# THE MONTY HALL PROBLEM

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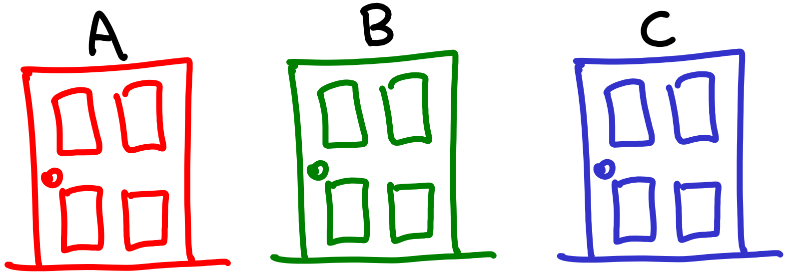

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The other 2 hide goats.

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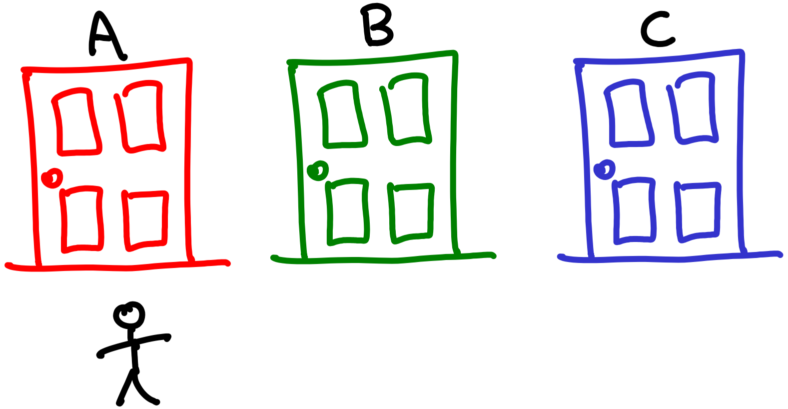

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You get to pick a door. You randomly pick A.

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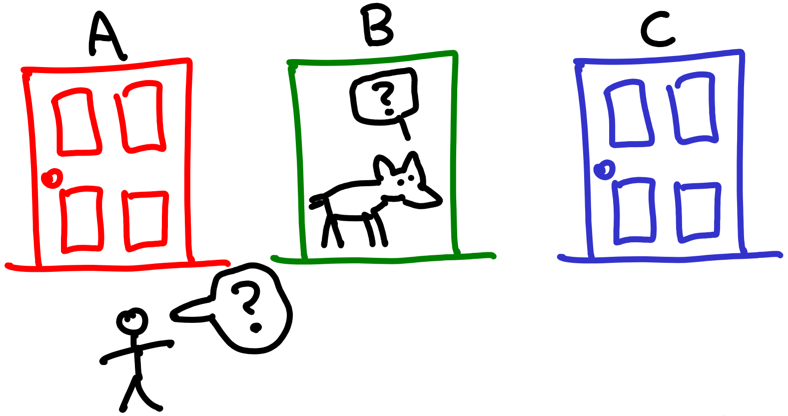

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You get to pick a door. You randomly pick A.

Then a door you didn't pick is opened (say, B) revealing a goat.

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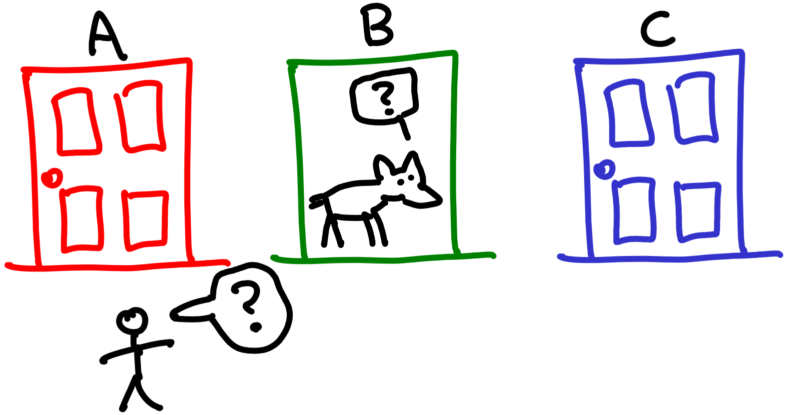


1 of these 3 doors hides a car.
The other 2 hide goats.

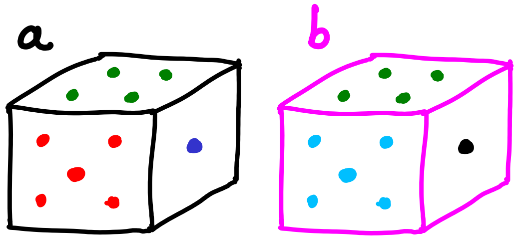
You get to pick a door. You randomly pick A.

Then a door you didn't pick is opened (say, B) revealing a goat.

You're given the choice: KEEP YOUR DOOR OR SWITCH ?

# CONDITIONAL PROBABILITY

Roll 2 dice ...



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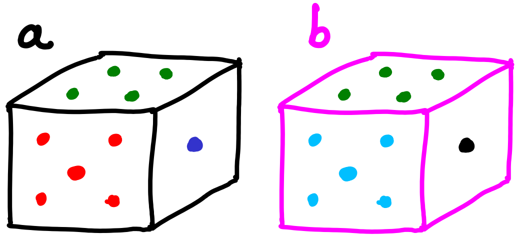

$P(A) = P(\text{sum} = 8)$

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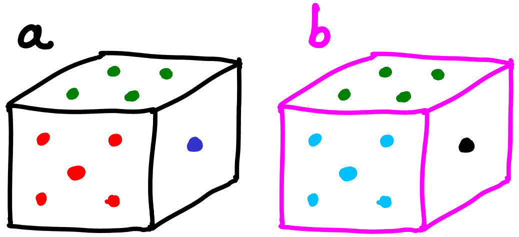

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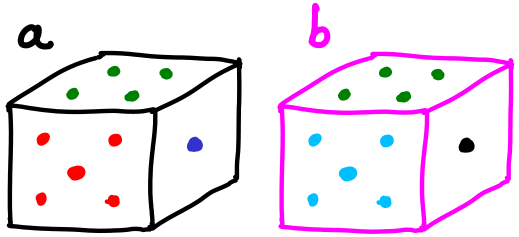

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$P(A|B)$

↑

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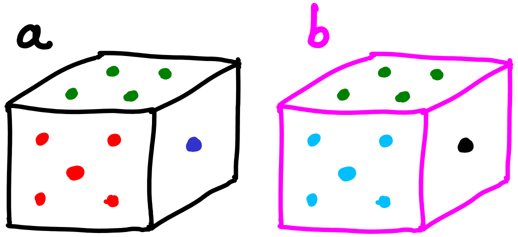

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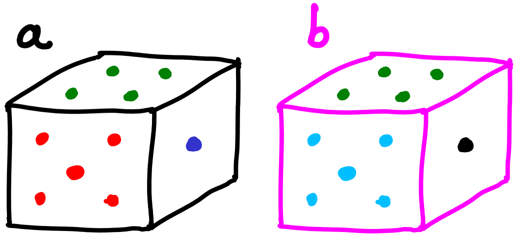

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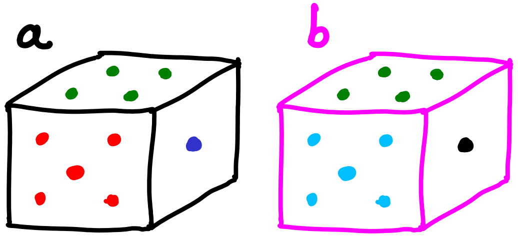

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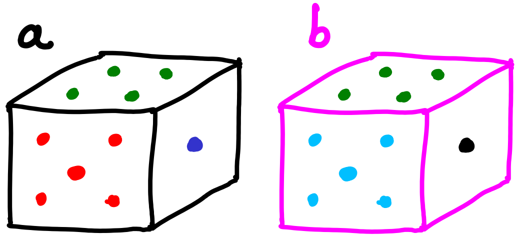

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$P(A \mid B) = \frac{3}{9}$

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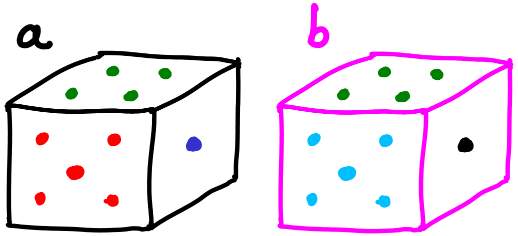

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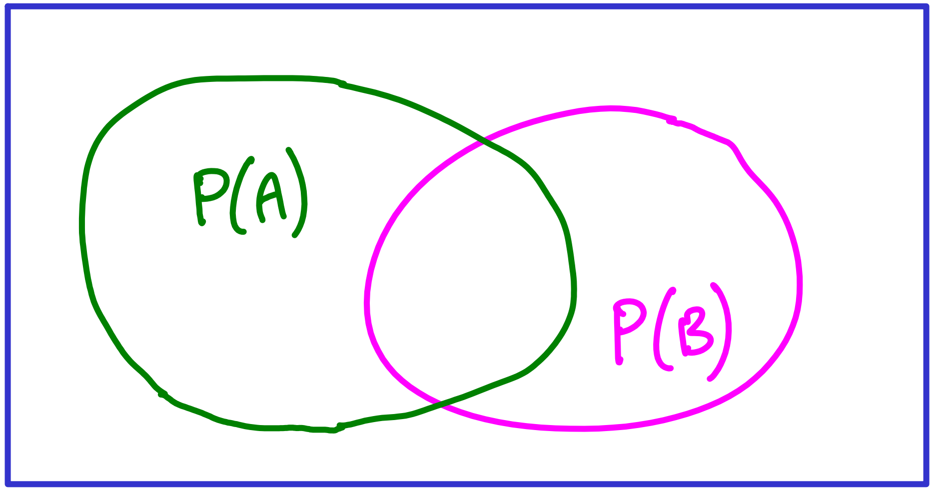
$\neq P(A)$

in this example

P(A) = P(sum = 8)

P(B) = P(both are even)



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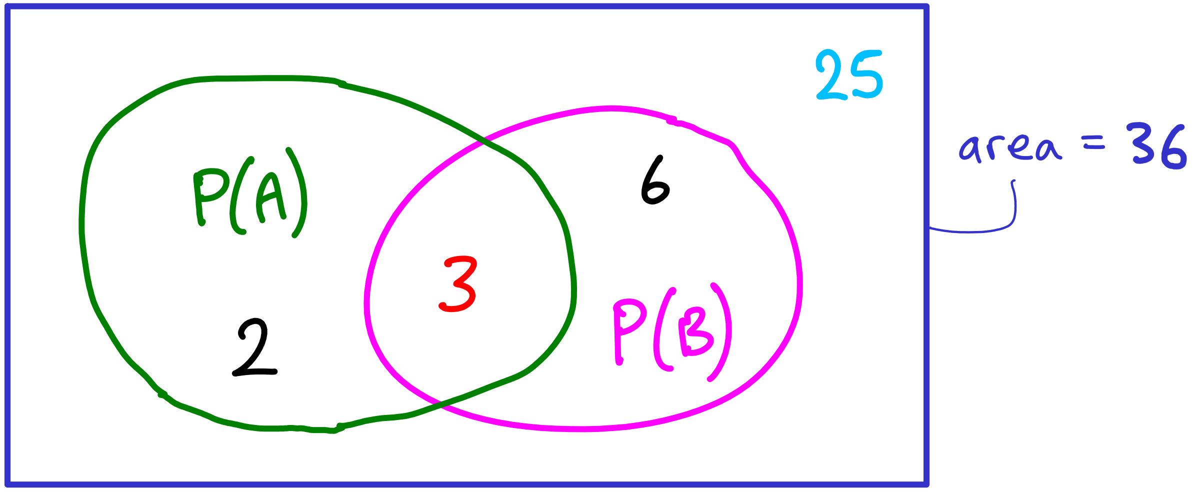

(if you want the universe to have area = 1, divide all by 36.)

