The hat-check problem (a.k.a. coat-check)
The hat-check problem (a.k.a. coat-check)

- \( n \) people leave their hats with an attendant, & get a ticket = number for retrieval.
The hat-check problem (a.k.a. coat-check)

- $n$ people leave their hats with an attendant, 
  & get a ticket = number for retrieval.

- The attendant loses all ticket info 
  & gives hats back randomly.
The hat-check problem (a.k.a. coat-check)

- \( n \) people leave their hats with an attendant, & get a ticket = number for retrieval.

- The attendant loses all ticket info & gives hats back randomly.

How many people do we expect to get their own hats back?
$X = \# \text{ people who get their own hat back}$
$X = \# \text{ people who get their own hat back}$

$X_k = ?$
INDICATOR RANDOM VARIABLES

\[ X = \text{\# people who get their own hat back} \]

\[ X_k = \begin{cases} 1 & \text{if person } k \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases} \]
**INDICATOR RANDOM VARIABLES**

\[
X = \# \text{ people who get their own hat back} = \sum_{k=1}^{n} X_k
\]

\[
X_k = \begin{cases} 
1 & \text{if person } k \text{ gets their own hat back} \\
0 & \text{otherwise}
\end{cases}
\]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people who get their own hat back} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 
1 & \text{if person } k \text{ gets their own hat back} \\
0 & \text{otherwise} 
\end{cases} \]

\[ E(X_k) = ? \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ people who get their own hat back} = \sum_{k=1}^{\hat{n}} X_k \]

\[ X_k = \begin{cases} 1 & \text{if person } k \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{n} \quad \text{(random)} \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people who get their own hat back} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 1 & \text{if person } k \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{n} \quad \text{(random)} \]

\[ E(X) = ? \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ people who get their own hat back} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 1 & \text{if person } k \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{n} \quad \text{(random)} \]

\[ E(X) = E\left( \sum_{k=1}^{n} X_k \right) = ? \]
**Indicator Random Variables**

\[ X = \# \text{ people who get their own hat back} = \sum_{k=1}^{\hat{n}} X_k \]

\[ X_k = \begin{cases} 1 & \text{if person } k \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{n} \] (random)

\[ E(X) = E\left( \sum_{k=1}^{\hat{n}} X_k \right) = \sum_{k=1}^{\hat{n}} E(X_k) \] linearity of expectation
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people who get their own hat back} = \sum_{k=1}^{\hat{n}} X_k \]

\[ X_k = \begin{cases} 1 & \text{if person } k \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{n} \quad \text{(random)} \]

\[ E(X) = E\left( \sum_{k=1}^{\hat{n}} X_k \right) = \sum_{k=1}^{\hat{n}} E(X_k) \quad \text{linearity of expectation} \]

\[ = \sum_{k=1}^{\hat{n}} \frac{1}{n} = \left( \frac{\hat{n}}{n} \right) \]

\[ = \left( \frac{1}{\frac{n}{\hat{n}}} \right) = 1 \]
The hiring problem: you need one assistant.
The hiring problem: you need one assistant. 

\( n \) candidates, interviewed in random order.
INDICATOR RANDOM VARIABLES

The hiring problem: you need one assistant.

- n candidates, interviewed in random order.
- No 2 equally skilled.
The hiring problem: you need one assistant.

- $n$ candidates, interviewed in random order.
- No 2 equally skilled.
- Any time you interview someone better than all previous, you hire the new person & fire the current assistant.
The hiring problem: you need one assistant.

- $n$ candidates, interviewed in random order.
- No two equally skilled.
- Any time you interview someone better than all previous, you hire the new person & fire the current assistant.

How many people do you expect to hire?
X = # people do you expect to hire
INDICATOR RANDOM VARIABLES

$X = \# \text{ people do you expect to hire}$

$X_k = ?$
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people do you expect to hire} \quad ? \]

\[ X_k = \begin{cases} 
1 & \text{if you hire candidate } k \\
0 & \text{otherwise} 
\end{cases} \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 
1 & \text{if you hire candidate } k \\
0 & \text{otherwise} 
\end{cases} \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 
1 & \text{if you hire candidate } k \\
0 & \text{otherwise} 
\end{cases} \]

\[ E(X_k) = ? \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 1 & \text{if you hire candidate } k \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{k} \quad (k \text{ is hired iff better than all } k-1 \text{ previous}) \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 1 & \text{if you hire candidate } k \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{k} \quad (k \text{ is hired } \iff \text{better than all } k-1 \text{ previous}) \]

