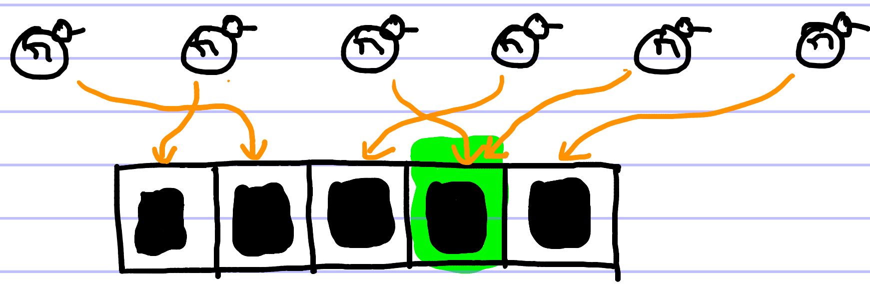
PIGEONHOLE PRINCIPLE

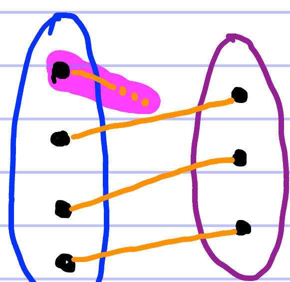
[Dirichlet 1834]

Thm: If p pigeons and h pigeonholes, with $p>h$, then some hole contains two or more pigeons.



Thm: Let P, H be two finite sets and $f: P \to H$.

If $|P| > |H|$, then f is not one-to-one.



P H

Some $(p_1,h),(p_2,h) \in f$ with $p_1 \neq p_2$.
The p_1 and p_2 are the pigeons in a common pigeonhole h.

Thm: Let $n \in \mathbb{N}$. Then $\exists\, a,b \in \mathbb{Z}$, $1 \le a,b$, $a \neq b$ such that $10 \mid n^a - n^b$.

Example: $n=3$. Then $3^5 - 3^1 = 243 - 3 = 240 = 24 \cdot 10$.
$n=17$. Then $17^6 - 17^2 = 24{,}137{,}280 = 2{,}413{,}728 \cdot 10$.
$n=4$. Then $4^5 - 4^3 = 1024 - 64 = 960 = 96 \cdot 10$.

Consider $3^1 = 3$ $3^2 = 9$ $3^3 = 27$ $3^4 = 81$ $3^5 = 243$.

Since 3^1 has 3 in one's place, as does 3^5, $3^5 - 3^1$ has a 0 in one's place.
A number is divisible by 10 iff it has a 0 in one's place.
So $10 \mid 3^5 - 3^1$.

$7^1=7$ $7^2=49$ $7^3=343$ $7^4=2041$ $7^5=16807$ power pigeons

0	1	2	3	4	5	6	7	8	9

one's place holes

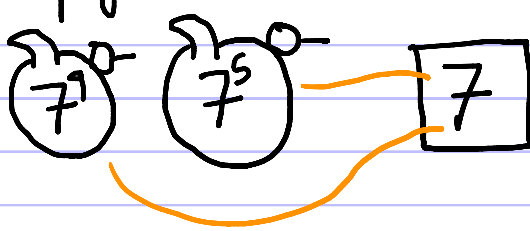
$7^6=117{,}649$ $7^7=823{,}543$ $7^8=5764801$ $7^9=40{,}353{,}607$

$7^{10}=282{,}475{,}249$ $7^{11}=1{,}977{,}326{,}743$

11 power pigeons: $7^1, 7^2, \ldots, 7^{11}$

10 one's place holes: $0, 1, \ldots, 9$

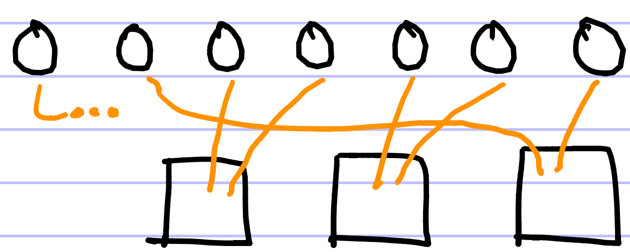
By PHP [Two power pigeons in same hole $\Rightarrow$ difference is divisible by 10.



