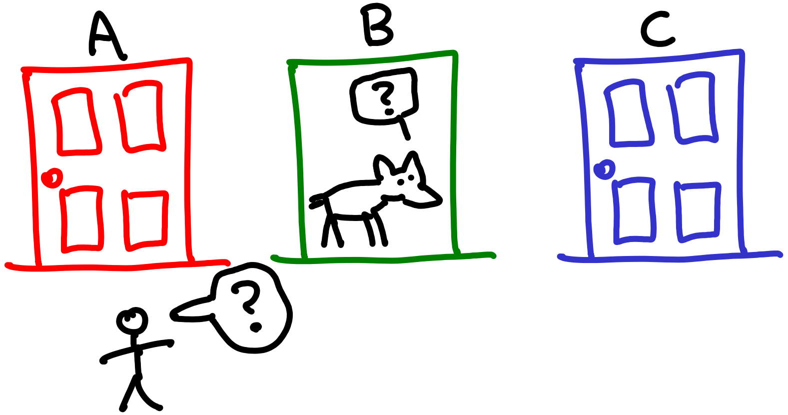
THE MONTY HALL PROBLEM



1 of these 3 doors hides a car.
The other 2 hide goats.

You get to pick a door. You randomly pick A.

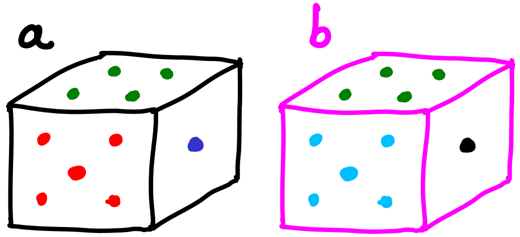
Then a door you didn't pick is opened (say, B) revealing a goat.

You're given the choice: KEEP YOUR DOOR OR SWITCH

?

CONDITIONAL PROBABILITY

Roll 2 dice ...



$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$

"prob. A given B"

$P(A \mid B)$

If we knew that both are even, then what is the probability that the sum is 8?

