MIDTERM 1 RETROSPECTIVE

Hardest problem: #14 "$3 \le x \Rightarrow x^2 - 2x + 1$" by contradiction.

Average score: $35/44 = 80\%$

Median score: $37/44 = 84\%$

Grade partition: $\underbrace{\text{24 As}}_{40-44}$ $\underbrace{\text{27 Bs}}_{35-44}$ $\underbrace{\text{10 Cs}}_{31-35}$ $\underbrace{\text{17 Ds/Es}}_{\le 30}$

Chance of curve: 0.

An Experiment: Bonus Problems.

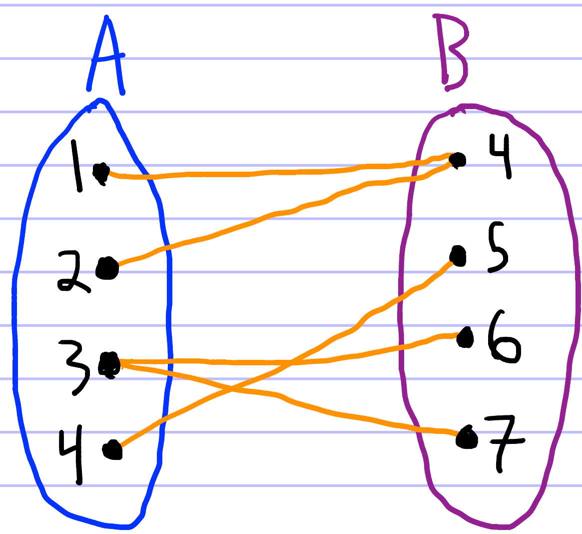
#1. Let $a_n = a_{n-1} + a_{n-\sqrt{n}}$, $a_0 = 100$.

Prove or disprove $\exists c > 1$ such that $\forall n \in \mathbb{N}$, $a_n \ge c^n$.

#2. Find a bijection between $\mathbb{Z}$ and $\mathbb{Q}$.

FUNCTIONS

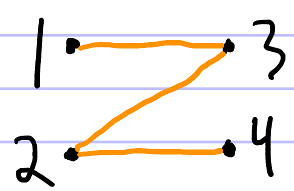
Recall that a relation R can not only be on a set A ($R \subseteq A \times A$) but also from a set A to a set B ($R \subseteq A \times B$):



Orange lines are the elements of the relation: a — b means $(a,b) \in R$.

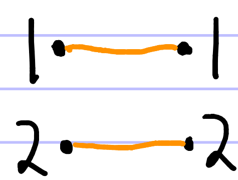
$$R = \{(1,4),(2,4),(3,6),(3,7),(4,5)\}$$

Defn: A relation f from A to B is a function provided $(a,b),(a,c) \in f \Rightarrow b = c$.

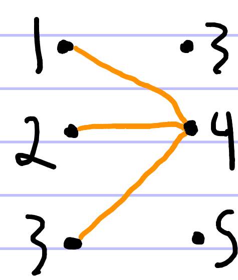


NOT A FUNCTION

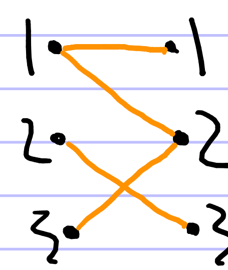
$(2,3),(2,4) \in f$ but $b \neq c$.



A FUNCTION



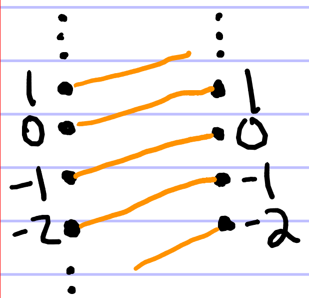
A FUNCTION

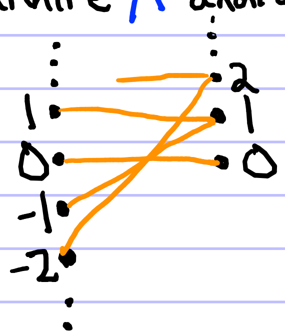


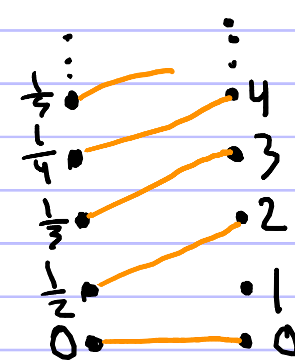
NOT A FUNCTION

$(1,1),(1,2) \in f$ but $1 \neq 2$.

