INDICATOR RANDOM VARIABLES

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- n people leave their hats with an attendant, & get a ticket = number for retrieval.
- The attendant loses all ticket info & gives hats back randomly.

How many people do we expect to get their own hats back?

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X = # people who get their own hat back

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$X_k = ?$

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The hiring problem: you need one assistant.

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How many people do you expect to hire?

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$= \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1) < \ln n + 1$

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The birthday problem (new variant)

How many people do we need in a room so that we expect to have (at least) one birthday match?

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X = # birthday matches among n people

What should our I.R.V. be ? $X_?$

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$X_{ij} = \ ?$

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$$X = \#\text{ birthday matches among } n \text{ people} = \underbrace{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n}}_{\text{all } \binom{n}{2} \text{ pairs}} X_{ij}$$

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$$E(X) = \ ?$$

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$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365}$$

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we said we want $E[X]=1$

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NOT 23