A bet: I randomly select half of the class...

If any 2 people in that group have the same birthday

you give me a dollar.

If no birthday match is found

I give you a dollar

Another bet :

I randomly select 10 people born in the same month

Same deal as before

My chances of winning : $>80\%$

One last bet ?

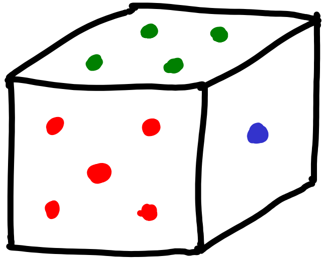
I randomly select 7 people

If any 2 people in that group have birthdays within a week of each other...

I win 60% of the time. (52% if within 6 days)

DISCRETE PROBABILITY

Roll a die ... Possible outcomes : $\overbrace{\{1, 2, 3, 4, 5, 6\}}^{\text{sample space}}$



each has a probability : $\underbrace{\{\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\}}_{\text{sum to } 1}$

Roll 2 dice ... sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$

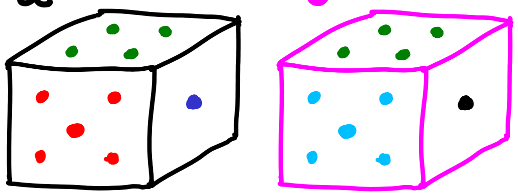
(a, b)

$(2,1), (2,2), \ldots$

$\vdots$

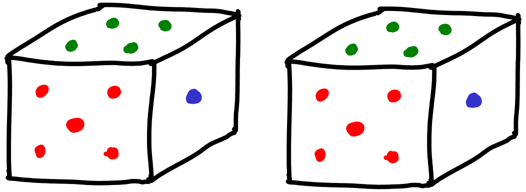
$(6,1), (6,2), \ldots \ldots (6,6)\}$

a b



prob. $\rightarrow \frac{1}{36}$

Roll 2 indistinguishable dice ...



sample space $\rightarrow$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,3), (3,4), (3,5), (3,6),$
$(4,4), (4,5), (4,6),$
$(5,5), (5,6),$
$(6,6)\}$

prob. $\neq \frac{1}{36}$

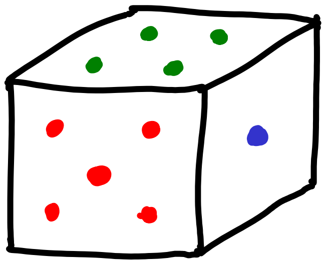
if (a,a) then $\frac{1}{36}$ $\quad \left(6 \cdot \frac{1}{36} = \frac{6}{36}\right)$

$+ \longrightarrow 1$

if (a,b) then $\frac{2}{36}$ $\quad \left(15 \cdot \frac{2}{36} = \frac{30}{36}\right)$

$a \neq b$

Roll a die



$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$P(\underbrace{\text{roll even}}_{\text{event}}) = P(\{2,4,6\}) = P(2) + P(4) + P(6) = \frac{1}{2}$$

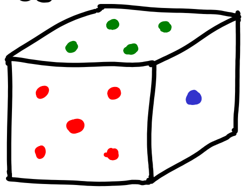
Roll 2 dice...

$$P(\overbrace{\text{sum} = 7}) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\})$$

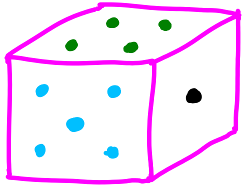
$$= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)$$

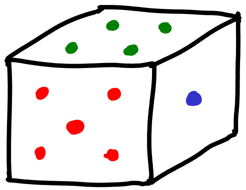
$$= 6 \cdot \frac{1}{36} = \frac{1}{6}$$

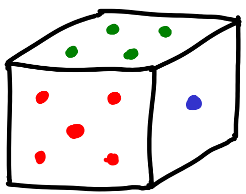
a



b

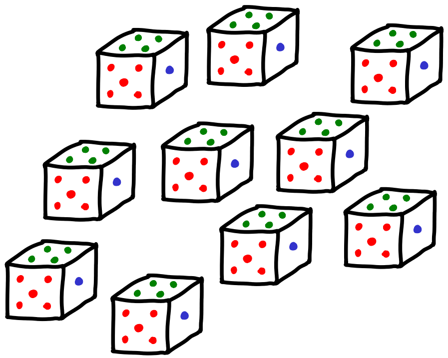






$$P(\text{sum} = 7) = P(\{(1,6),(2,5),(3,4)\}) = 3 \cdot \frac{2}{36} = \frac{1}{6}$$