$\Rightarrow 10 \mid 7^9 - 7^5$

Thm: If p pigeons and h pigeonholes, some pigeonhole contains at least $\frac{p}{h}$ pigeons. "Generalized" pigeonhole principle.

$p>h \Rightarrow \frac{p}{h}>1 \Rightarrow 2$ in a hole.



$7=p$ $\frac{7}{3} \approx 2.33$ pigeons

$3=h$ $\Rightarrow 3$ pigeons

No fractional pigeons! So $p=n+1, h=n \Rightarrow 2$ pigeons in some hole

Intuition: some hole has at least the average number of pigeons.

If $\frac{p}{h} \notin \mathbb{Z}$, some hole contains $\left\lceil \frac{p}{h} \right\rceil$ pigeons. $\lceil 2.5 \rceil = 3$ $\lceil 1.01 \rceil = 2$

Thm: Every set of n non-zero integers has at least $\frac{n}{2}$ elements with the same sign. $+5, -7, -4, +9$

Example: $\{-7, +5, +1, -4\}$ has $2 \geq \frac{4}{2}$ positive

$\{-9, -6, +5, -1, +3\}$ has $3 \geq \frac{5}{2}$ negative

Pf: By pigeonhole $(p=n, h=2)$, some sign has at least $\frac{n}{2}$ elements with the same sign. ▨

Thm: $\forall S \subseteq \mathbb{Z}$ with $|S| = n+1$, $\exists x, y \in S, x \neq y$, with $n \mid x-y$. ($n \geq 1$)

Pf: Write each integer $x_i \in S$ as $x_i = a_i n + b_i$, $0 \leq b_i < n$.

By pigeonhole $(p=|S|, h=n)$ there must be $x_i, x_j \in S$ with $x_i \neq x_j$ such that $b_i = b_j$. That is, $x_i = a_i n + b_i$, $x_j = a_j n + b_i$

So $x_i - x_j = a_i n + b_i - (a_j n + b_j) = (a_i - a_j)n + (b_i - b_j) = (a_i - a_j)n$.

So $n \mid x_i - x_j$. ▨

Thm: Let $S \subseteq \{i \in \mathbb{Z} : 1 \leq i \leq n\}$, $|S| \geq 3$. Then $\exists a, b \in S, a \neq b$, with $|a-b| < \frac{n}{2}$.

$\underbrace{1 \quad 2 \quad 3 \quad 4}, \underbrace{5 \quad 6 \quad 7 \quad 8}, \quad |5-8| = 3 \leq \frac{7}{2} = \frac{8}{2}$

$\underbrace{1 \quad 2 \quad 3 \quad 4}, \underbrace{5 \quad 6 \quad 7 \quad 8 \quad 9} \quad |5-9| = 4 \leq \frac{9}{2}$

Pf: First, we define two sets $S_1 = \{i \in S : 1 \leq i \leq \frac{n}{2}\}$, $S_2 = \{i \in S : \frac{n}{2} < i \leq n\}$

So $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \{\}$. By pigeonhole, either S_1 or S_2 contains at least two elements $a, b \in S$ with $a \neq b$.

So $|a-b| \leq \frac{n}{2} - 1$ and $|a-b| < \frac{n}{2}$. ▨

in S_1 in S_2

Exercise:

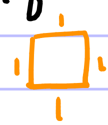
Prove that $\forall S \subseteq \{i \in \mathbb{Z} : 1 \leq i \leq 100\}$ with $|S| > 5$, $\exists a,b \in S$ with $|a-b| \leq 19$.

