TESTING FOR A DISEASE

- Suppose 1% of the population has a disease
- there is a diagnostic test, that finds it 80% of the time (assuming the subject has it)
- the test also produces false positives, at a rate of 9.6% (you're fine, but the test says you're not)

If someone tests positive,
what are the odds that they have the disease?

4 events

	1% Have disease	99% Don't have disease
Test ☹	80%	9.6%
Test ☺	20%	90.4%

$$P(\text{disease} \mid \text{test ☹}) = \frac{P(\text{disease} \cap \text{test ☹})}{P(\text{test ☹})} = \frac{P(\text{test ☹} \mid \text{disease}) \cdot P(\text{disease})}{P(\text{test ☹})}$$

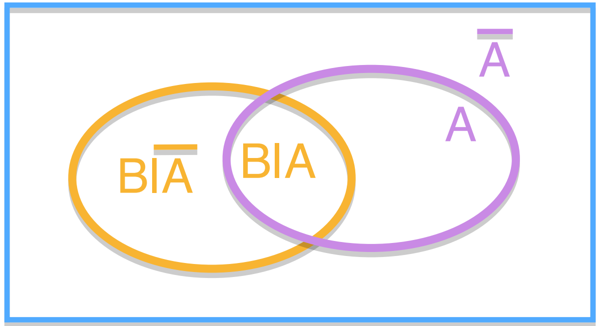
$$P(\text{test ☹}) = \begin{cases} P(\text{test ☹} \mid \text{disease}) \cdot P(\text{disease}) = 0.8 \cdot 0.01 = 0.008 \ (0.8\%) \\ + \\ P(\text{test ☹} \mid \text{no disease}) \cdot P(\text{no disease}) = 0.096 \cdot 0.99 \simeq 0.095 \ (9.5\%) \end{cases}$$

$$P(\text{disease} \mid \text{test ☹}) = \frac{0.008}{0.008 + 0.095} \sim 7.8\%$$

Bayes theorem

$$P(A \cap B) = P(A|B) \cdot P(B)$$
$$= P(B|A) \cdot P(A)$$

$$P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)}$$



$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})}$$

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B|A)\cdot P(A) + P(B|\bar{A})\cdot P(\bar{A})}$$

A: have disease

B: test :(

1% A	99% $\bar{A}$
$B|A$ = 80%	$B|\bar{A}$ = 9.6%

$P(A) = 0.01$

$P(\bar{A}) = 0.99$

$P(B|A) = 0.8$

$P(B|\bar{A}) = 0.096$

$$P(A|B) = \frac{0.8\cdot 0.01}{0.8\cdot 0.01 + 0.096\cdot 0.99}$$

RANDOM VARIABLES

A quote from your textbook:

"A random variable is neither random nor variable"

We have been using random variables, implicitly.

ex: roll 2 dice, examine probability that sum $= k$, or $=$ even.

↳ define random variable X : sum of two dice rolls.

So, $X[(1,2)] = 3$

$X[(5,5)] = 10$

↳ define random variable Y : parity of two dice rolls.

So, $Y[(1,2)] = 1$ or $= 13$

$Y[(5,5)] = 0$ or $= -22$

$\left.\begin{matrix} =13 \\ =-22 \end{matrix}\right\}$ (arbitrary)

Think of a r.v. as a function,

mapping sample space to $\underbrace{\text{whatever you like}}_{\text{usually a number}}$

Then we can express questions neatly: $P(X<3) = \frac{1}{36}$ ← sum (2 dice)

$P(Y=1) = \frac{1}{2}$ ← parity

We can also eliminate absurd events, e.g., $P(X=13)=0$

Section 33 discusses in more detail: not required for our class

EXPECTATION : the very basics

As mentioned, a r.v. X can have several values.

It is based on outcomes that result from a random process.

So we don't know what value it will have.