Roll 10 dice (or 1 die 10 times)



Sample space size : 6^{10} >60 million

P(observe no 1's) ? $\left(\frac{5}{6}\right)^{10}$

How many outcomes have no 1's ? → 5^{10}

or, say that each roll/die is independent
so for each roll, P(no 1) = $\frac{5}{6}$ ⇒ $\left(\frac{5}{6}\right)^{10}$ } to be discussed further

Poker: 52 cards (4×13 types); select 5.

P(4 of a kind) = P(4 of 5 are of same type)

e.g. 3, 3, 3, 3, 7 or 8, 8, 8, J, 8

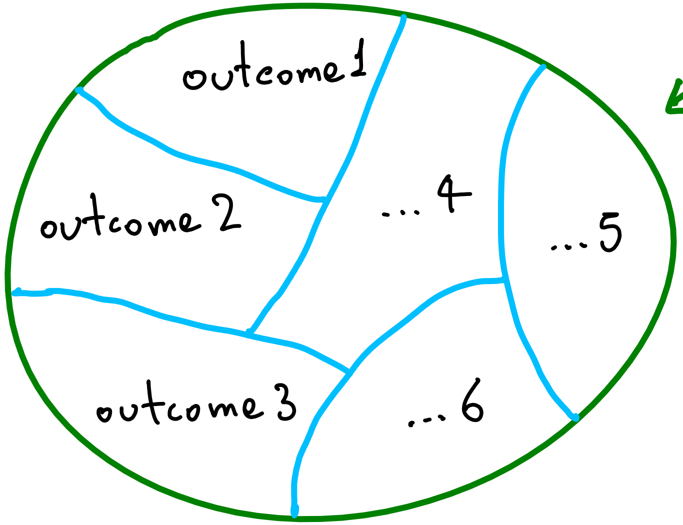
ans: $\dfrac{\text{\# 4-of-a-kinds}}{\text{\# possible outcomes}}$

\# 4-of-a-kinds → $\left.\begin{matrix} AAAA \\ 2222 \\ \vdots \\ KKKK \end{matrix}\right\}$ 13 types $\times$ 48 choices of remaining card

\# possible outcomes → $\binom{52}{5}$ } order doesn't matter

$$\frac{13 \cdot 48}{\binom{52}{5}} = \frac{1}{4165} \sim 0.00024$$

PROBABILITY VISUALIZATION



sample space → area = 1

$P(\text{outcome } i) = \text{area}(i)$

example: roll one die → all areas equal $= \frac{1}{6}$

$P(\text{event}) = \text{sum of appropriate areas}$

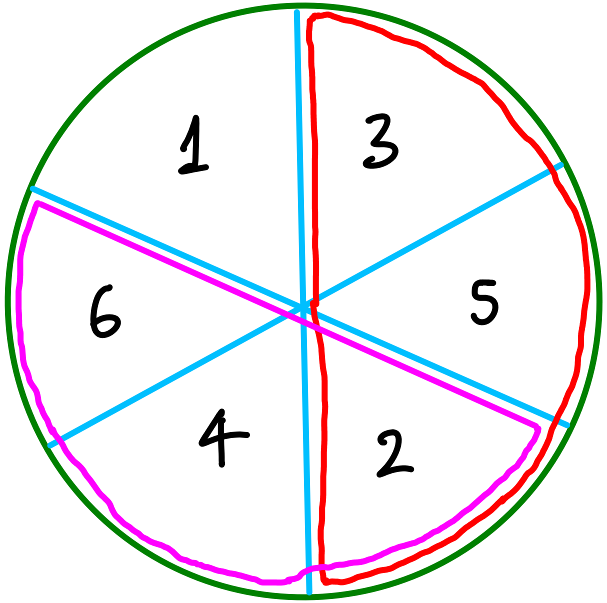
e.g. $P(\underbrace{\text{roll prime \#}}_{2,\,3,\,5} \text{ OR } \underbrace{\text{even \#}}_{2,\,4,\,6})$