$P(A) = P(\text{sum} = 8)$

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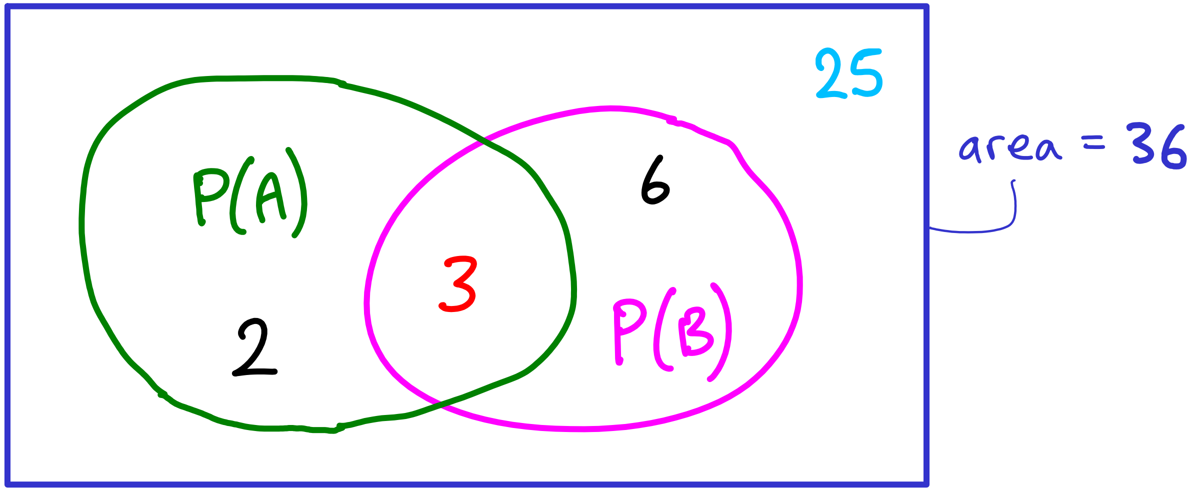
$$P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36}$$



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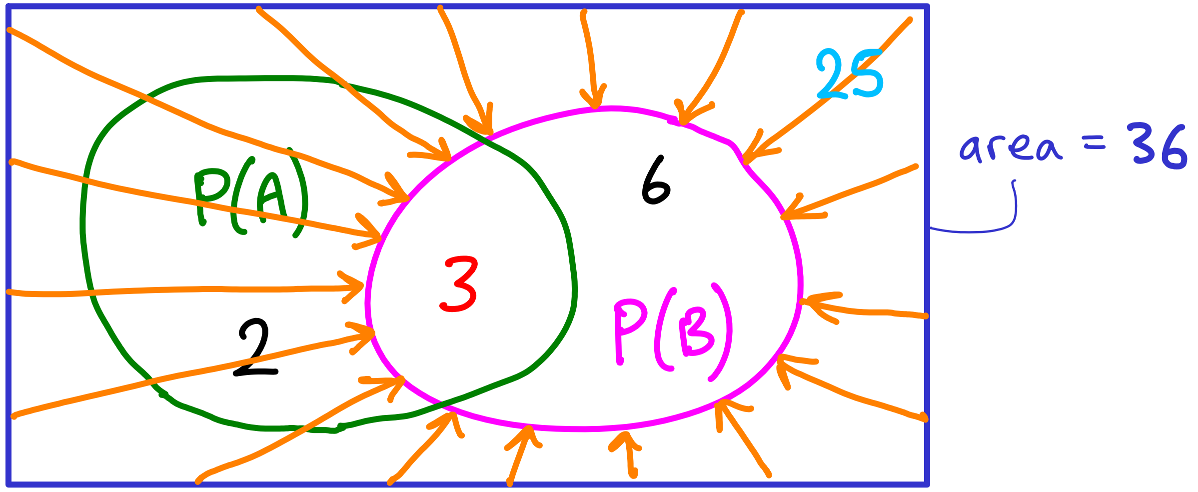

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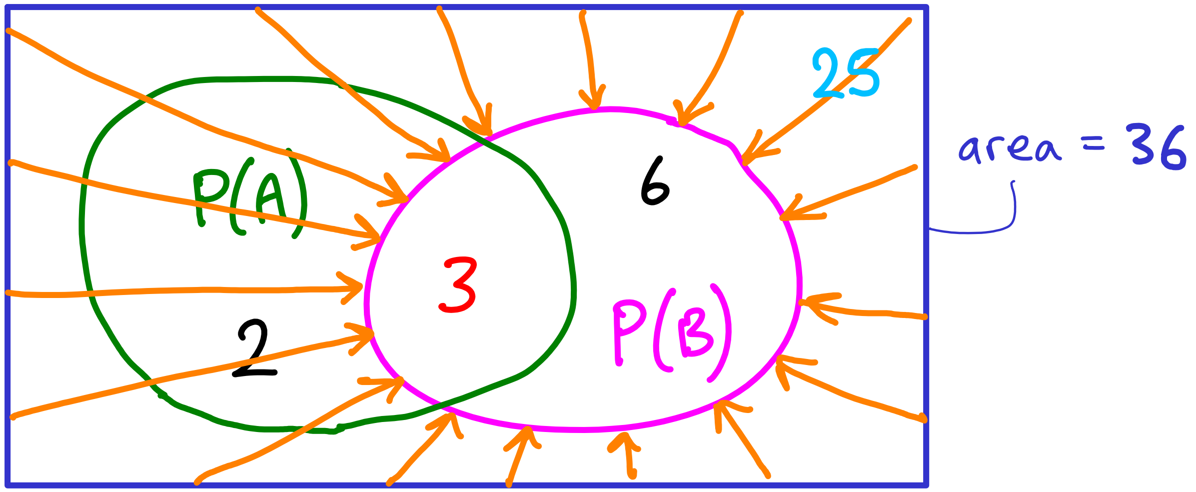

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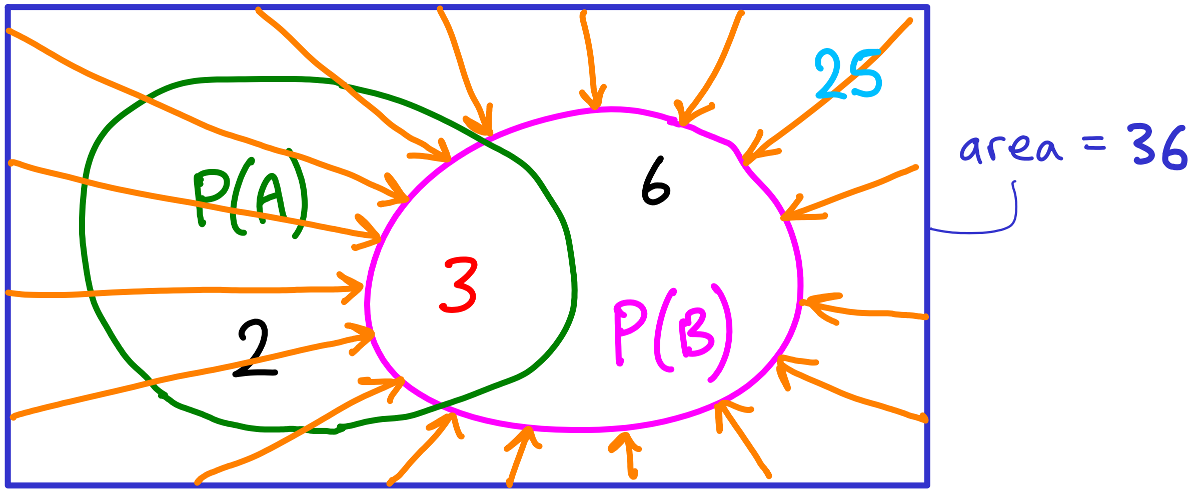

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$$P(A|B) = \ ?$$

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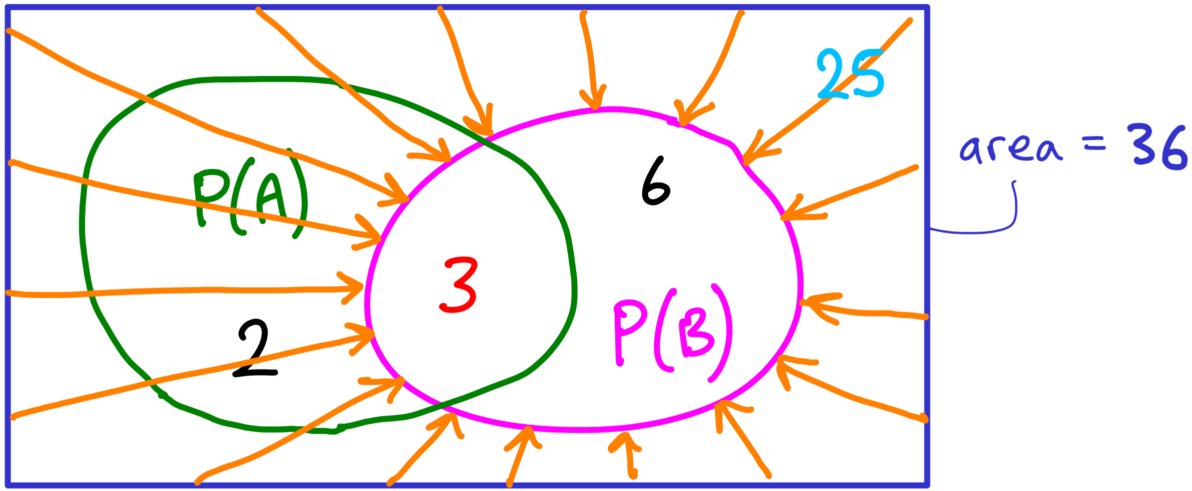

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$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{3+6}$$

another example

Flip a coin 5 times. $P(\text{1st flip} = T) = \frac{1}{2}$

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$$P(3 \cdot H) = \frac{\binom{5}{3}}{2^5}$$

→ Ways to choose 3 positions for H.

→ Sample space.

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$$P(3\cdot H) = \frac{\binom{5}{3}}{2^5} \begin{matrix} \rightarrow \text{Ways to choose 3 positions for H.} \\ \rightarrow \text{Sample space.} \end{matrix} = \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}$$

$$P[(\text{1st flip} = T) \cap (3\cdot H)] : \left.\begin{matrix} T\ HHHT \\ T\ HHTH \\ T\ HTHH \\ T\ THHH \end{matrix}\right\} \frac{4}{32}$$

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→ Ways to choose 3 positions for H.