Can also have infinite A and/or B.







Let f be a function from A to B.
If $\exists a \in A$ such that $(a,b) \in f$ for some $b \in B$, then we denote this by $f(a) = b$.

just for writing. It's a definition for some notation.

$A = \{1,2,3\}$, $B = \{4,5,6\}$, $R = \{(1,3),(2,6)\}$

$f(1) = 3$

$f(2) = 6$

$f(3)$ is undefined.

Since there is at most one $(a,b) \in R$ for each $a \in A$, if $a \in A$, then $f(a)$ is well-defined.

only one edge leaving a, so b is well-defined.

what can x be?

A familiar "function" like $f(x) = x^2$, is a function in the way we're defined "function", with some added care.

$A = \mathbb{Z}$, $B = \mathbb{Z}$.

$f = \{\ldots(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\ldots\}$

$f = \{(x,y) : x \in \mathbb{Z}, y \in \mathbb{Z}, x^2 = y\}$

$A = \mathbb{R}$, $B = \mathbb{R}$

$f = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x^2 = y\}$] Why a function?

$(a,b),(a,c) \in f \Rightarrow$
$a^2 = b, a^2 = c \Rightarrow$
$b = c$ by transitivity of $=$

EXERCISE

absolute value

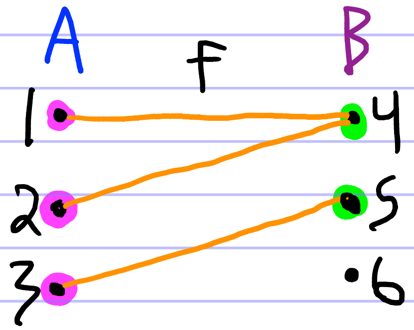
Write $f(x) = |x|$ as a relation (that is also a function) without using absolute value.

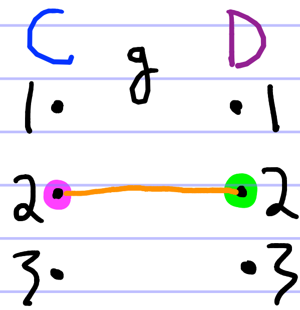
Assume $x \in \mathbb{R}$. Then F is a relation from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$F = \{(a,a) : a \in \mathbb{R}, a \geq 0\} \cup \{(a,-a) : a \in \mathbb{R}, a < 0\}$$

DOMAIN & IMAGE

Defn: Let f be a function on sets A and B.
The <u>domain</u> of f is the set $\{a \in A : \exists b \in B \text{ such that } (a,b) \in f\}$
The <u>image</u> of f is the set $\{b \in B : \exists a \in A \text{ such that } (a,b) \in f\}$





$\text{dom } f = \{1,2,3\} = A$ $\quad$ $\text{im } f = \{4,5\}$ $\qquad$ $\text{dom } g = \{2\}$ $\quad$ $\text{im } g = \{2\}$

Defn: Let f be a function on sets A and B. We say that f is a <u>function from A to B</u>, written <u>$f: A \to B$</u>, provided $\text{dom } f = A$ and $\text{im } f \subseteq B$.

$f: A \to B$ $\qquad$ ~~$g: C \to D$~~

$f: A \to \{4,5\}$ $\qquad$ $g: \{2\} \to D$ $\quad$ $g: \{2\} \to \{1,2\}$

$g: \{2\} \to \{2\}$ $\quad$ $g: \{2\} \to \{2,3\}$

If $f: A \to B$, then:

Conditions
1. f is a function (a relation from A to B with $(a,b),(a,c) \in f \Rightarrow b = c$)
2. $\text{dom } f = A$.
3. $\text{im } f \subseteq B$.

Thm: Let $A = \{1,2,3\}$, $B = \{4,5,6\}$. Let $f = \{(1,6),(2,4),(3,4)\}$. $f: A \to B$.