$P(\text{prime}) = \frac{3}{6}$ $\quad$ $P(\text{even}) = \frac{3}{6}$

$\frac{5}{6}$

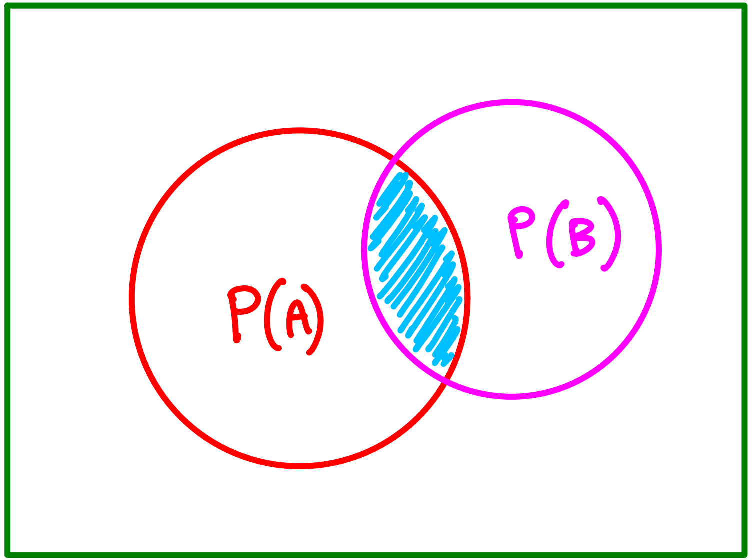
avoid doublecounting

NOT $\frac{3}{6} + \frac{3}{6}$

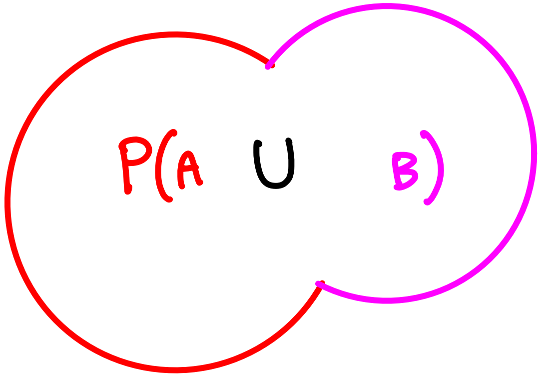


$$P(\underbrace{\text{roll prime \#}}_{2,3,5} \text{ OR } \underbrace{\text{even \#}}_{2,4,6}) \rightarrow P(A \cup B)$$