→ Sample space.

$$= \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}$$

$$\rightarrow \frac{2/16}{5/16} = \boxed{\frac{2}{5}}$$

$P[(\text{1st flip} = T) \cap (3\cdot H)]$ :

T HHHT
T HHTH
T HTHH
T THHH

$$\left.\begin{matrix}\text{T HHHT}\\ \text{T HHTH}\\ \text{T HTHH}\\ \text{T THHH}\end{matrix}\right\} \frac{4}{32}$$

OR $\frac{1}{2} \cdot \frac{\binom{4}{3}}{2^4} = \frac{1}{8}$

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But what if you know that 3 of the 5 flips were H ?

think of having a biased coin : 60-40 vs 50-50

then the first flip has 40% for T $\rightarrow \frac{2}{5}$

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

$= 1 - P(\text{all } k \text{ have distinct birthdays})$

$\hookrightarrow P(\text{2nd person has different bday than 1st}) = \dfrac{364}{365} = P(A)$

$\cdot\ P(\underbrace{\text{3rd} \ldots \quad \ldots\ldots \quad \ldots \text{ 1st \& 2nd}}_{B}) \rightarrow \underbrace{\dfrac{363}{365} = P(B|A)}_{\text{assuming 1st \& 2nd differ}}$

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$\cdot\ P($4th $\ldots \ \ldots \ \ldots$ $(1-3)) \rightarrow \frac{362}{365} = P(C | (A \cap B))$

etc

$= P(A) \cap P(B|A) \cap P(C|(A \cap B)) \ldots$

Flip a coin x3 : $P(3\text{rd} = T \mid 1\text{st} = H) = ?$

Flip a coin x3 : $P(3rd = T \mid 1st = H) =$

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$$= \frac{P[(3\text{rd} = T) \cap (1\text{st} = H)]}{P(1\text{st} = H)} = \frac{\frac{2 \longrightarrow \text{\# outcomes}}{8 \longrightarrow \text{sample space}}}{1/2} = \frac{1}{2}$$

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Notice $P(3\text{rd} = T) = \frac{1}{2}$

Flip a coin x3 : $P(\text{3rd} = T \mid \text{1st} = H) =$

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$2 \longrightarrow$ # outcomes

$8 \longrightarrow$ sample space

Notice $P(\text{3rd} = T) = \frac{1}{2}$ so knowledge of $(\text{1st} = H)$ was useless.

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A & B are *independent* if $P(A) = P(A \mid B)$

if $P(B) = P(B \mid A)$ [equivalent]

$$\underbrace{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}_{\text{always}}$$

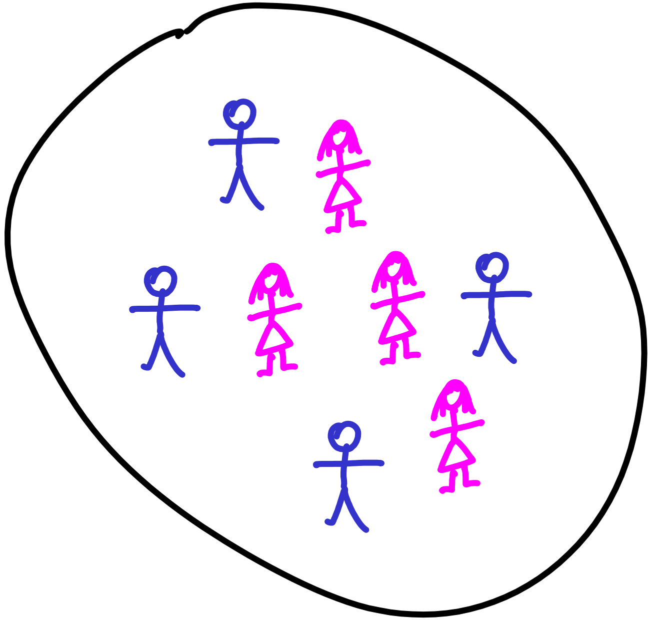
$$\underbrace{\underbrace{P(A|B) = \frac{P(A \cap B)}{P(B)}}_{\text{always}} = P(A)}_{\text{if } A \text{ \& } B \text{ independent}}$$

$$\underbrace{\underbrace{P(A|B) = \frac{P(A \cap B)}{P(B)}}_{\text{always}} = P(A)}_{\text{if } A \text{ \& } B \text{ independent}}$$

alternate definition: A & B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

# another example



4 boys & 4 girls

Select 2 people from this group

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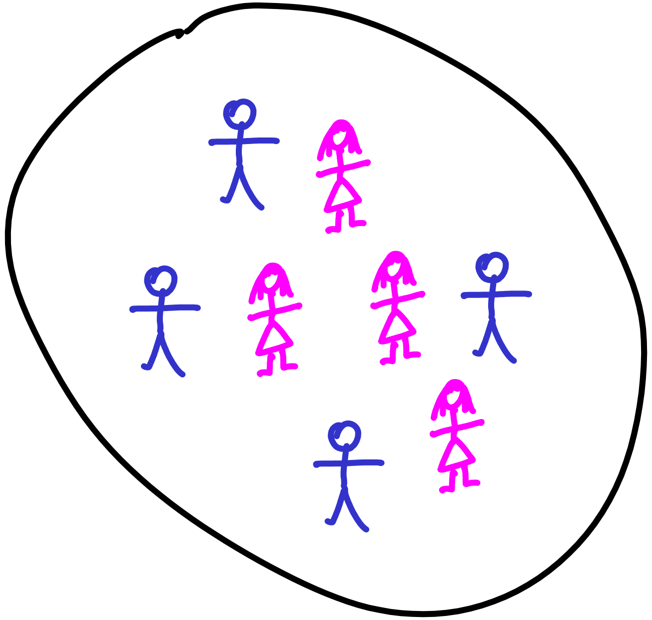

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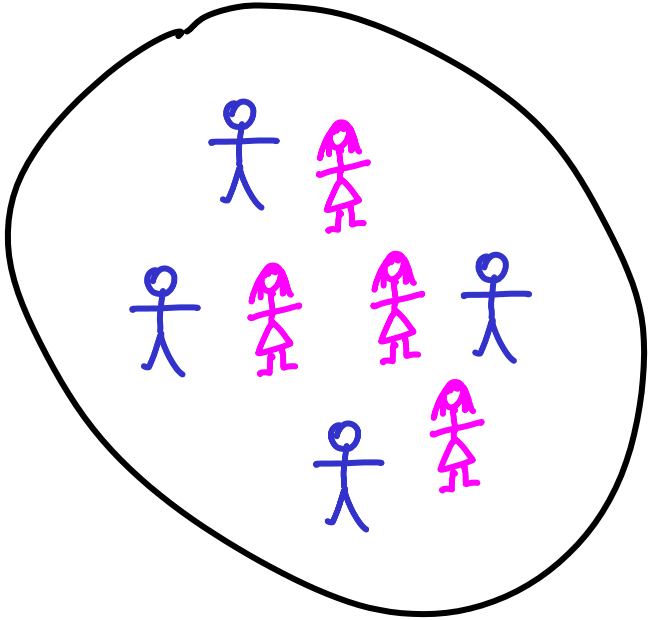

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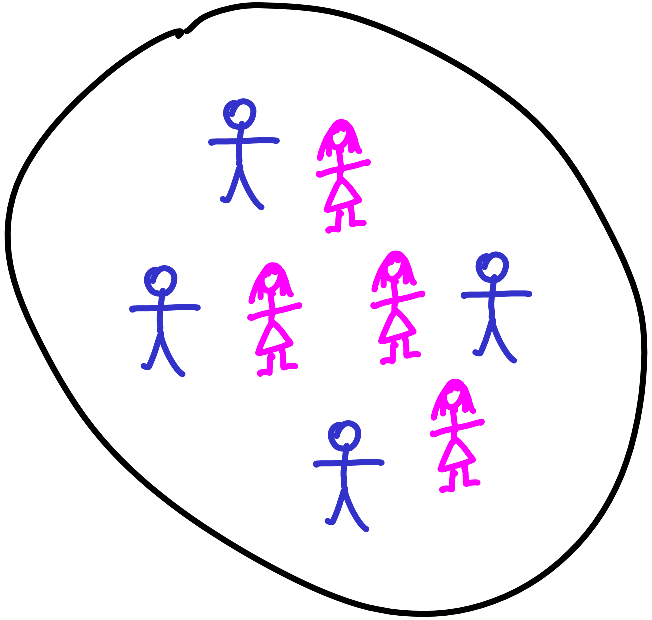

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$P(A) = \frac{4}{8}$

$P(B) =$

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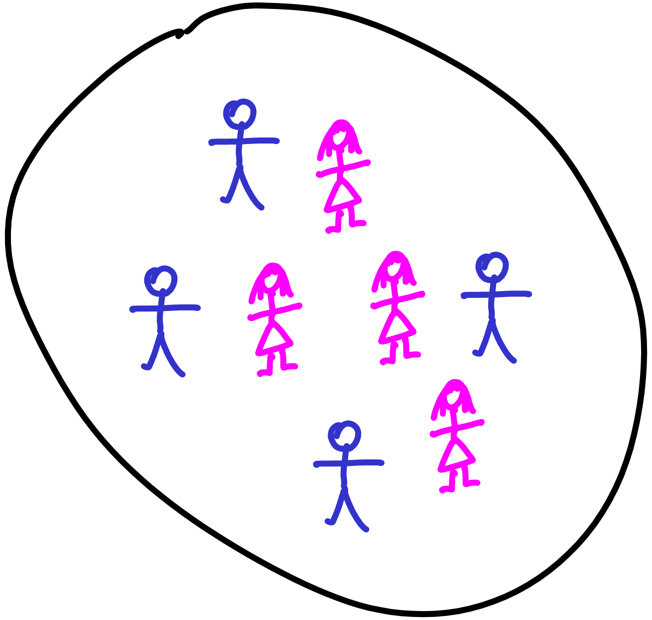

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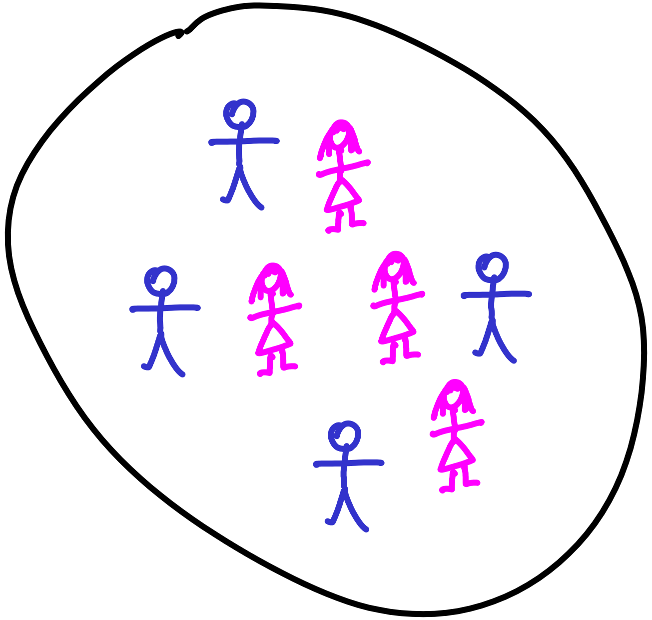
$P(A) = \frac{4}{8}$

$P(B) = \frac{1}{2}$ (by symmetry)

(or : sample space = 8·7
& for each girl=2nd, #outcomes =7)