Pf: $f \subseteq A \times B$, so f is a relation. $\forall (1,b),(1,c)$, $b=c=6$, $\forall (2,b),(2,c)$, $b=c=4$, $\forall (3,b),(3,c)$, $b=c=4$.
So f is a function. (Condition 1)
$\text{dom } f = \{1,2,3\} = A$ (Condition 2)
$\text{im } f = \{6,4\} \subseteq B$. (Condition 3)
So $f: A \to B$. $\boxed{/\!/}$

INVERSES OF FUNCTIONS

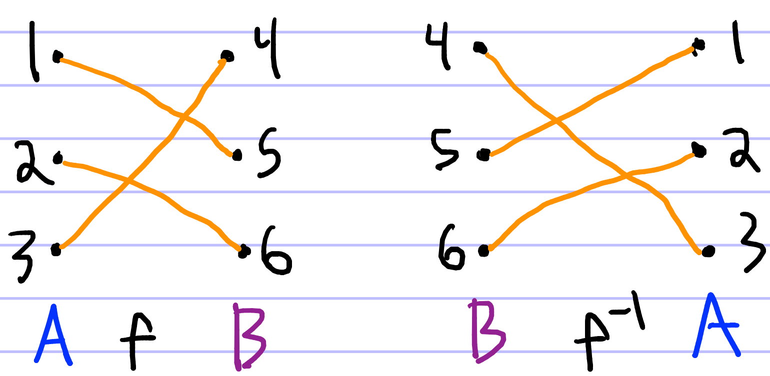
Recall for any relation R, $R^{-1} = \{(b,a) : (a,b) \in R\}$. Inverse of R
Since functions are special cases of relations, this is defined for them too.

Thm: The inverse of any relation R from A to B is a relation R^{-1} from B to A.

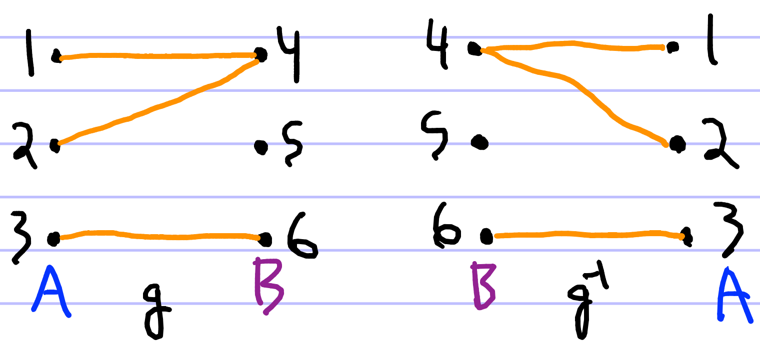
Pf: Let R be a relation from A to B. So $R \subseteq A \times B$.

$$\text{So } R^{-1} = \{(b,a) : (a,b) \in R\} = \{(b,a) : a \in A \wedge b \in B\}$$
$$= \{(b,a) : b \in B \wedge a \in A\} \subseteq B \times A \qquad \blacksquare$$

Conjecture: The inverse of any function f from A to B is a function f^{-1} from B to A.



If $A = \{1,2,3\}$, $B = \{4,5,6\}$,
$f = \{(1,5),(2,6),(3,4)\}$,
$f^{-1} = \{(5,1),(6,2),(4,3)\}$
the conjecture holds!



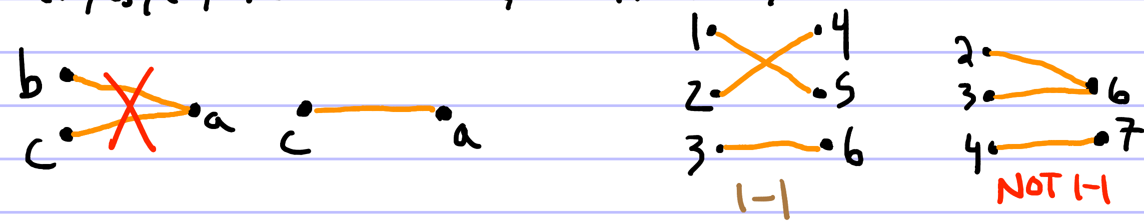
If $A = \{1,2,3\}$, $B = \{4,5,6\}$,
$f = \{(1,4),(2,4),(3,6)\}$,
$f^{-1} = \{(4,1),(4,2),(6,3)\}$,
the conjecture does not hold,
since $5 \in B$, $5 \notin \operatorname{dom} f^{-1}$

Not every function's inverse is a function... but some are.

Can we describe/classify when a functions inverse is also a function?

