The Monty Hall Problem

1 of these 3 doors hides a car. The other 2 hide goats.

You get to pick a door. You randomly pick A.

Then a door you didn't pick is opened (say, B) revealing a goat.

You're given the choice: KEEP YOUR DOOR OR SWITCH?
**Conditional Probability**

Roll 2 dice...

- \( P(A) = P(\text{sum} = 8) \)
- \( P(B) = P(\text{both are even}) \)

"prob. A given B"

\[
P(A | B) \uparrow
\]

If we knew that both are even, then what is the probability that the sum is 8?

\[
A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}
\]

\[
B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}
\]

\[
P(A|B) = \frac{3}{9}
\]

\[\neq P(A) \text{ in this example}\]
\[ P(A) = P(\text{sum} = 8) \]

\[ P(B) = P(\text{both are even}) \]

\[ P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36} \]

Divide all these numbers by 36, if you prefer the universe to have area 1.
\[ P(A) = P(\text{sum} = 8) \]
\[ P(B) = P(\text{both are even}) \]

\[
P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36}
\]

When we establish \( B \) then the universe shrinks.

The probability that \( A \) holds is normalized:

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{3+6}
\]
another example

Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(1\text{st flip} = T | 3\cdot H) = \frac{P[(1\text{st flip} = T) \cap (3\cdot H)]}{P(3\cdot H)}
\]

\[
P(3\cdot H) = \frac{{5 \choose 3}}{2^5} \quad \rightarrow \text{Ways to choose 3 positions for H.}
\]

\[
\rightarrow \text{Sample space.} \quad = \frac{5!}{3!2!} = \frac{5}{16} \quad \rightarrow \frac{2/16}{5/16} = \frac{2}{5}
\]

\[
P[(1\text{st flip} = T) \cap (3\cdot H)]:
\{ T \ HHHHT
T \ HHHHT
T \ HTHHT
T \ HTHHT
T \ THHHT \} \quad \frac{4}{32} \quad \text{OR} \quad \frac{1}{2} \cdot \frac{{4 \choose 3}}{2} = \frac{1}{8}
\]
another example  Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

Think of having a biased coin: 60-40 vs 50-50

Then the first flip has 40\% for T \( \rightarrow \frac{2}{5} \)
\[ P(\geq 2 \text{ people in a group of } k \text{ have same birthday}) \]

\[ = 1 - P(\text{all } k \text{ have distinct birthdays}) \]

\[ \downarrow P(\text{2nd person has different bday than 1st}) \]

\[ \cdot P(\text{3rd} \ldots \ldots \ldots \text{1st & 2nd}) \]

\[ \cdot P(\text{4th} \ldots \ldots \ldots (1-3)) \]

\[ \rightarrow \frac{364}{365} = P(A) \]

\[ \rightarrow \frac{363}{365} = P(B|A) \]

assuming 1st & 2nd differ

\[ \rightarrow \frac{362}{365} = P(C|A \cap B) \]

\[ \text{etc} \]

\[ = P(A) \land P(B|A) \land P(C|A \cap B) \ldots \]
**INDEPENDENCE**

Flip a coin \(x3\) : \[ P(3\text{rd} = T \mid 1\text{st} = H) = \]

\[
= \frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)} \quad \quad \frac{\frac{2}{8}}{\frac{1}{2}} \quad \quad \text{# outcomes} \quad \quad \text{sample space} \quad \quad = \frac{1}{2}
\]

Notice \(P(3\text{rd} = T) = \frac{1}{2}\) so knowledge of \((1\text{st} = H)\) was useless.