## another example



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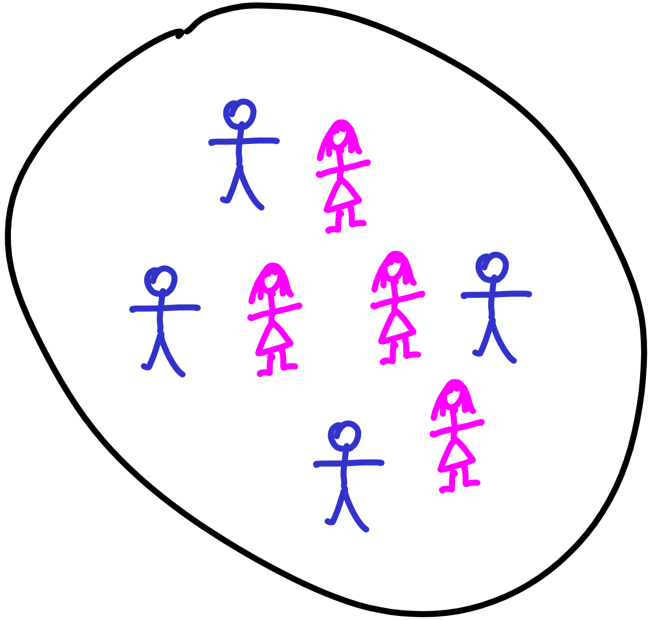
(or: sample space $= 8 \cdot 7$
& for each girl=2nd, #outcomes = 7)

$P(B|A) =$

# another example



4 boys & 4 girls

Select 2 people from this group

A: 1st person is a girl

B: 2nd person is a girl

dependent

$P(A) = \frac{4}{8}$

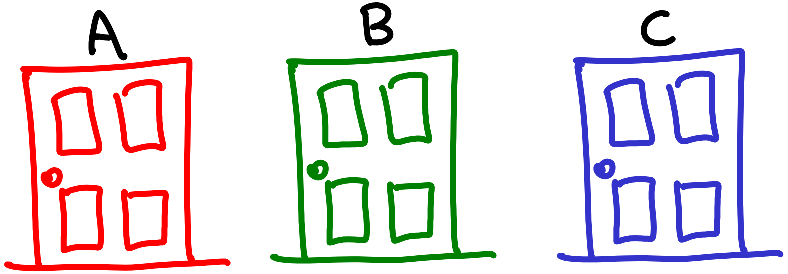
$P(B) = \frac{1}{2}$ (by symmetry)

(or: sample space $= 8 \cdot 7$
& for each girl=2nd, #outcomes =7)

$\neq$

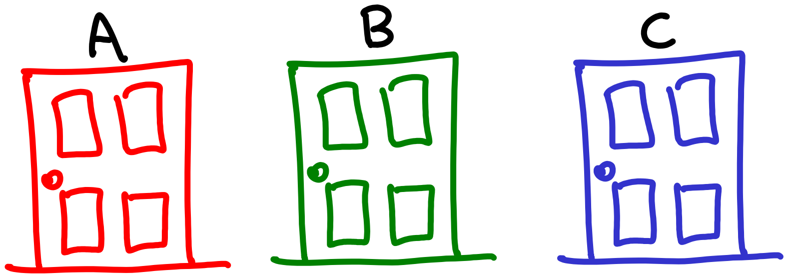
$P(B|A) = \frac{3}{7}$

# BACK TO MONTY HALL



$$P(\text{car} @ A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

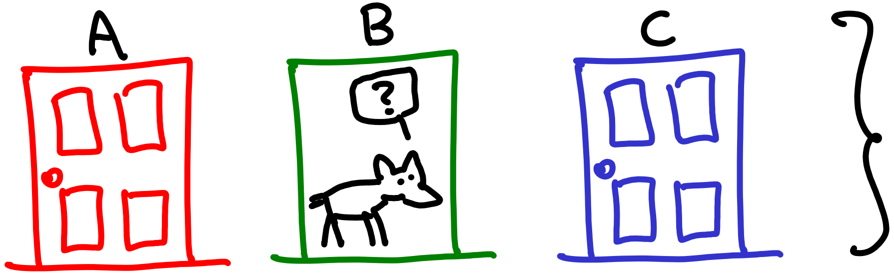
# BACK TO MONTY HALL



$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$

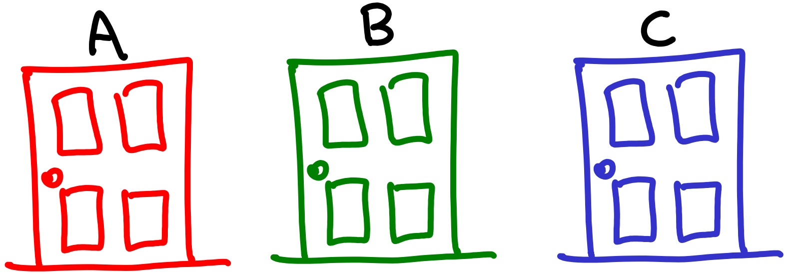
Now we know the car is not at B
What is the probability it's at A?

↓



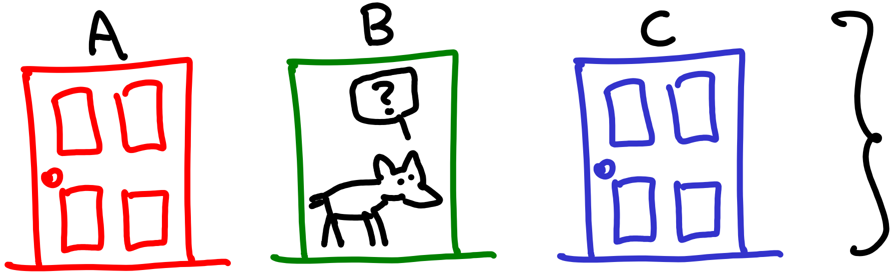
$P(A|\overline{B})$

# BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

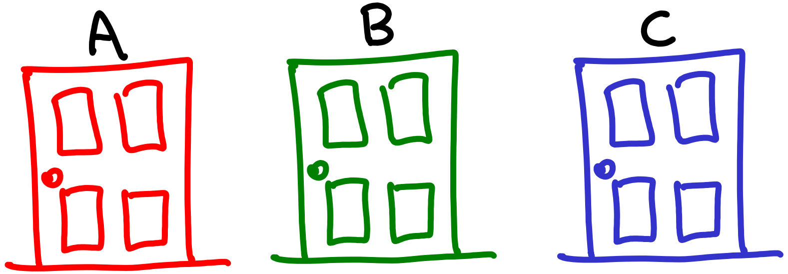
Now we know the car is not at B
What is the probability it's at A?



↓

$$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})}$$

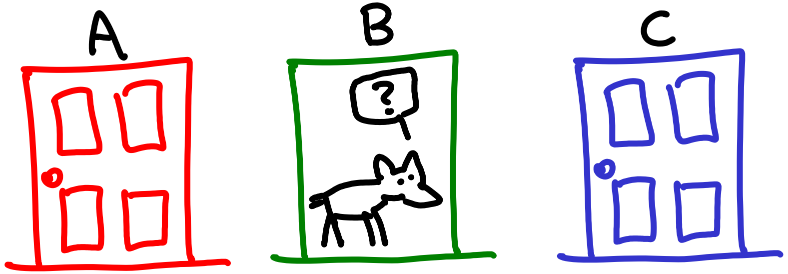
# BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(c)$$

Now we know the car is not at B
What is the probability it's at A?

↓

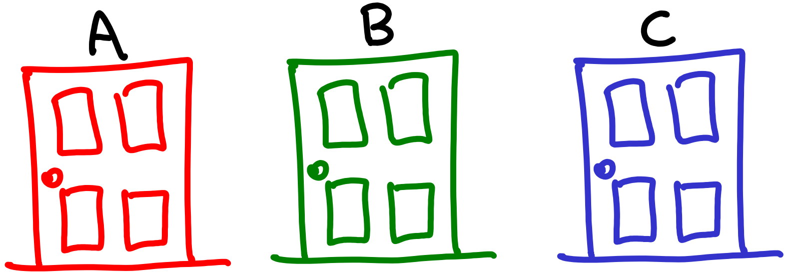


$$P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

...because $P(\bar{B}|A) = \frac{P(A \cap \bar{B})}{P(A)}$ ...

$$= \frac{P(\bar{B}|A) \cdot P(A)}{P(\bar{B})}$$

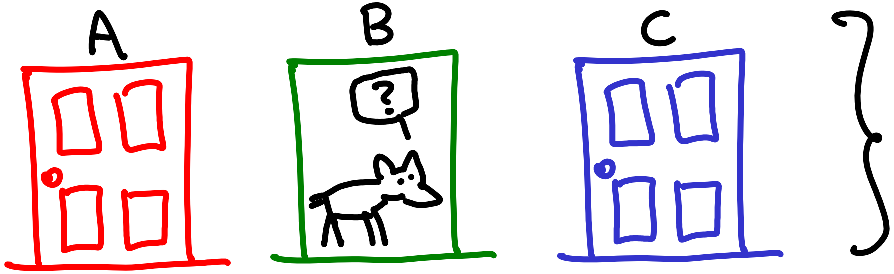
# BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

Now we know the car is not at B
What is the probability it's at A?

↓

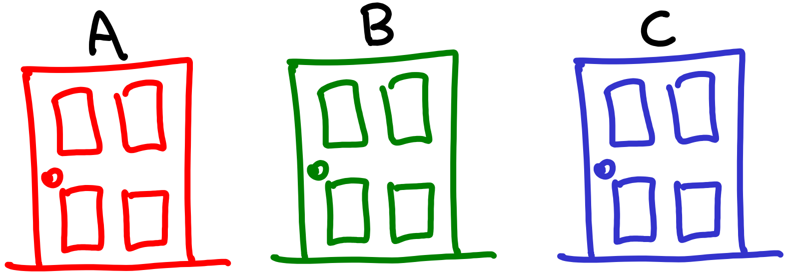


$$P(A \mid \bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

...because $P(\bar{B} \mid A) = \frac{P(A \cap \bar{B})}{P(A)}$ ...

$$= \frac{P(\bar{B} \mid A) \cdot P(A)}{P(\bar{B})} = \frac{1 \cdot P(A)}{P(\bar{B})}$$

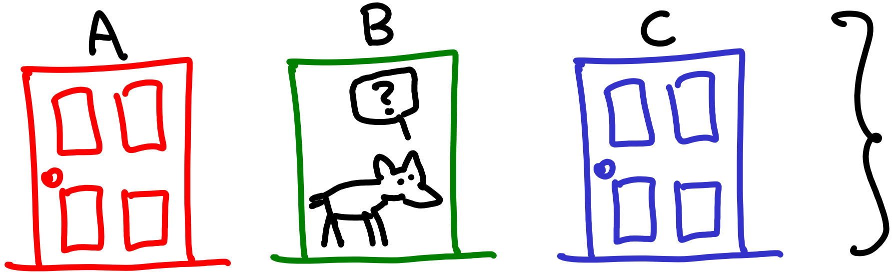
# BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

Now we know the car is not at B
What is the probability it's at A?

