations of the string.

"hello", $\Sigma = \{h, e, l, o\}$,

$$\frac{n!}{\prod_{c\in\Sigma} f(c)!} = \frac{5!}{1!\cdot 1!\cdot 2!\cdot 1!} = \frac{5!}{2}$$

"mississippi", $\Sigma = \{m, i, s, p\}$

$$\frac{n!}{\prod_{c\in\Sigma} f(c)!} = \frac{11!}{1!\cdot 4!\cdot 4!\cdot 2!} = \frac{11!}{4!\cdot 4!\cdot 2!}$$

Exercise:

Compute the number of permutations of "aardvark".

Can also count constrained permutations:

How many permutations of "sandstorm" with substrings "and", "tor"

Treat and, tor as single symbols: "s $\boxed{\text{and}}$ s $\boxed{\text{tor}}$ m"

"$\boxed{\text{tor}}$ s $\boxed{\text{and}}$ sm", "m $\boxed{\text{and}}$ $\boxed{\text{tor}}$ ss", "m $\boxed{\text{tor}}$ ss $\boxed{\text{and}}$", ...

Then $n = 5$, $\Sigma = \{s, \boxed{\text{and}}, \boxed{\text{tor}}, m\}$ and:

$$\frac{n!}{\prod_{c\in C}[f(c)!]} = \frac{5!}{2!\cdot 1!\cdot 1!\cdot 1!} = 60$$

same as "hello"

How may permutations of "aardvark" with substrings "ark" and "rad"?

v$\boxed{\text{ark}}$a$\boxed{\text{rad}}$, $\boxed{\text{rad}}\boxed{\text{ark}}va, av\boxed{\text{ark}}\boxed{\text{rad}}$, ...

Use previous theorem with $n=4$, $\Sigma=\{a, v, \boxed{\text{ark}}, \boxed{\text{rad}}\}$

$$\frac{n!}{\prod_{c\in C}[f(c)!]} = \frac{4!}{\underbrace{1!\cdot 1!\cdot 1!\cdot 1!}_{}} = 4!$$

no 3! for the 3 "a"s
no 2! for the 2 "r"s

Why?

When we combine symbols "a","r","k" to form $\boxed{\text{ark}}$, a unique "r" and "a" are chosen. No other "r" or "a" can trade places, so no overcounting.

The formula $\dfrac{n!}{\prod_{c\in C}[f(c)!]}$ can be derived using multiplication rule.

$$\#\text{permutations} = \frac{\#\text{symbol permutations}}{\#\text{ways to assign symbol copies to locations}}$$

"mississippi": 11! symbol permutations.

permutation	m	i	s	s	i	s	s	i	p	p	i	
choices	1	×4	×4	×3	×3	×2	×1	×2	×2	×1	×1	$=1!\times 4!\times 4!\times 2!$

permutation	i	i	p	s	p	s	i	s	i	s	m	
choices	4	×3	×2	×4	×1	×3	×2	×2	×1	×1	×1	$=1!\times 4!\times 4!\times 2!$

$$\#\text{permutations} \times \overbrace{\#\text{ways to assign}}^{1!\times 4!\times 4!\times 2!} = \overbrace{\#\text{symbol permutations}}^{11!}$$