\[ E(X) = ? \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{n} X_k \]

\[ X_k = \begin{cases} 1 & \text{if you hire candidate } k \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{k} \quad (k \text{ is hired iff better than all } k-1 \text{ previous}) \]

\[ E(X) = E\left( \sum_{k=1}^{n} X_k \right) = ? \]
INDICATOR RANDOM VARIABLES

\[ X = \text{# people do you expect to hire} = \sum_{k=1}^{\tilde{n}} X_k \]

\[ X_k = \begin{cases} 
1 & \text{if you hire candidate } k \\
0 & \text{otherwise} 
\end{cases} \]

\[ E(X_k) = \frac{1}{k} \quad (k \text{ is hired iff better than all } k-1 \text{ previous}) \]

\[ E(X) = E\left( \sum_{k=1}^{\tilde{n}} X_k \right) = \sum_{k=1}^{\tilde{n}} E(X_k) \quad \text{linearity of expectation} \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{\hat{n}} X_k \]

\[ X_k = \begin{cases} 1 & \text{if you hire candidate } k \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{k} \quad (k \text{ is hired iff better than all } k-1 \text{ previous}) \]

\[ E(X) = E\left( \sum_{k=1}^{\hat{n}} X_k \right) = \sum_{k=1}^{\hat{n}} E(X_k) \text{ linearity of expectation} \]

\[ = \sum_{k=1}^{\hat{n}} \frac{1}{k} \quad \ldots ? \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ people do you expect to hire} = \sum_{k=1}^{\hat{n}} X_k \]

\[ X_k = \begin{cases} 1 & \text{if you hire candidate } k \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_k) = \frac{1}{k} \quad (k \text{ is hired iff better than all } k-1 \text{ previous}) \]

\[ E(X) = E\left( \sum_{k=1}^{\hat{n}} X_k \right) = \sum_{k=1}^{\hat{n}} E(X_k) \text{ linearity of expectation} \]

\[ = \sum_{k=1}^{\hat{n}} \frac{1}{k} = \ln n + O(1) < \ln n + 1 \]
The birthday problem (new variant)

How many people do we need in a room so that we expect to have (at least) one birthday match?
$X = \# \text{ birthday matches among } n \text{ people}$

What should our I.R.V. be? $X$?
$X = \# \text{ birthday matches among } n \text{ people}$

$X_{ij} = ?$
INDICATOR RANDOM VARIABLES

\[ X = \text{\# birthday matches among n people} \]

\[ X_{ij} = \begin{cases} 
1 & \text{if persons i & j match} \\
0 & \text{otherwise} 
\end{cases} \]
X = # birthday matches among n people = ?

\[ X_{ij} = \begin{cases} 
1 & \text{if persons } i \& j \text{ match} \\
0 & \text{otherwise}
\end{cases} \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \text{ & } j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ birthday matches among } n \text{ people} \]

\[ X_{ij} = \begin{cases} 
1 & \text{if persons } i \& j \text{ match} \\
0 & \text{otherwise} 
\end{cases} \]

\[ E(X_{ij}) = ? \]
**Indicator Random Variables**

\[ X = \# \text{ birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \& j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = ? \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ birthday matches among } n \text{ people} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \text{ & } j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = E\left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right) = ? \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \]
\[ X = \text{\# birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \text{ & } j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = E\left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \text{ linearity of expectation} \]
INDICATOR RANDOM VARIABLES

\[ X = \# \text{ birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \text{ & } j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = E\left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \text{ linearity of expectation} \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365} \]
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \& j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = E\left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \quad \text{linearity of expectation} \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365} = \frac{n \cdot (n-1)}{2}, \frac{1}{365} \]
**INDICATOR RANDOM VARIABLES**

\[ X = \text{# birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \text{ & } j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = E(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \text{ (linearity of expectation)} \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365} = \frac{n \cdot (n-1)}{2} \cdot \frac{1}{365} = 1 \]

We said we want \( E[X] = 1 \).
**INDICATOR RANDOM VARIABLES**

\[ X = \# \text{ birthday matches among } n \text{ people} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \]

\[ X_{ij} = \begin{cases} 1 & \text{if persons } i \& j \text{ match} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_{ij}) = \frac{1}{365} \]

\[ E(X) = E\left( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{ij}) \text{ linearity of expectation} \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365} = \frac{n \cdot (n-1)}{2} \cdot \frac{1}{365} = 1 \Rightarrow n \approx 28 \]