↓

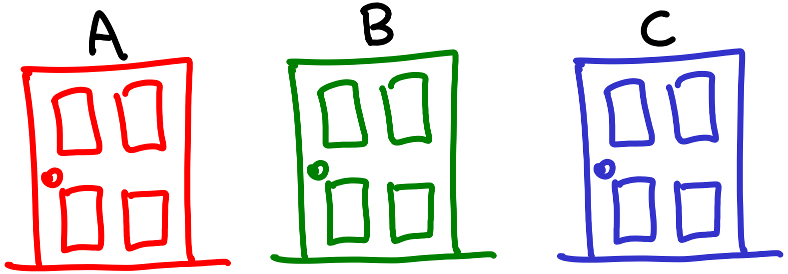


$$P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

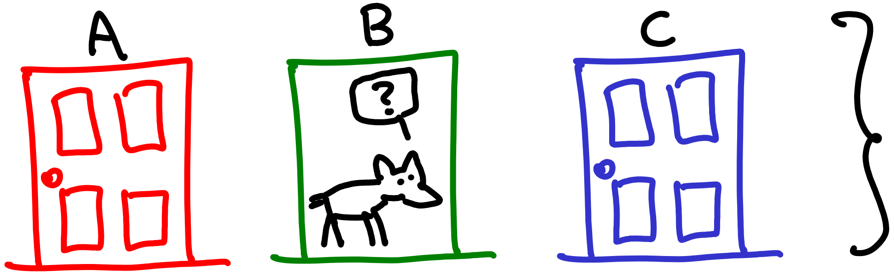
...because $P(\bar{B}|A) = \frac{P(A \cap \bar{B})}{P(A)}$ ...

$$= \frac{P(\bar{B}|A) \cdot P(A)}{P(\bar{B})} = \frac{1 \cdot P(A)}{P(\bar{B})} = \frac{1/3}{2/3} = \frac{1}{2}$$

# BACK TO MONTY HALL

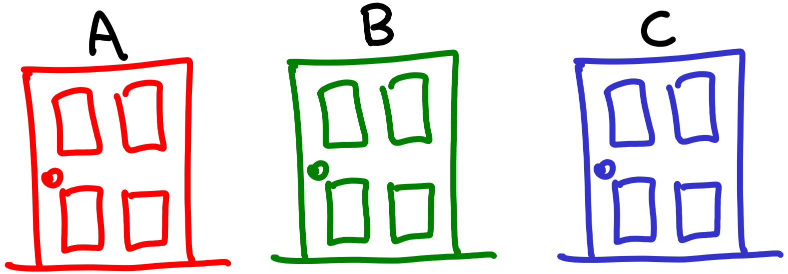


$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$

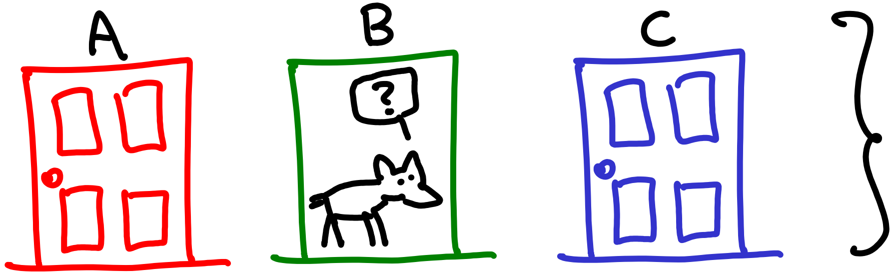


$P(A|\overline{B})$ ← back up

# BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

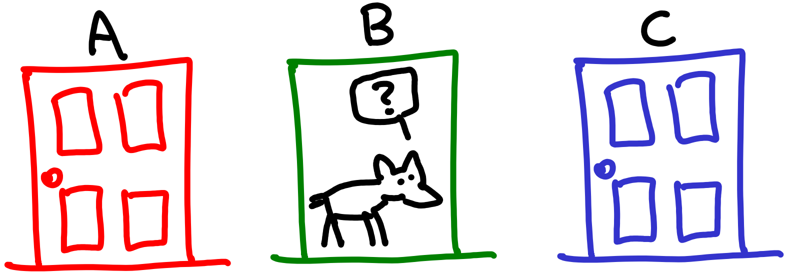


$\xcancel{P(A \mid \overline{B})}$ ← back up

What we actually want is

$$P(A \mid (\underbrace{\text{door B was opened}}_{\text{not} = \overline{B}} \cap \underbrace{\text{we chose A}}_{\text{extra info}}))$$

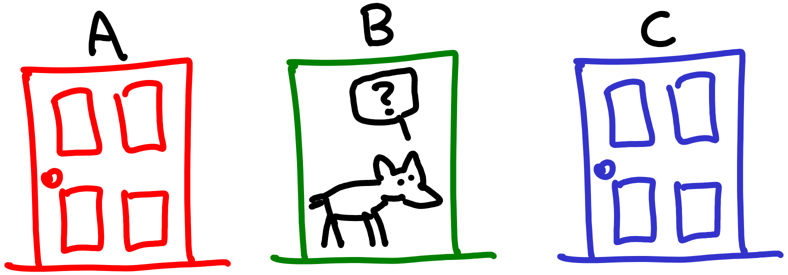
# BACK TO MONTY HALL : intuition



$$P(\text{car} @ A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

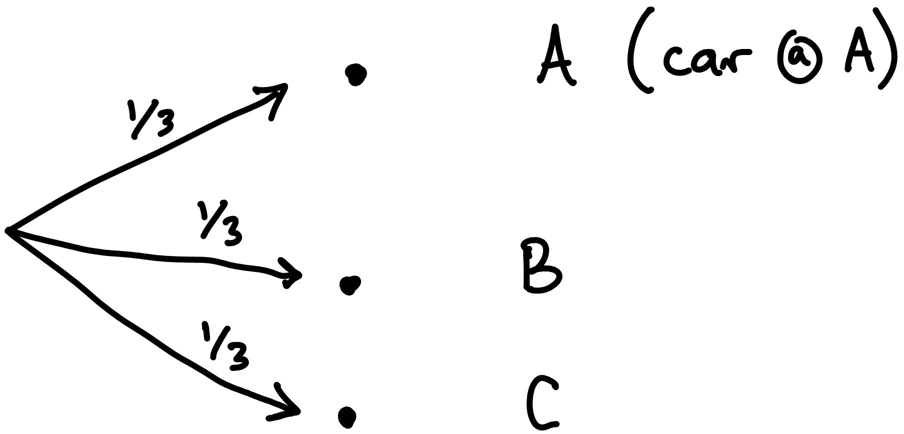
w.l.o.g. guess A
& B is shown.

# BACK TO MONTY HALL : intuition

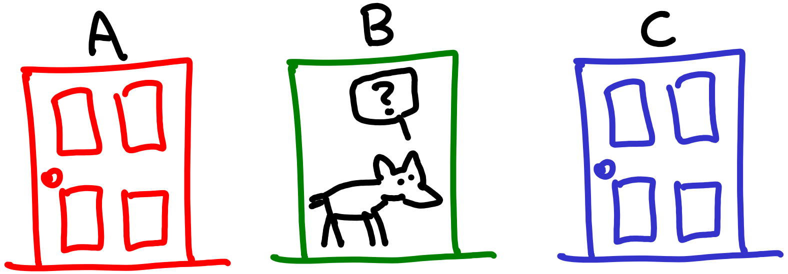


$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$

w.l.o.g. guess A
& B is shown.

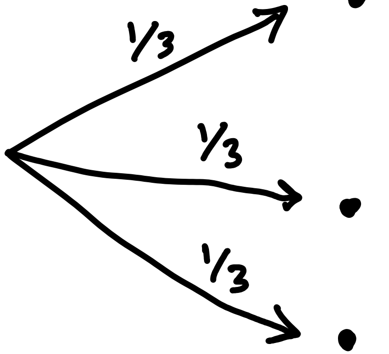


# BACK TO MONTY HALL : intuition



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

w.l.o.g. guess A
& B is shown.

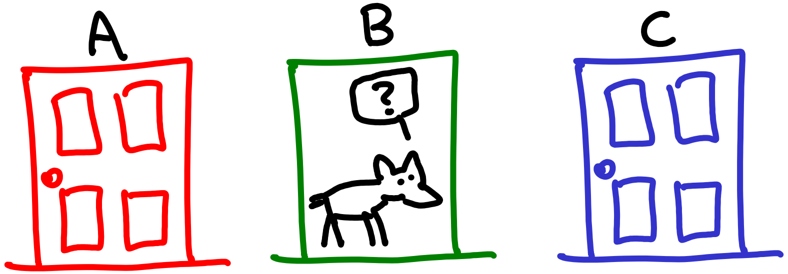


A (car @ A)

B

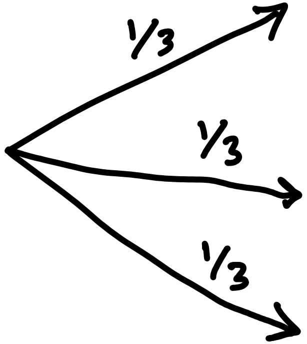
if C, then B revealed → switch to win

# BACK TO MONTY HALL : intuition



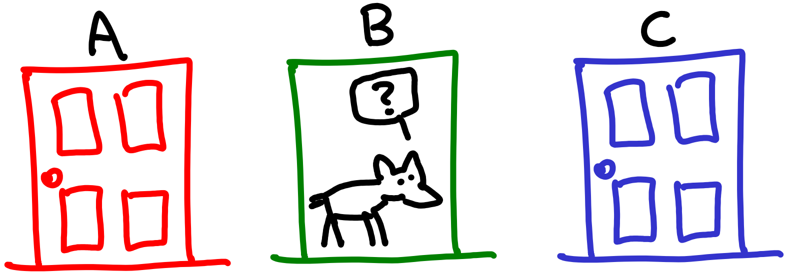
$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

w.l.o.g. guess A
& B is shown.



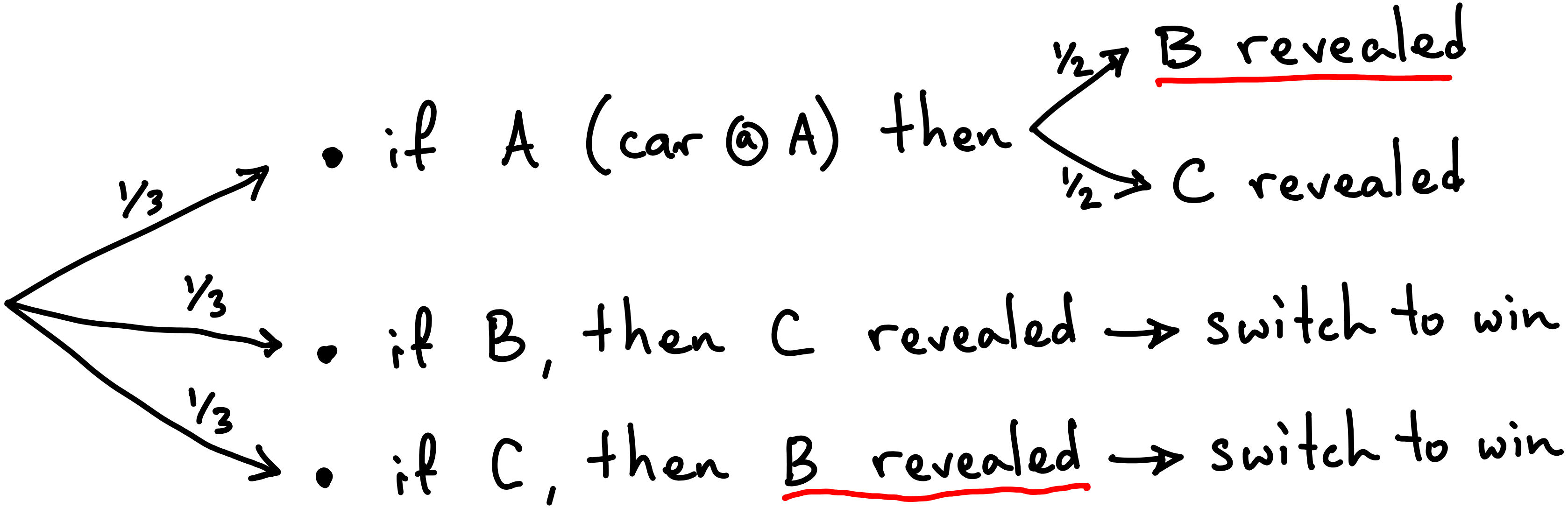
- A (car @ A)
- if B, then C revealed → switch to win
- if C, then B revealed → switch to win

# BACK TO MONTY HALL : intuition



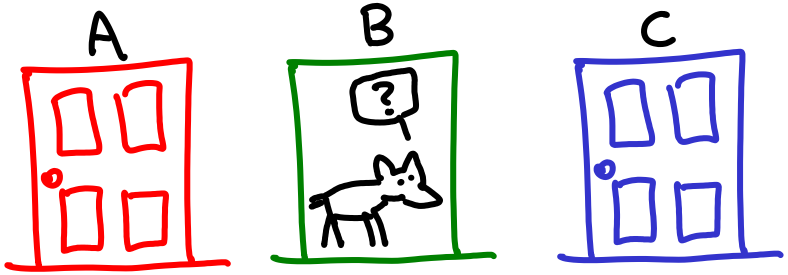
$P(\text{car} @ A) = P(A) = \frac{1}{3} = P(B) = P(C)$

w.l.o.g. guess A
& B is shown.