Let $S_j = \{i \in \mathbb{Z} : 20(j-1)+1 \leq i \leq 20j\}$. Then $\bigcup_{k=1}^{5} S_k = \{i \in \mathbb{Z} : 1 \leq i \leq 100\}$

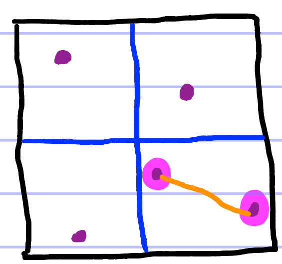
So by pigeonhole, $\exists a,b \in S$ with $a,b \in S_k$ for some k.

So $|a-b| \leq |20(k-1)+1-20k| = 19$. $\blacksquare$

Thm: Every set of five points in a unit square has two points with distance at most $\frac{\sqrt{2}}{2}$.

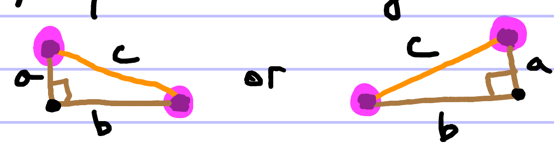


Pf:



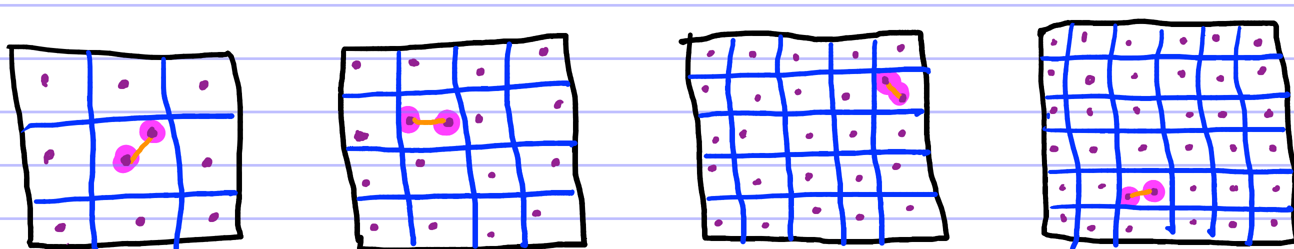
By pigeonhole, two points lie in the same quadrant of the square.

They form a right triangle:



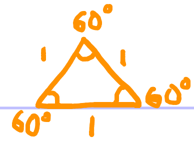
In either case, $a,b \leq \frac{1}{2}$. So $c = \sqrt{a^2+b^2} \leq \sqrt{\frac{1}{4}+\frac{1}{4}} \leq \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$ by the Pythagorean Theorem, and so is the distance between these points. $\blacksquare$

Generalization: k^2+1 points have a pair with distance at most $\frac{\sqrt{2}}{k}$.



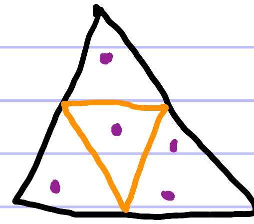
$k=3$ $\quad$ $k=4$ $\quad$ $k=5$ $\quad$ $k=6$

Exercise:

Unit triangle 

Prove that any set of five points in a unit equilateral triangle has two points with distance at most $\frac{1}{2}$.

Partition unit triangle into 4 $\frac{1}{2}$-triangles:

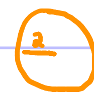


By pigeonhole, some $\frac{1}{2}$-triangle contains two points.

These two points have distance at most $\frac{1}{2}$. $\blacksquare$

The pigeonhole principle is not as rigid as induction.

Thm: Any set of 13 unit squares in a radius-2 circle has two squares that overlap.



Pf: Suppose, for the sake of contradiction, that a set of 13 non-overlapping squares in a radius-2 circle exists.

Then the area of these squares, 13, is less than or equal to the area of the radius-2 circle. But the area of a radius-2 circle is $\pi \cdot 2^2 = 4\pi \approx 12.6 < 13$, a contradiction.

So no such set exists and all sets contain an overlapping pair of squares. $\blacksquare$

Thm: Let R be an irreflexive and symmetric relation on a finite set A. Then $\exists a,b \in A$ with $a \neq b$ such that

$$|\{(a,x)\in R\}| = |\{(b,x)\in R\}|$$

Example: At any party, there are two people that have the same number of friends at the party.