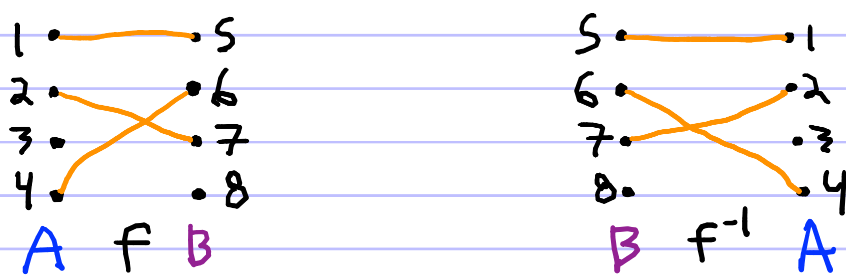
ONE-TO-ONE FUNCTIONS

Defn: A function f is one-to-one (aka 1-1) provided that
$(b,a),(c,a) \in f \Rightarrow b=c$, i.e. if $b \neq c$, then $f(b) \neq f(c)$.



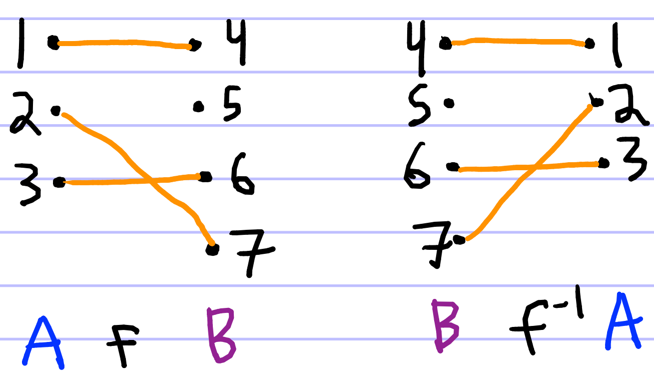
By definition of the inverse of a relation:

$\underbrace{(b,a),(c,a) \in f \Rightarrow b=c}_{f \text{ is a 1-1 function}}$ $\qquad$ $\underbrace{(a,b),(a,c) \in f^{-1} \Rightarrow b=c}_{f^{-1} \text{ is a function}}$



Thm: Let f be a function. Then f^{-1} is a function iff f is one-to-one.

Now suppose $f: A \to B$. f^{-1} is a function, but it might not be the case that $f^{-1}: B \to A$:



TRUE: $f: A \to B$

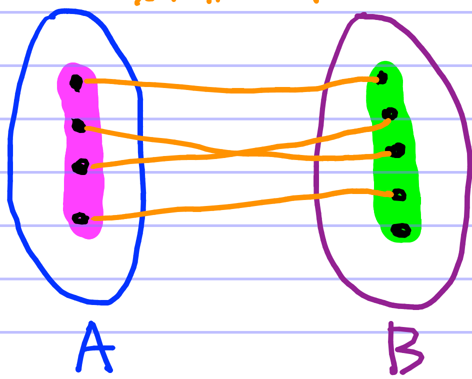
TRUE: f^{-1} is a function

FALSE: $f^{-1}: B \to A$, since $\operatorname{dom} f^{-1} \neq B$.

Can we describe/classify when a function $f: A \to B$ has an inverse $f^{-1}: B \to A$?

Thm: For any 1-1 function f, $\operatorname{im} f = \operatorname{dom} f^{-1}$ and $\operatorname{im} f^{-1} = \operatorname{dom} f$.

so f^{-1} is a function



dom f | im f

dom f^{-1} | im f^{-1}

If $f: A \to B$, then $f^{-1}: B \to A$ provided:

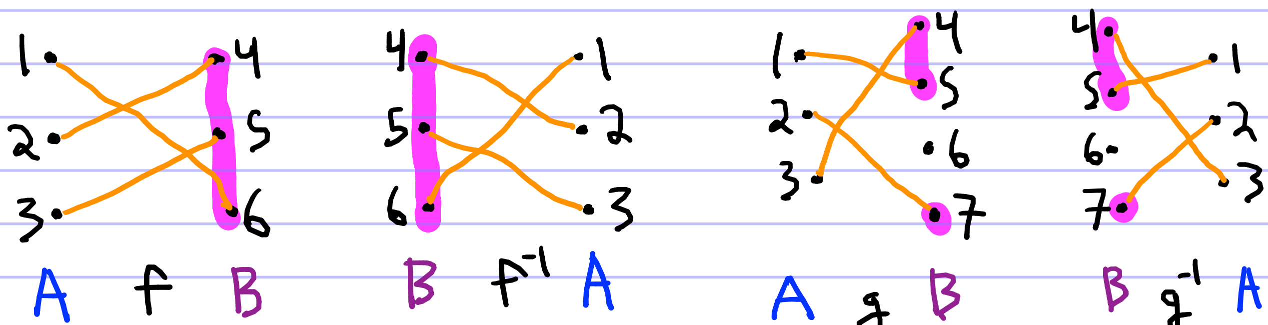
Definition of a function from B to A

1. f^{-1} is a function.] True iff f is 1-1
2. $\operatorname{dom} f^{-1} = B$.
3. $\operatorname{im} f^{-1} \subseteq A$. True since $\operatorname{im} f^{-1} = \operatorname{dom} f = A$ by prev thm

What about condition #2? True iff $\operatorname{im} f = B$.