- if A (car @ A) then
  - B revealed
  - C revealed
- if B, then C revealed → switch to win
- if C, then B revealed → switch to win

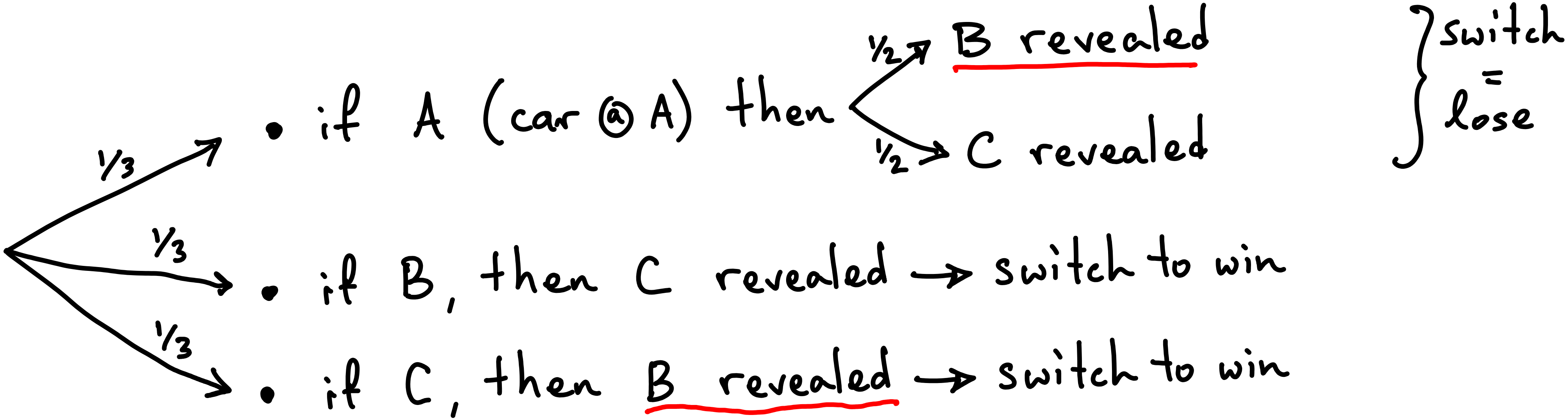
# BACK TO MONTY HALL : intuition



$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$

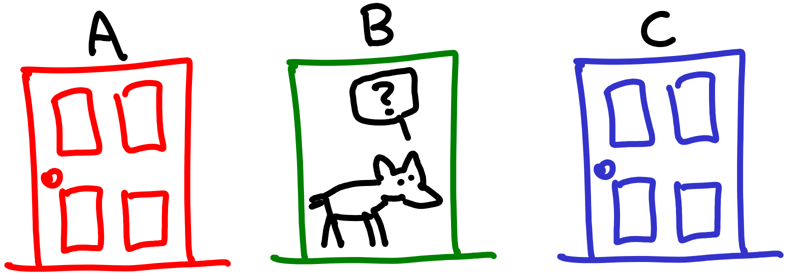
w.l.o.g. guess A
& B is shown.

4 events



- if A (car @ A) then $\xrightarrow{1/2}$ B revealed, $\xrightarrow{1/2}$ C revealed } switch = lose
- if B, then C revealed → switch to win
- if C, then B revealed → switch to win

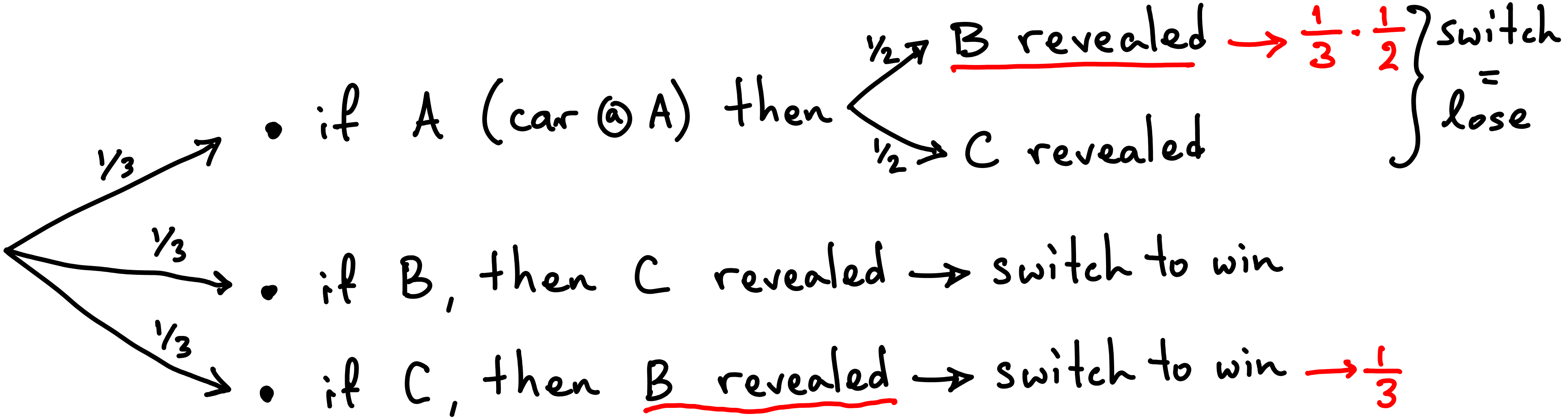
# BACK TO MONTY HALL : intuition



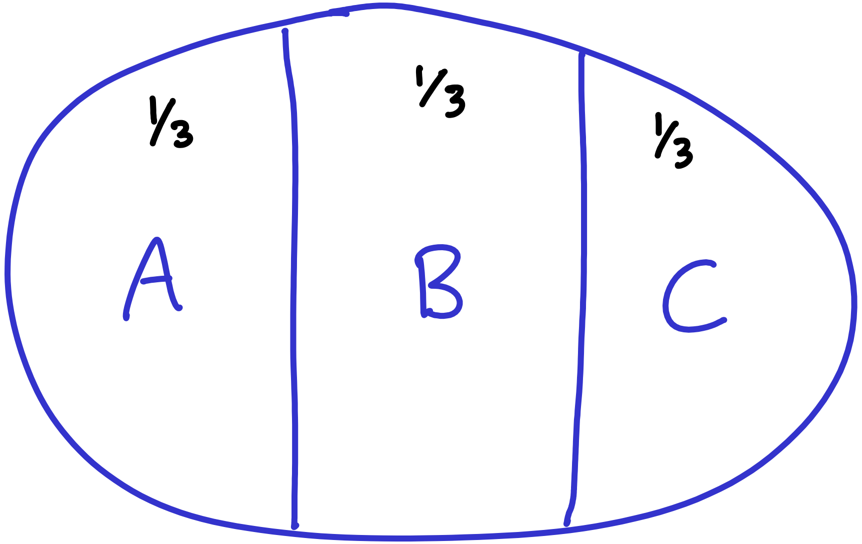
$P(\text{car} \,@\, A) = P(A) = \frac{1}{3} = P(B) = P(C)$

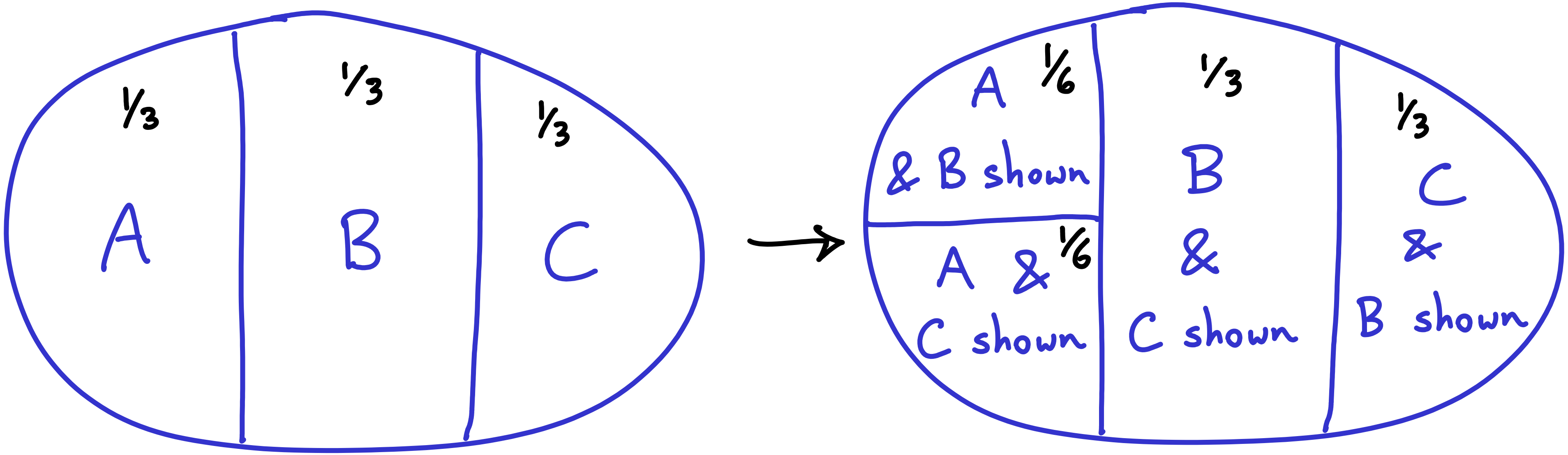
w.l.o.g. guess A
& B is shown.