$A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$

$P(A|B) = \frac{3}{9}$

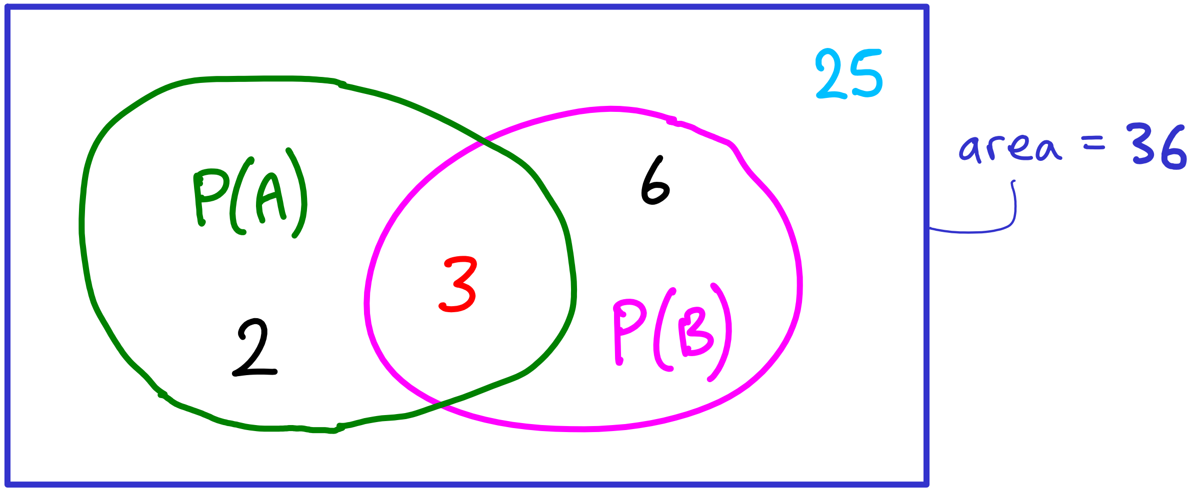
$\neq P(A)$

in this example

$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$

$$P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36}$$

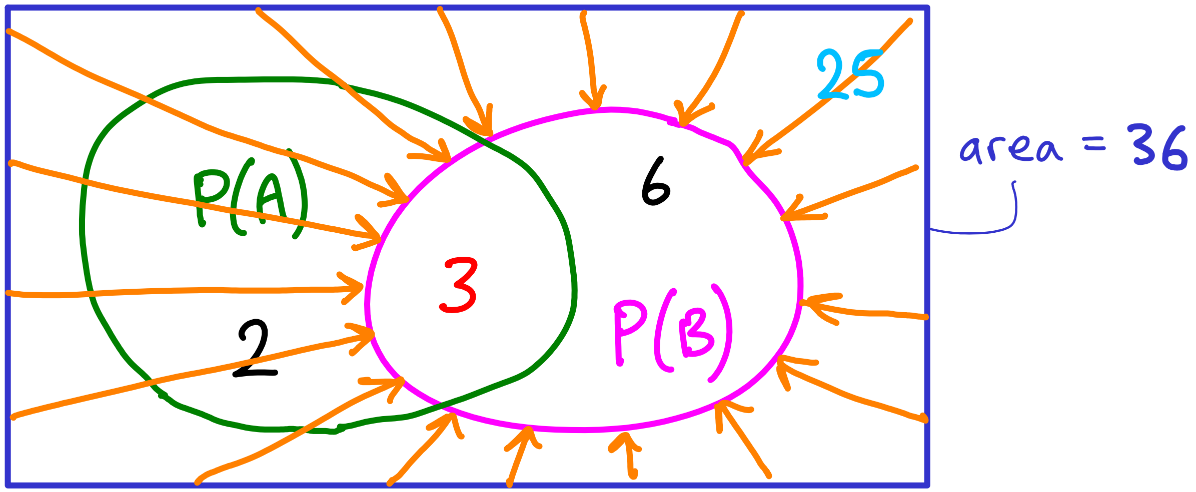


Divide all these numbers by 36, if you prefer the universe to have area 1.

$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$

$$P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36}$$



When we establish B then the universe shrinks.

The probability that A holds is normalized: $\dfrac{\text{remaining valid green area}}{\text{new universe (pink area)}}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{3+6}$$

another example

Flip a coin 5 times. $P(\text{1st flip} = T) = \frac{1}{2}$

But what if you know that 3 of the 5 flips were H?

$$P(\text{1st flip} = T \mid 3 \cdot H) = \frac{P[(\text{1st flip} = T) \cap (3 \cdot H)]}{P(3 \cdot H)} = \frac{2/16}{5/16} = \boxed{\frac{2}{5}}$$

$P(3 \cdot H) = \dfrac{\binom{5}{3}}{2^5}$ → Ways to choose 3 positions for H. → Sample space.

$$= \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}$$

$P[(\text{1st flip} = T) \cap (3 \cdot H)]$:

$$\left.\begin{matrix} T\ HHHT \\ T\ HHTH \\ T\ HTHH \\ T\ THHH \end{matrix}\right\} \frac{4}{32}$$

OR $\frac{1}{2} \cdot \dfrac{\binom{4}{3}}{2^4} = \frac{1}{8}$

another example

Flip a coin 5 times. $P(\text{1st flip} = T) = \frac{1}{2}$

But what if you know that 3 of the 5 flips were H ?

think of having a biased coin : 60-40 vs 50-50

then the first flip has 40% for T $\rightarrow \frac{2}{5}$

$P(\geq 2$ people in a group of k have same birthday$)$

$= 1 - P($all k have distinct birthdays$)$

↳ $P($2nd person has different bday than 1st$) = \frac{364}{365} = P(A)$

• $P(\underbrace{\text{3rd } \ldots \quad \ldots\ldots \quad \ldots \text{ 1st \& 2nd}}_{B}) \rightarrow \underbrace{\frac{363}{365} = P(B|A)}_{\text{assuming 1st \& 2nd differ}}$

• $P($4th $\ldots \quad \ldots \quad \ldots$ $(1-3)) \rightarrow \frac{362}{365} = P(C|(A \cap B))$

etc

$= P(A) \cap P(B|A) \cap P(C|(A \cap B)) \ldots$

INDEPENDENCE

Flip a coin x3 : $P(3rd = T \mid 1st = H) =$

$$= \frac{P[(3rd = T) \cap (1st = H)]}{P(1st = H)} = \frac{\frac{2}{8}}{\frac{1}{2}} = \frac{1}{2}$$

$2 \rightarrow$ # outcomes

$8 \rightarrow$ sample space

Notice $P(3rd = T) = \frac{1}{2}$ so knowledge of $(1st = H)$ was useless.