Defn: A function $f: A \to B$ is onto provided $\operatorname{im} f = B$.

Thm: Let $f: A \to B$. Then $f^{-1}: B \to A$ iff f is onto.



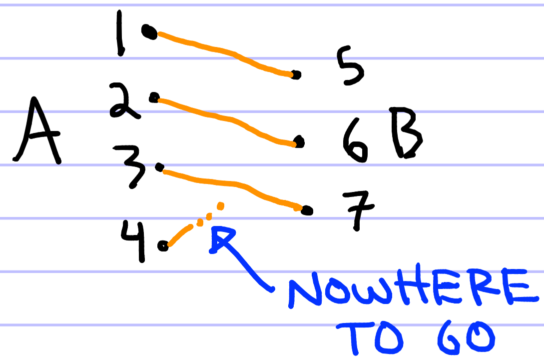
f onto | $f^{-1}: B \to A$ | f not onto | not $f^{-1}: B \to A$

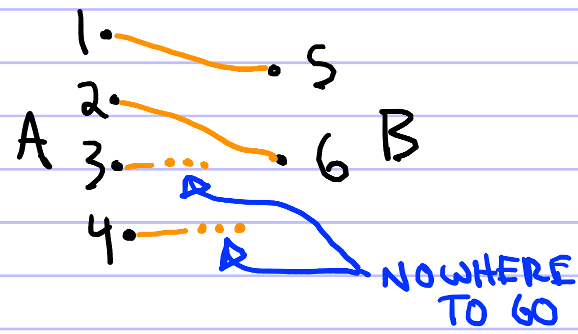
Other names for functions that are 1-1, onto, or both:

adjectives	one-to-one	onto	one-to-one & onto
	injective	surjective	bijective
noun	injection	surjection	bijection

Let A, B be finite sets.

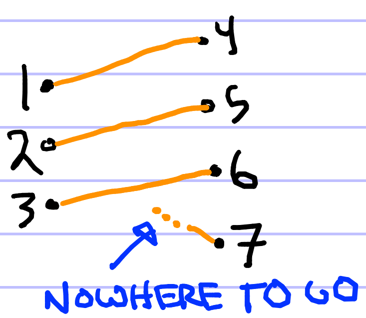
Thm: If $f: A \to B$ and $|A| > |B|$, then f is not one-to-one.

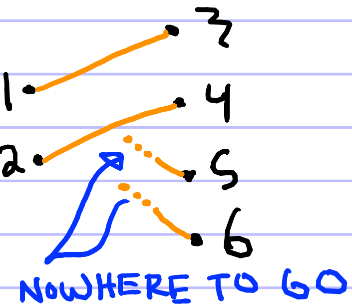


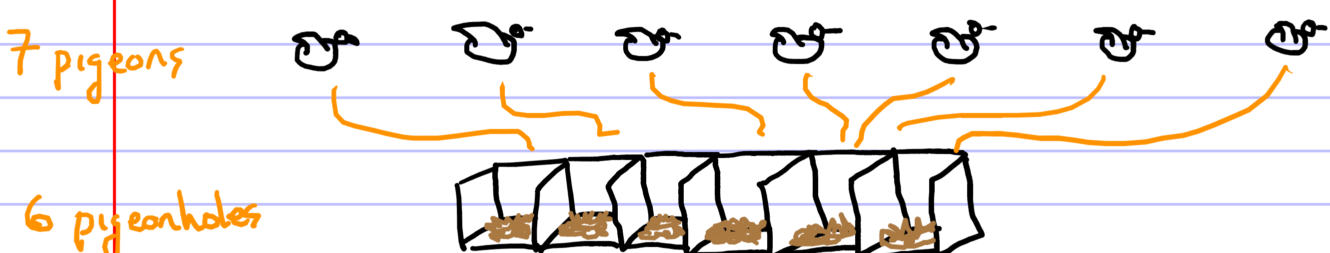


$|A| - |B|$ 'NOWHERE TO GO's

Thm: If $f: A \to B$ and $|A| < |B|$, then f is not onto.