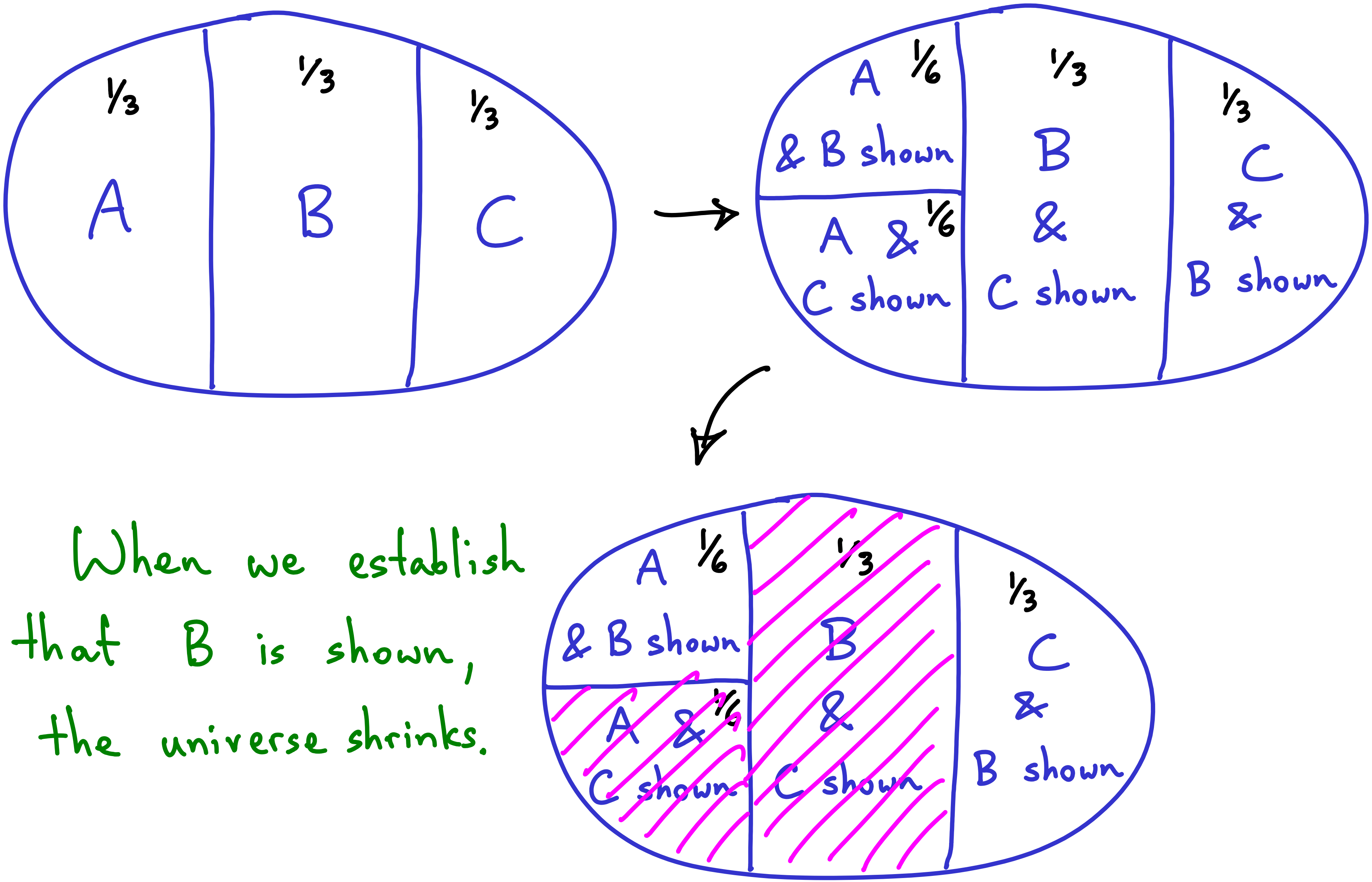
$\underbrace{\text{4 events}}$



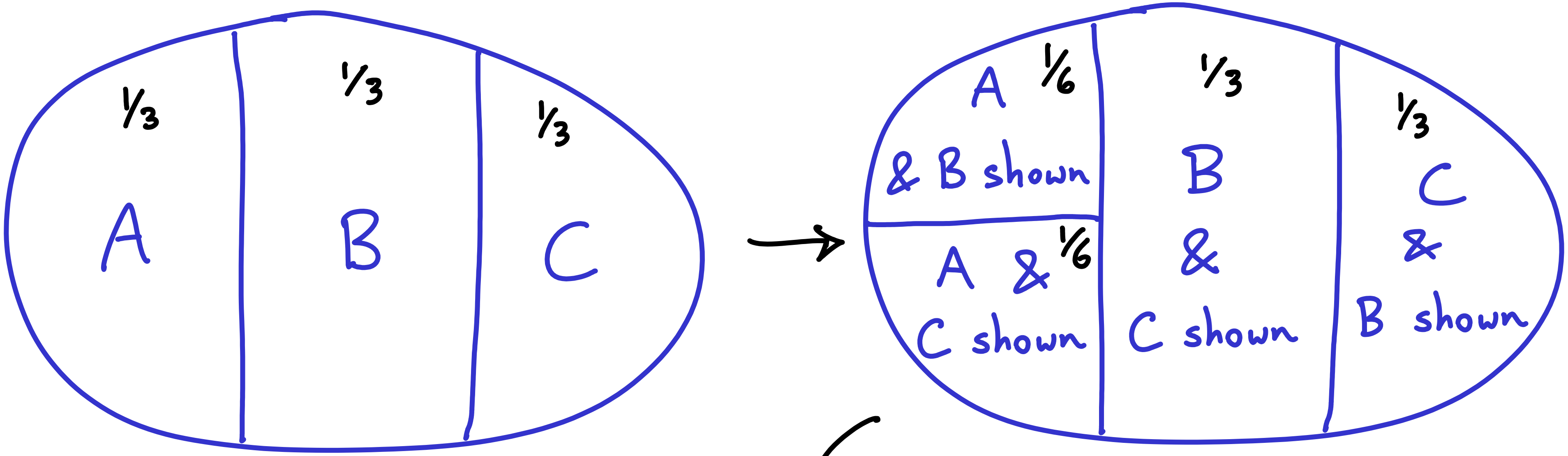
- if A (car @ A) then
  - ½ → B revealed → $\frac{1}{3}\cdot\frac{1}{2}$
  - ½ → C revealed
  - } switch = lose
- if B, then C revealed → switch to win
- if C, then B revealed → switch to win → $\frac{1}{3}$


1/3
1/3
1/3
A
B
C


1/3
A
1/3
B
1/3
C
A & B shown 1/6
A & C shown 1/6
1/3
B & C shown
1/3
C & B shown



When we establish that B is shown, the universe shrinks.



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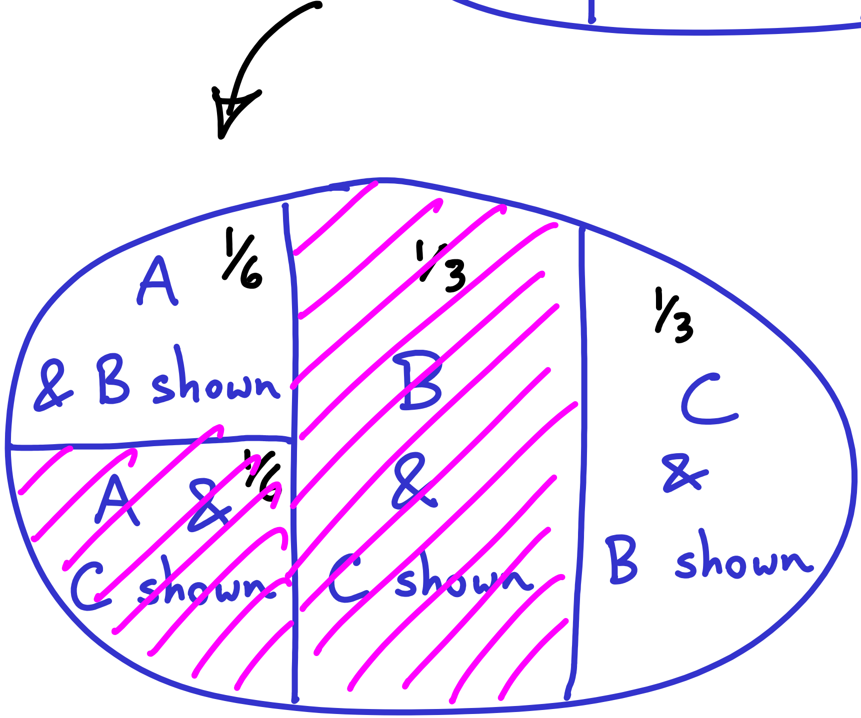
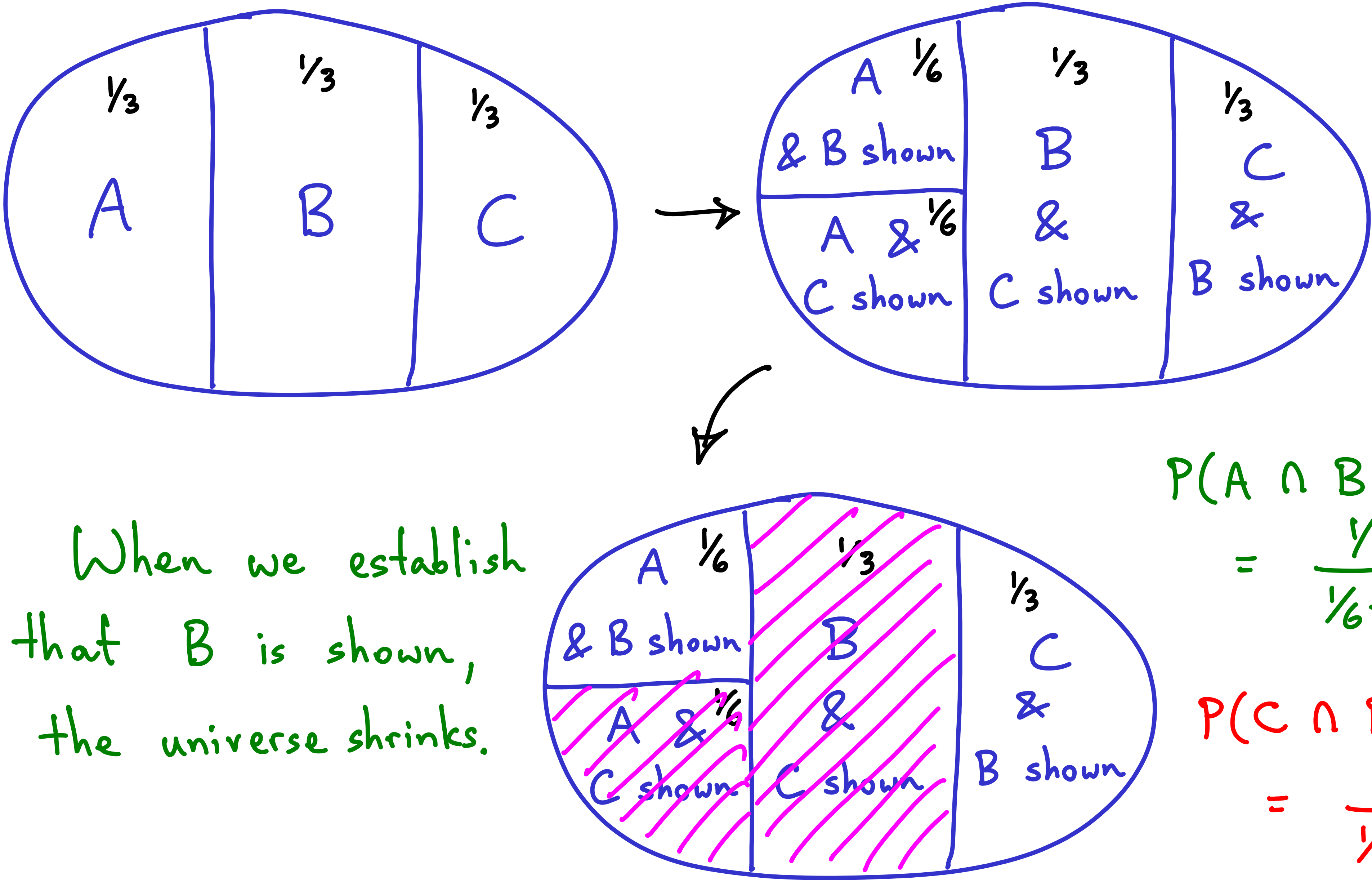


$$P(A \cap B\text{ shown}) =$$

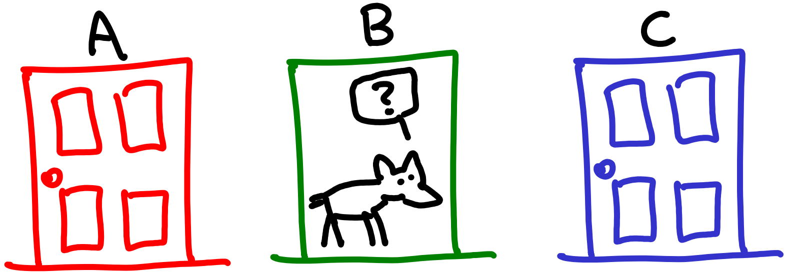
$$= \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$