A & B are independent if $P(A) = P(A \mid B)$

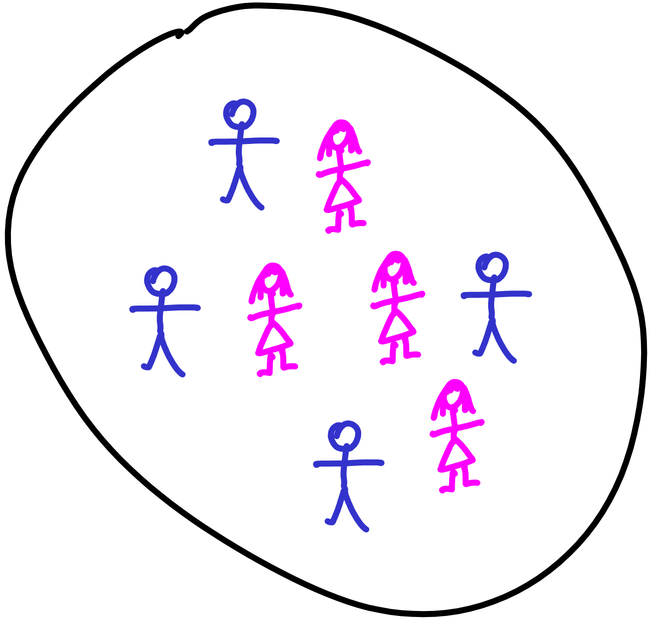
if $P(B) = P(B \mid A)$ [equivalent]

$$\underbrace{\underbrace{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}_{\text{always}} = P(A)}_{\text{if } A \text{ \& } B \text{ independent}}$$

alternate definition: A & B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

another example



4 boys & 4 girls

Select 2 people from this group

$$\left.\begin{array}{l} A: \text{ 1st person is a girl} \\ B: \text{ 2nd person is a girl} \end{array}\right\} \text{dependent}$$

$$P(A) = \frac{4}{8}$$

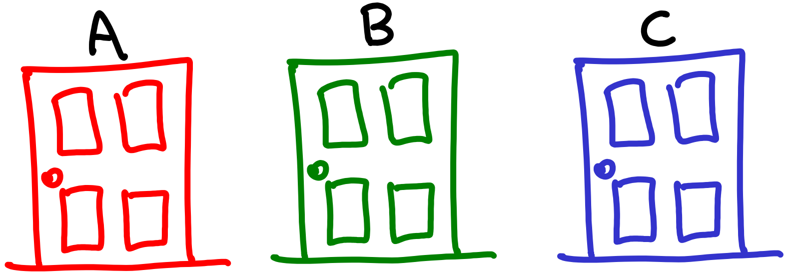
$$P(B) = \frac{1}{2}$$ (by symmetry)

(or : sample space $= 8 \cdot 7$
& for each girl = 2nd, # outcomes = 7)

$$P(B) \neq P(B|A)$$

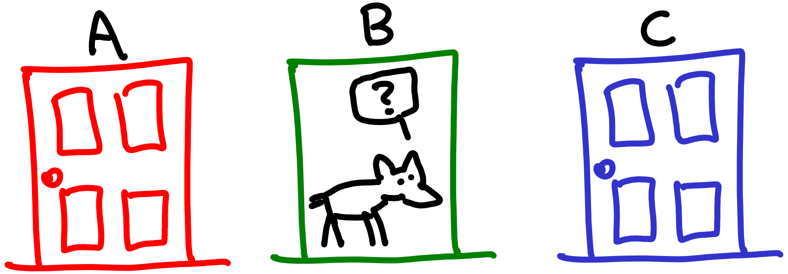
$$P(B|A) = \frac{3}{7}$$

BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

Now we know the car is not at B
What is the probability it's at A?