A = people at the party R = "is a friend of" - irreflexive - symmetric

Pf: $\forall a \in A \quad 0 \le |\{(a,x)\in R\}| \le n-1.$

someone with no friends no one with everyone a friend

Also, $\exists a \in A$ with $0 = |\{(a,x)\in R\}|$ iff $\nexists b \in A$ with $n-1 = |\{(x,b)\in R\}|$

iff $\nexists b \in A$ with $n-1 = |\{(b,x)\in R\}|$

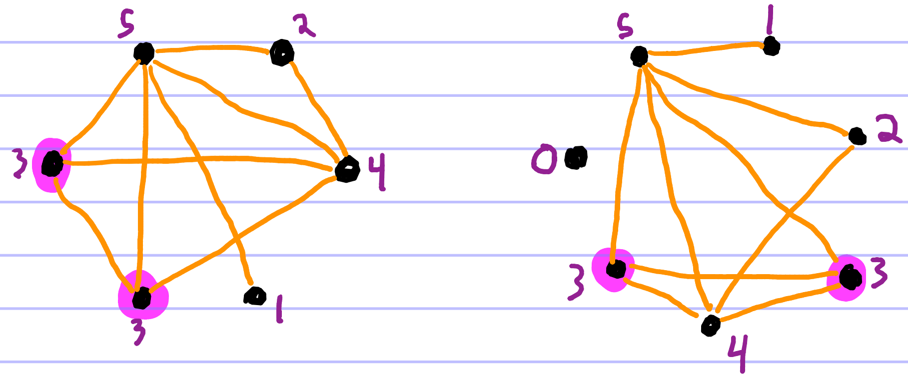
So either: $\forall a \in A \quad 0 \le |\{(a,x)\in R\}| \le n-2$ or

$\forall a \in A \quad 1 \le |\{(a,x)\in R\}| \le n-1$

In either case, there are $|A| = n$ elements of A (people at the party) and $n-1$ values of $|\{(a,x)\in R\}|$ (quantities of friends).

So by pigeonhole, $\exists a,b \in R$, $a \neq b$, with $|\{(a,x)\in R\}| = |\{(b,x)\in R\}|$. ▨

two people same number of friends



These drawings of a relation on a set are also called graphs.

Consider "zipping" files on your computer:

$$\underset{10\text{ Mb}}{\text{my\_cats.gif}} \xrightarrow[\text{zip}]{} \underset{0.5\text{ Mb}}{\text{my\_cats.zip}} \xrightarrow[\text{unzip}]{} \underset{10\text{ Mb}}{\text{my\_cats.gif}}$$

$$\underset{20\text{ Mb}}{\text{my\_dogs.txt}} \xrightarrow[\text{zip}]{} \underset{0.7\text{ Mb}}{\text{my\_dogs.zip}} \xrightarrow[\text{unzip}]{} \underset{20\text{ Mb}}{\text{my\_dogs.zip}}$$

Thm: For every $n \in \mathbb{N}, n \geq 1$, some file of n bits has a zipped version with n bits or more.

$$\underset{n\text{ bits}}{\text{average\_file.txt}} \xrightarrow[\text{zip}]{} \underset{\geq n\text{ bits}}{\text{average\_file.zip}} \xrightarrow[\text{unzip}]{} \underset{n\text{ bits}}{\text{average\_file.zip}}$$

Pf: There are 2^n files with n bits by the multiplication rule.
By HW#4 $\sum_{i=0}^{n-1} 2^i = 2^n - 1$, so the number of files with fewer than n bits is less than the number of files with n bits.

Suppose, for the sake of contradiction, that all zipped versions of files with n bits have $n-1$ bits or less.

Then by pigeonhole, two files must have the same zipped version.
So some files cannot be unzipped, and zip is broken, a contradiction.

Probably

So some file must have a zipped version with at least n bits. ▨