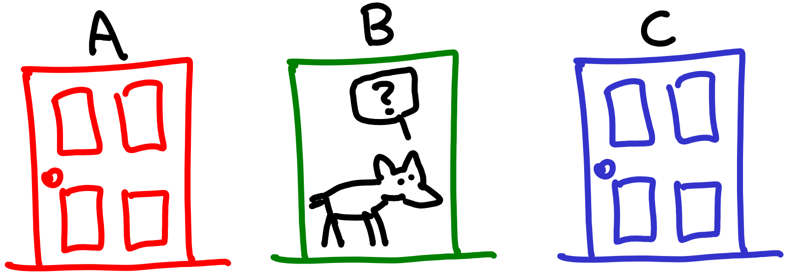
When we establish that B is shown, the universe shrinks.

$$P(A \cap B \text{ shown}) = \\ = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

$$P(C \cap B \text{ shown}) = \\ = \frac{1/3}{1/6 + 1/3} = \frac{2}{3}$$



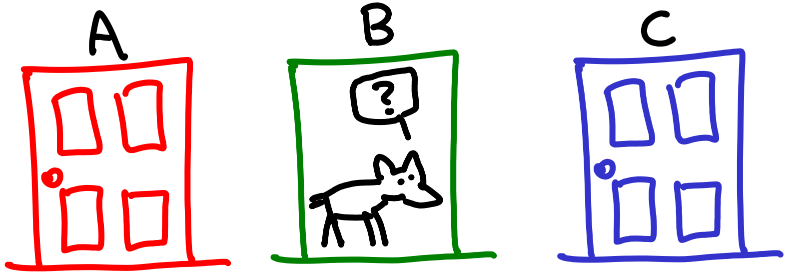
$$P\left[A \mid \left(\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$



$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

apply $P(x|y) = \dfrac{P(x \cap y)}{P(y)}$

$$= \frac{P\left[A \cap \left(\text{we chose A} \cap \text{door B was opened}\right)\right]}{P\left(\text{we chose A} \cap \text{door B was opened}\right)}$$



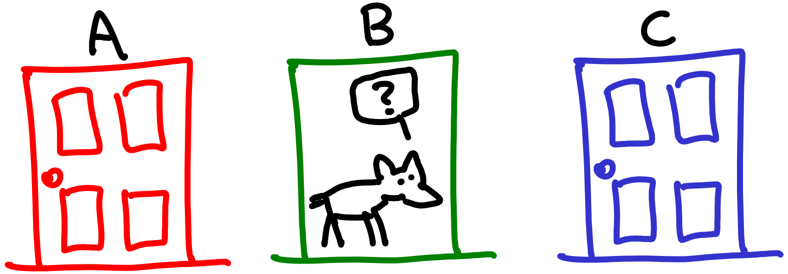
$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{P\left[A \cap \left(\text{we chose A} \cap \text{door B was opened}\right)\right]}{P\left(\text{we chose A} \cap \text{door B was opened}\right)}$$

apply $P(x|y) = \frac{P(x \cap y)}{P(y)}$

$$= \frac{P\left[\left(A \cap \text{we chose A}\right) \cap \text{door B was opened}\right]}{P\left(\text{we chose A} \cap \text{door B was opened}\right)}$$

just moving parentheses



$$P\left[A \,\middle|\, (\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}})\right]$$

apply $P(x|y) = \frac{P(x \cap y)}{P(y)}$

$$= \frac{P\left[A \cap (\text{we chose } A \cap \text{door } B \text{ was opened})\right]}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

just moving parentheses

$$= \frac{P\left[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\right]}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

again, for numerator

$$= \frac{P\left[\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)\right] \cdot P(A \cap \text{we chose } A)}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$

$$= \frac{P\left[\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)\right] \cdot P(A \cap \text{we chose } A)}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

So far,

$$P\left[A \mid (\underline{\text{we chose } A} \cap \underline{\text{door B was opened}})\right]$$

$$= \frac{P\left[\text{door B was opened} \mid (A \cap \text{we chose } A)\right] \cdot P(A \cap \text{we chose } A)}{P(\text{we chose } A \cap \text{door B was opened})}$$

$$= \frac{\overbrace{1/2}^{\substack{\text{host} \\ \text{picks randomly}}} \cdot \overbrace{P(A) \cdot P(\text{we chose } A)}^{\text{independent}}}{P(\text{we chose } A \cap \text{door B was opened})}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$

$$= \frac{P\left[\text{door } B \text{ was opened} \mid \left(A \cap \text{we chose } A\right)\right] \cdot P\left(A \cap \text{we chose } A\right)}{P\left(\text{we chose } A \cap \text{door } B \text{ was opened}\right)}$$

$$= \frac{\overbrace{1/2}^{\text{host picks randomly}} \cdot \overbrace{P(A) \cdot P(\text{we chose } A)}^{\text{independent}}}{P\left(\text{we chose } A \cap \text{door } B \text{ was opened}\right)}$$

$$= \frac{1/2 \cdot \left(1/3 \cdot 1/3\right)}{P\left(\text{we chose } A \cap \text{door } B \text{ was opened}\right)}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{\tfrac{1}{2} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}}{P(\text{we chose A} \cap \text{door B was opened})}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}}{P(\text{we chose A} \cap \text{door B was opened})}$$

could not have
door B opened
AND
car at B

$$= \frac{1/18}{\left[\begin{array}{l} P(\text{we chose A} \cap \text{door B was opened} \cap A) \\ + P(\text{we chose A} \cap \text{door B was opened} \cap C) \end{array}\right]}$$

The 2 terms in the denominator are "mutually exclusive", which means that they have no intersection. We can see that because one asks for A to happen, but the other asks for C to happen (among other things). In fact we should also include the term +P(we chose A AND door B was opened AND B) but this term is equal to zero.

Here's what is going on. There is an event, X. In our case, X = (we chose A AND door B was opened). We are interested in P(X), as shown in orange above. We can say that P(X) = P(X and A) + P(X and B) + P(X and C), if A, B, C are mutually exclusive and cover all possibilities. Each of those 3 terms can have different values, but they will sum to P(X). Think of each term as a subset of X. (it makes sense that they are subsets of X, because they are more restrictive, and maintain all requirements of X). In fact not only are they subsets, but the 3 terms partition X. If X happens, then either (X and A) happens, or (X and B) happens, or (X and C) happens. In our case we have some additional information; X and B cannot happen.

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{\tfrac{1}{2} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}}{P\left(\text{we chose A} \cap \text{door B was opened}\right)}$$

$$\left.\begin{array}{c}\text{could not have}\\ \text{door B opened}\\ \text{AND}\\ \text{car at B}\end{array}\right\} \qquad = \frac{\tfrac{1}{18}}{\left[\begin{array}{l}P\left(\text{we chose A} \cap \text{door B was opened} \cap A\right)\\ +P\left(\text{we chose A} \cap \text{door B was opened} \cap C\right)\end{array}\right]}$$

$$P\left(\text{open B} \mid \left(C \cap \text{choose A}\right)\right) = 1$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose A} \cap \text{door B was opened})}$$

could not have door B opened
AND
car at B

$$= \frac{\frac{1}{18}}{\left[\begin{array}{l} P(\text{we chose A} \cap \text{door B was opened} \cap A) \\ + P(\text{we chose A} \cap \text{door B was opened} \cap C) \end{array}\right]}$$

$$P(\text{open B} \mid (C \cap \text{choose A})) = 1$$

$$= \frac{P(C \cap \text{open B} \cap \text{choose A})}{P(C \cap \text{choose A})}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{1/2 \cdot 1/3 \cdot 1/3}{P(\text{we chose A} \cap \text{door B was opened})}$$

could not have
door B opened
AND
car at B

$$= \frac{1/18}{\left[\begin{array}{l} P(\text{we chose A} \cap \text{door B was opened} \cap A) \\ + P(\text{we chose A} \cap \text{door B was opened} \cap C) \end{array}\right]}$$

$$P\left(\text{open B} \mid (C \cap \text{choose A})\right) = 1$$

$$= \frac{P(C \cap \text{open B} \cap \text{choose A})}{P(C \cap \text{choose A})}$$

$$= \frac{1/18}{P(\text{we chose A} \cap \text{door B was opened} \cap A) + \frac{1}{3} \cdot \frac{1}{3}}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{1/18}{P(\text{we chose A} \cap \text{door B was opened} \cap A) + \frac{1}{3}\cdot\frac{1}{3}}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose A}} \cap \underline{\text{door B was opened}}\right)\right]$$

$$= \frac{1/18}{P(\text{we chose A} \cap \text{door B was opened} \cap A) + \frac{1}{3}\cdot\frac{1}{3}}$$

$$= \frac{1/18}{P\left[(A \cap \text{we chose A}) \cap \text{door B was opened}\right] + 1/9}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3}\cdot\frac{1}{3}}$$

$$= \frac{1/18}{P\left[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\right] + 1/9}$$

$$= \frac{1/18}{P\left[\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)\right] \cdot P(A \cap \text{we chose } A) + 1/9}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose}}\ A \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3} \cdot \frac{1}{3}}$$

$$= \frac{1/18}{P\left[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\right] + 1/9}$$

$$= \frac{1/18}{P\left[\underbrace{\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A}_{\text{host's random choice}})\right] \cdot P\underbrace{(A \cap \text{we chose } A)}_{\text{independent}} + 1/9}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3}\cdot\frac{1}{3}}$$

$$= \frac{1/18}{P\left[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\right] + 1/9}$$

$$= \frac{1/18}{P\left[\underbrace{\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)}_{\text{host's random choice}}\right] \cdot \underbrace{P(A \cap \text{we chose } A)}_{\text{independent}} + 1/9}$$

$$= \frac{1/18}{\underline{1/2} \cdot \underline{(1/3 \cdot 1/3)} + \frac{1}{9}}$$

So far,

$$P\left[A \mid \left(\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}}\right)\right]$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3} \cdot \frac{1}{3}}$$

$$= \frac{1/18}{P\left[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\right] + 1/9}$$

$$= \frac{1/18}{P\left[\underbrace{\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)}_{\text{host's random choice}}\right] \cdot \underbrace{P(A \cap \text{we chose } A)}_{\text{independent}} + 1/9}$$

$$= \frac{1/18}{\underline{1/2} \cdot \underline{(1/3 \cdot 1/3)} + \frac{1}{9}} = \frac{1/18}{1/18 + 2/18} = \boxed{\frac{1}{3}}$$