\(A\) & \(B\) are independent if \(P(A) = P(A \mid B)\)

\(\text{if } P(B) = P(B \mid A) \quad [\text{equivalent}]\)
\[ P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A) \]

always

\[ \text{if } A \text{ & } B \text{ independent} \]

alternate definition: A & B are independent if

\[ P(A \cap B) = P(A) \cdot P(B) \]
A: 1st person is a girl
B: 2nd person is a girl

4 boys & 4 girls
Select 2 people from this group

P(A) = \( \frac{4}{8} \)

P(B) = \( \frac{1}{2} \) (by symmetry)

P(B|A) = \( \frac{3}{7} \)

(or: sample space = 8 \( \cdot \) 7
& for each girl=2nd, #outcomes = 7)
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

Now we know the car is not at B
What is the probability it's at A?

\[ P(A | \overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} \]

...because \[ P(\overline{B} | A) = \frac{P(A \cap \overline{B})}{P(A)} ... \]

\[ = \frac{P(\overline{B} | A) \cdot P(A)}{P(\overline{B})} = \frac{1 \cdot P(A)}{\frac{2}{3}} = \frac{1}{2} \]
**BACK TO MONTY HALL**

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

What we actually want is:

\[ P(A \mid (\text{door B was opened } \cap \text{ we chose A})) \]

\( \not= \bar{B} \)  
\( \text{extra info} \)
BACK TO MONTY HALL: intuition

A
w.l.o.g. guess A & B is shown.

B

C

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

\begin{itemize}
  \item if A (car @ A) then \[ \frac{1}{3} \rightarrow B \text{ revealed} \rightarrow \frac{1}{3} \cdot \frac{1}{2} \}\text{ switch lose} \\
  \item if B, then C revealed \rightarrow \text{ switch to win} \\
  \item if C, then B revealed \rightarrow \text{ switch to win} \rightarrow \frac{1}{3} \\
\end{itemize}
When we establish that B is shown, the universe shrinks.

\[
P(A \cap B \text{ shown}) = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3}
\]

\[
P(C \cap B \text{ shown}) = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3}
\]
\[
P[A \mid (\text{we chose } A \land \text{ door } B \text{ was opened})] = \frac{P[A \land (\text{we chose } A \land \text{ door } B \text{ was opened})]}{P(\text{we chose } A \land \text{ door } B \text{ was opened})}
\]

apply \(P(x \mid y) = \frac{P(x \cap y)}{P(y)}\)

just moving parentheses

again, for numerator

\[
P[\text{door } B \text{ was opened} \mid (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A)
\]

\[
= \frac{P[\text{door } B \text{ was opened} \mid (A \land \text{ we chose } A)]}{P(\text{we chose } A \land \text{ door } B \text{ was opened})}
\]
So far,

\[ P[A \mid (\text{we chose } A \land \text{ door } B \text{ was opened})] = \frac{P[\text{door } B \text{ was opened} \mid (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A)}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]

\[ = \frac{1/2 \cdot P(A) \cdot P(\text{we chose } A)}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]

\[ = \frac{1/2 \cdot (1/3 \cdot 1/3)}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]
So far,

The 2 terms in the denominator
are "mutually exclusive", which means that they have no intersection. We can see that because one asks for A to happen, but the other asks for C to happen (among other things). In fact we should also include the term +P(we chose A AND door B was opened AND B) but this term is equal to zero.

See full notes for a little more discussion on this.

\[
P[A | (\text{we chose A} \land \text{door B was opened})] = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose A} \land \text{door B was opened})}
\]

\[
= \frac{\frac{1}{18}}{P(\text{we chose A} \land \text{door B was opened} \land A)} + P(\text{we chose A} \land \text{door B was opened} \land C)
\]

\[
P(\text{open B} | (C \land \text{choose A})) = \frac{P(C \land \text{open B} \land \text{choose A})}{P(C \land \text{choose A})} = \frac{1}{18}
\]
So far, \[
P[A | (\text{we chose } A \land \text{ door } B \text{ was opened})] = \frac{1/18}{P(\text{we chose } A \land \text{ door } B \text{ was opened} \land A) + \frac{1}{3} \cdot \frac{1}{3}}
\]
\[
= \frac{1/18}{P[(A \land \text{ we chose } A) \land \text{ door } B \text{ was opened}]} + \frac{1}{9}
\]
\[
= \frac{1/18}{P[\text{door } B \text{ was opened} | (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A) + \frac{1}{9}}
\]
with the host's random choice independent.
\[
= \frac{1/18}{\frac{1}{2} \cdot (\frac{1}{3} \cdot \frac{1}{3}) + \frac{1}{9}} = \frac{1/18}{\frac{1}{18} + \frac{2}{18}} = \frac{1}{3}
\]