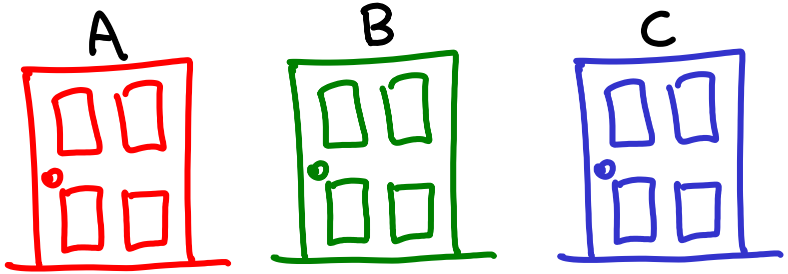


$$P(A \mid \bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

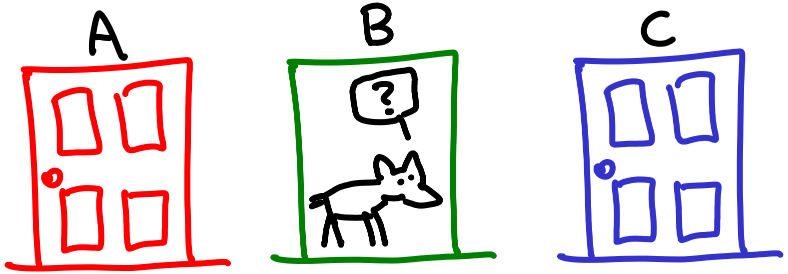
...because $P(\bar{B} \mid A) = \dfrac{P(A \cap \bar{B})}{P(A)}$ ...

$$= \frac{P(\bar{B} \mid A) \cdot P(A)}{P(\bar{B})} = \frac{1 \cdot P(A)}{P(\bar{B})} = \frac{1/3}{2/3} = \frac{1}{2}$$

BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

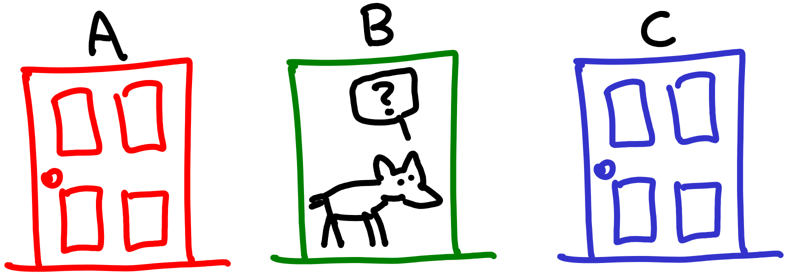


$\cancel{P(A \mid \overline{B})}$ ← back up

What we actually want is

$$P\Big(A \mid \big(\underbrace{\text{door B was opened}}_{\text{not } = \overline{B}} \cap \underbrace{\text{we chose A}}_{\text{extra info}}\big)\Big)$$

BACK TO MONTY HALL : intuition



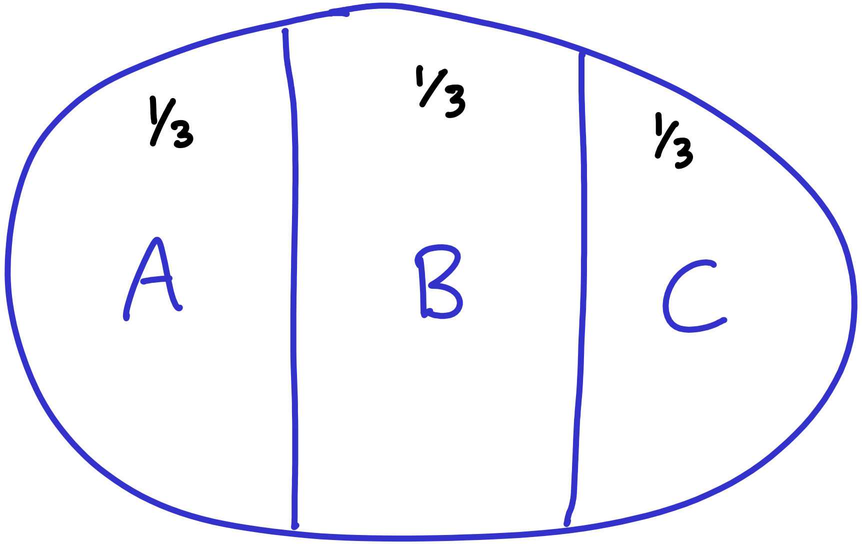
$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$