Pigeonhole principle: If more pigeons than pigeonholes and each pigeon is in some pigeonhole, some pigeonhole must contain at least two pigeons.

We will revisit this in a later lecture.

PROOFS ON FUNCTIONS

- Proving $f: A \to B$.
- Proving $f: A \to B$ is one-to-one.
- Proving $f: A \to B$ is onto.

Thm: Let $A = B = \mathbb{Z}$ and $f = \{(x,y) : x,y \in \mathbb{Z}, y = 3x+4\}$. Then $f: A \to B$.

Pf: Let $(x,y), (x,y') \in f$. Then $y = 3x+4 = y'$. So $y = y'$. (Condition 1)

Let $x \in \mathbb{Z}$. Then $3x+4 \in \mathbb{Z}$. So $(x, 3x+4) \in f$.
So $\forall x \in A, \exists y \in B$ such that $(x,y) \in f$. So $\operatorname{dom} f = A$. (Condition 2)

Since $\forall x \in A, x \in \mathbb{Z}$ and $3x+4 \in \mathbb{Z}$.
So $\forall (x,y) \in f, y \in \mathbb{Z}$. So $\operatorname{im} f \subseteq \mathbb{Z} = B$. (Condition 3)

So $f: A \to B$ $\blacksquare$

Is $f: A \to B$ if $A = B = \mathbb{Q}$? If $A = \mathbb{Q}$ and $B = \mathbb{Z}$?

No, $\operatorname{dom} f = \mathbb{Z} \neq \mathbb{Q}$ No, for the same reason.

Thm: Let $f = \{(x,y) : x,y \in \mathbb{Z}, y = 3x+4\}$. Then f is one-to-one.

Pf: Let $x \neq x'$. Then $3x \neq 3x'$ and $3x+4 \neq 3x'+4$. $\blacksquare$

$x \neq x' \Rightarrow f(x) \neq f(x')$ [Direct]

Pf: Let $f(x) = f(x')$. Then $3x+4 = 3x'+4$ and $3x = 3x'$ and $x = x'$. $\blacksquare$

$f(x) = f(x') \Rightarrow x = x'$ [Contrapositive]

Pf: Let $x \neq x'$ and $3x+4 = 3x'+4$. So $3(x-x') = 0$ and $x - x' \neq 0$.
So $3 = 0$, a contradiction. $\blacksquare$

$x \neq x' \wedge f(x) = f(x') \Rightarrow F$ [Contradiction]

$f: \mathbb{Q} \to \mathbb{Q}$

Thm: Let $f = \{(x,y) : x, y \in \mathbb{Q}, y = 3x+4\}$. Then f is onto.

Pf: Let $y \in \mathbb{Q}$. So $y = \frac{a}{b}$ for some $a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0$.

Let $x = \frac{a-4b}{3b} \in \mathbb{Q}$. Then $3x+4 = 3\left(\frac{a-4b}{3b}\right)+4$

$$= \frac{3a-12b}{3b} + 4$$

$$= \frac{3a-12b+3b\cdot 4}{3b}$$

$$= \frac{3a}{3b} = \frac{a}{b} = y.$$

So $\forall y \in \mathbb{Q}, \exists x \in \mathbb{Q}$ such that $f(x) = y$. So $\operatorname{im} f \supseteq \mathbb{Q}$.
Also, $\operatorname{im} f \subseteq \mathbb{Q}$ by definition of f. So $\operatorname{im} f = \mathbb{Q}$. ▨

$\forall y \in \operatorname{im} f, \exists x \in \operatorname{dom} f$ such that $f(x) = y$.

EXERCISE

Let f be a relation from $\mathbb{R}$ to $\mathbb{N}$ defined by

$f = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{N}, y = x^2 - 1\}$. Prove f is onto.

Let $y \in \mathbb{N}$.

Then $x = \sqrt{y+1} \in \mathbb{R}$.

Also, $x^2 - 1 = (\sqrt{y+1})^2 - 1 = y+1-1 = y$.

So $(x,y) \in f$.

So $\operatorname{im} f \supseteq \mathbb{N}$.

Also, since $b \in \mathbb{N} \ \forall (a,b) \in \mathbb{N}$, $\operatorname{im} f \subseteq \mathbb{N}$.

So $\operatorname{im} f = \mathbb{N}$. So f is onto.