But we can expect it to have some value

Expected value = weighted average

(e.g. average parity for unfair coin $= \frac{1}{2}$; weighted average $\neq \frac{1}{2}$)

Expected value = weighted average

$$E(X) = \sum_{*} y \cdot P(X=y)$$ see example 34.3 in book (mainly p.237)

$\ast$ over all possible values y, compatible with X.
(however we only care about finitely many)

$$E(X) = \sum_{s \in S} \left[X(s) \cdot P(s) \right]$$

over all samples that define X. } a finite number

e.g. roll 1 die. X = number observed.

$$E(X) = \sum_{i=1}^{6} X(i) \cdot P(i)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \frac{1}{6} \cdot (1+2+3+4+5+6) = 3.5$$

$E(X) = \sum y \cdot P(X=y)$

typically use this when you have a good way of aggregating outcomes

example: roll 2 dice. $X = |\text{difference between the } 2|$

possible values of X → 0 1 2 3 4 5

outcomes supporting value → 6 5·2 4·2 3·2 2·2 1·2

(for probability, divide by 36)

$$E(x) = \frac{0 + 10 + 16 + 18 + 16 + 10}{36}$$

(~ 1.944)

EXPECTATION : PROPERTIES

LINEARITY OF EXPECTATION (important)

$c_1, c_2 \in \mathbb{R}$ $\qquad E(c_1 X + c_2 Y) = c_1 \cdot E(X) + c_2 \cdot E(Y)$

Generally, $E(c_1 X_1 + c_2 X_2 + \dots + c_n X_n) = c_1 E(X_1) + c_2 E(X_2) + \dots c_n E(X_n)$

$E\left(\sum c_i X_i\right) = \sum c_i E(X_i)$ → Does NOT assume independence

Independence: $P(X=a \;\&\; Y=b) = P(X=a) \cdot P(Y=b)$
for all a, b ...

(def. 33.6)

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2 dice, A, B. $\quad X =$ result of A. $\quad Y =$ result of B. $\quad Z = X + Y$

$E(Z) = E(X+Y) = E(X) + E(Y) = 2 \cdot 3.5 = 7$

EXPECTATION : PROPERTIES

$$E(X+Y) = E(X) + E(Y)$$

Linearity of expectation doesn't assume independence

but $E(X \cdot Y) \neq E(X) \cdot E(Y)$ in general.

If X & Y are independent, then $E(X \cdot Y) = E(X) \cdot E(Y)$

However, $E(X \cdot Y) = E(X) \cdot E(Y)$ does NOT imply X & Y are independent.

(see example 34.15)

- We are skipping the proofs of most statements in this section.

- You are not required to study these, but it would probably be beneficial.

- We are also skipping **variance**, which is an important concept to learn independently.

INDICATOR RANDOM VARIABLES

(taking value 0 or 1)

We already saw this: Y: parity of rolling one die.

Another example: flip a coin 10 times.

X = # times we see pattern HT

$$E(X) = ?$$

Try solving this "directly"

INDICATOR RANDOM VARIABLES

flip a coin 10 times. X = # times we see pattern HT

HT could appear at flips 1 & 2, or 2 & 3, ..., or 9 & 10

Define r.v. $X_i = \begin{cases} 1 & \text{if flips } i \text{ \& } i+1 \text{ produce HT} \\ 0 & \text{otherwise} \end{cases}$

Notice X_1 & X_2 are not independent. $P(X_i=1)=\frac{1}{4}$

$P(X_1 \wedge X_2) = 0$

$$X = X_1 + X_2 + \cdots + X_9$$

$$E(X) = E(X_1 + X_2 + \cdots + X_9)$$

linearity of expectation

$$= E(X_1) + E(X_2) + \cdots E(X_9)$$

$$E(X_i) = \cancel{0 \cdot P(X_i=0) + 1} \cdot \underbrace{P(X_i = 1)} = \frac{1}{4}$$

$$= 9 \cdot \frac{1}{4}$$

(p.240)