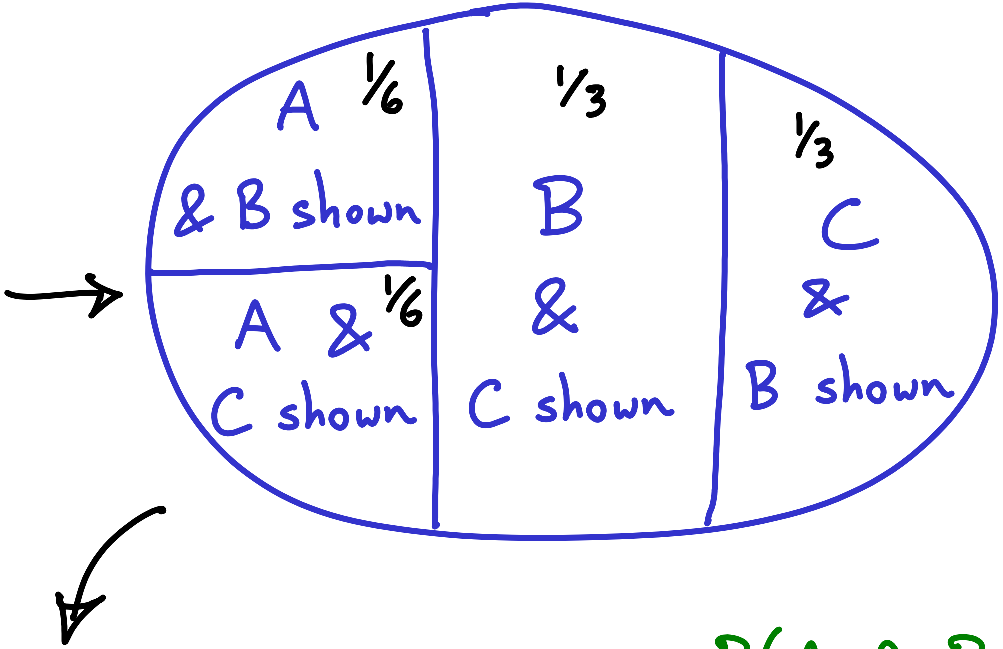
w.l.o.g. guess A
& B is shown.

4 events

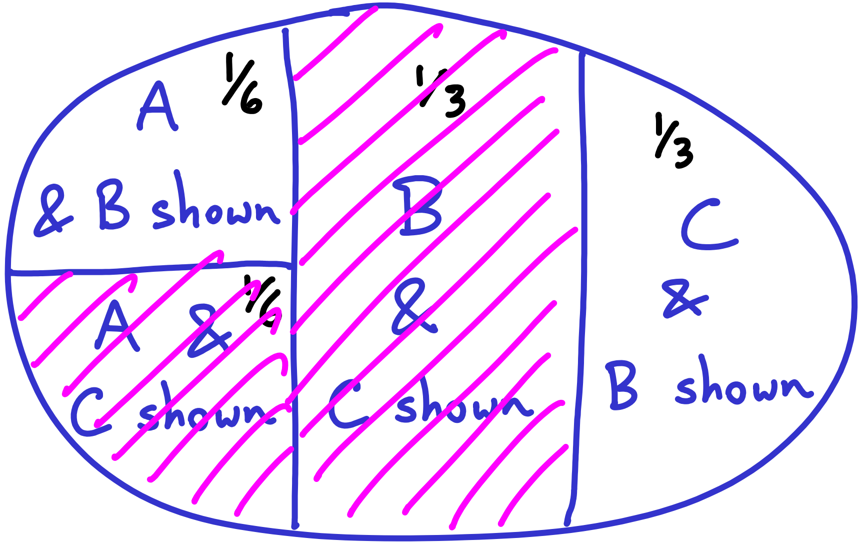
- if A (car @ A) then $\xrightarrow{1/2}$ B revealed $\rightarrow \frac{1}{3} \cdot \frac{1}{2}$ } switch = lose
  $\xrightarrow{1/2}$ C revealed
- if B, then C revealed $\rightarrow$ switch to win
- if C, then B revealed $\rightarrow$ switch to win $\rightarrow \frac{1}{3}$

(each branch: $\frac{1}{3}$)



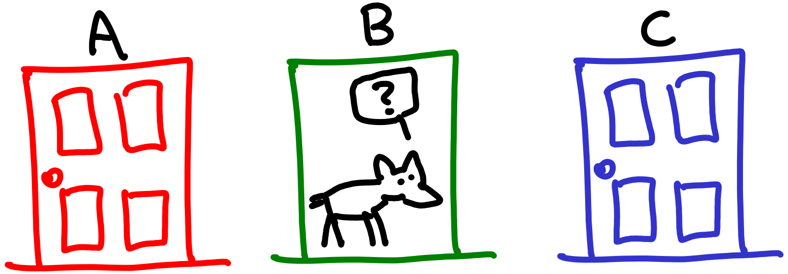


When we establish that B is shown, the universe shrinks.



$$P(A \cap B\text{ shown}) = \\ = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

$$P(C \cap B\text{ shown}) = \\ = \frac{1/3}{1/6 + 1/3} = \frac{2}{3}$$



$$P\big[A \mid (\text{we chose } A \cap \text{door } B \text{ was opened})\big]$$

apply $P(x|y) = \dfrac{P(x \cap y)}{P(y)}$

$$= \frac{P\big[A \cap (\text{we chose } A \cap \text{door } B \text{ was opened})\big]}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

just moving parentheses

$$= \frac{P\big[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\big]}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

again, for numerator

$$= \frac{P\big[\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)\big] \cdot P(A \cap \text{we chose } A)}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

So far,

$$P\left[A \mid \left(\text{we chose A} \cap \text{door B was opened}\right)\right]$$

$$= \frac{P\left[\text{door B was opened} \mid (A \cap \text{we chose A})\right] \cdot P(A \cap \text{we chose A})}{P(\text{we chose A} \cap \text{door B was opened})}$$

$$= \frac{\overbrace{\tfrac{1}{2}}^{\text{host picks randomly}} \cdot \overbrace{P(A) \cdot P(\text{we chose A})}^{\text{independent}}}{P(\text{we chose A} \cap \text{door B was opened})}$$

$$= \frac{\tfrac{1}{2} \cdot \left(\tfrac{1}{3} \cdot \tfrac{1}{3}\right)}{P(\text{we chose A} \cap \text{door B was opened})}$$

So far,

The 2 terms in the denominator are "mutually exclusive", which means that they have no intersection. We can see that because one asks for A to happen, but the other asks for C to happen (among other things). In fact we should also include the term +P(we chose A AND door B was opened AND B) but this term is equal to zero. See full notes for a little more discussion on this.

$$P\left[A \mid (\text{we chose } A \cap \text{door } B \text{ was opened})\right]$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

$$= \frac{1/18}{\left[\begin{array}{l} P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) \\ + P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap C) \end{array}\right]}$$

could not have door B opened AND car at B

$$P(\text{open } B \mid (C \cap \text{choose } A)) = 1 = \frac{P(C \cap \text{open } B \cap \text{choose } A)}{P(C \cap \text{choose } A)}$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3} \cdot \frac{1}{3}}$$

So far,

$$P[A \mid (\underline{\text{we chose } A} \cap \underline{\text{door } B \text{ was opened}})]$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3} \cdot \frac{1}{3}}$$

$$= \frac{1/18}{P[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}] + 1/9}$$

$$= \frac{1/18}{P[\underbrace{\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)}_{\text{host's random choice}}] \cdot \underbrace{P(A \cap \text{we chose } A)}_{\text{independent}} + 1/9}$$

$$= \frac{1/18}{\underline{1/2} \cdot \underline{(1/3 \cdot 1/3)} + \frac{1}{9}} = \frac{1/18}{1/18 + 2/18} = \boxed{\frac{1}{3}}$$