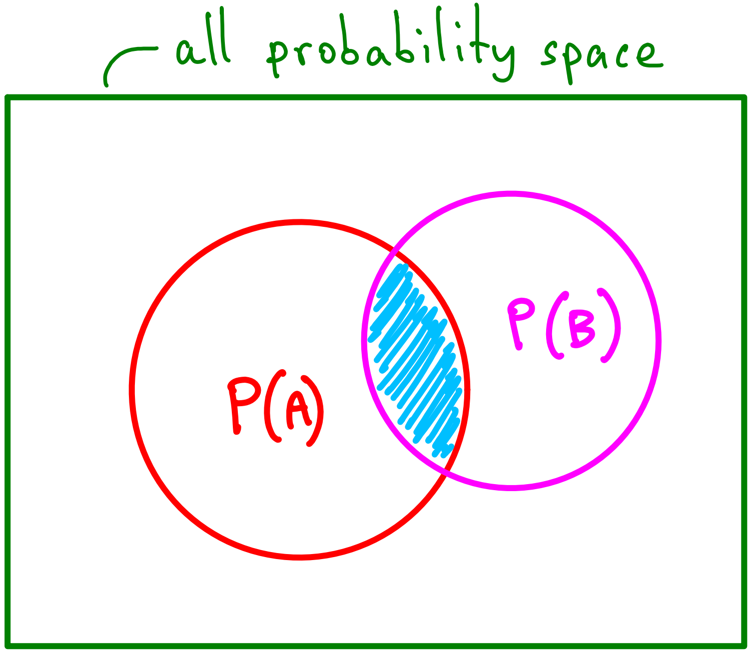
all probability space



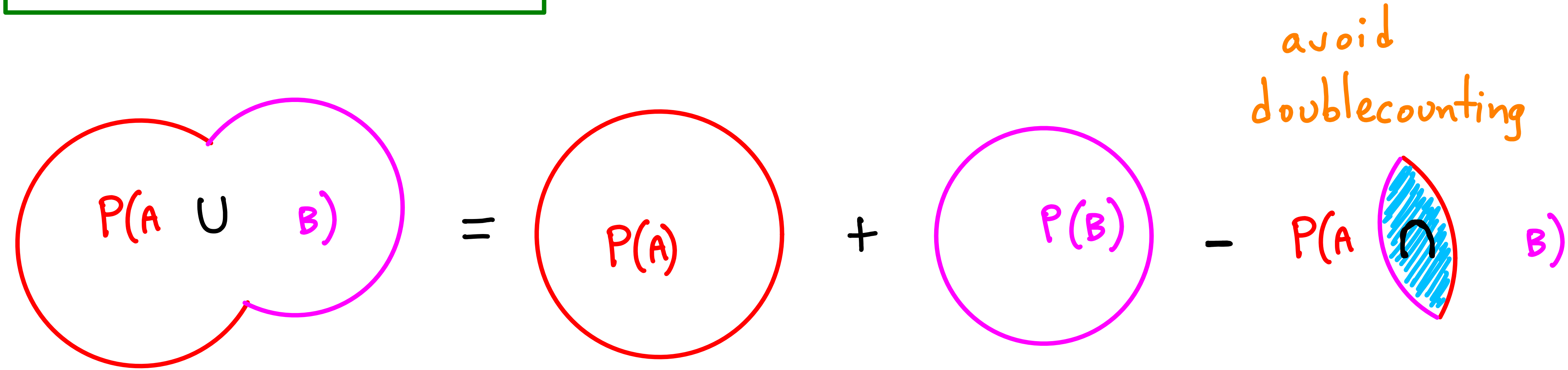
"OR"



"AND"



$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$



$P(\text{someone in class was born on Feb. 29})$

$= P[(\text{student 1 born on Feb. 29}) \cup (\text{student 2 born on Feb. 29}) \ldots$

$\ldots \cup (\text{student 3 born on Feb. 29}) \ldots \cup \ldots \underbrace{(\text{student } k \text{ born on Feb. 29})}_{\sim \underbrace{\frac{1}{365 \cdot 4 + 1}}_{1-\alpha} \sim 0.07\%}]$

awful but we could say it is $< \sum P(i) \sim 80 \cdot 0.07\%$
$\sim 5.6\%$

$= 1 - P(\text{nobody in class was born on Feb. 29})$

$= 1 - P[(\text{student 1 NOT born on Feb. 29}) \cap (\text{student 2 NOT born on Feb. 29})$
$\ldots \cap (\text{student } k \text{ NOT born on Feb. 29})]$

$= 1 - \alpha^k \qquad \alpha = P(\text{student } i \text{ NOT born on Feb. 29})$

P(someone in class was born on Feb. 29) (suppose $k=80$ students)

$= P\big[$(student 1 born on Feb. 29) $\cup$ (student 2 born on Feb. 29) ...

... $\cup$ (student 3 born on Feb. 29) ... $\cup$ ... $\underbrace{\text{(student } k \text{ born on Feb. 29)}}_{\sim \frac{1}{365\cdot 4+1} \sim 0.07\%}\big]$

$\underbrace{365\cdot 4+1}$ ↳ assuming all days equally likely & 1 leap year every 4.

↳ awful but we could say it is $< \sum P(i) \sim 80 \cdot 0.07\%$ (approximation) $\sim 5.6\%$

$= 1 -$ P(nobody in class was born on Feb. 29)

$= 1 - P\big[$(student 1 NOT born on Feb. 29) $\cap$ (student 2 NOT born on Feb. 29)

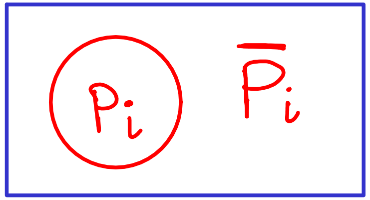
... $\cap$ (student k NOT born on Feb. 29)$\big]$

$$= 1 - \alpha^k = 1 - \left(\frac{365\cdot 4}{365\cdot 4+1}\right)^k$$ exactly

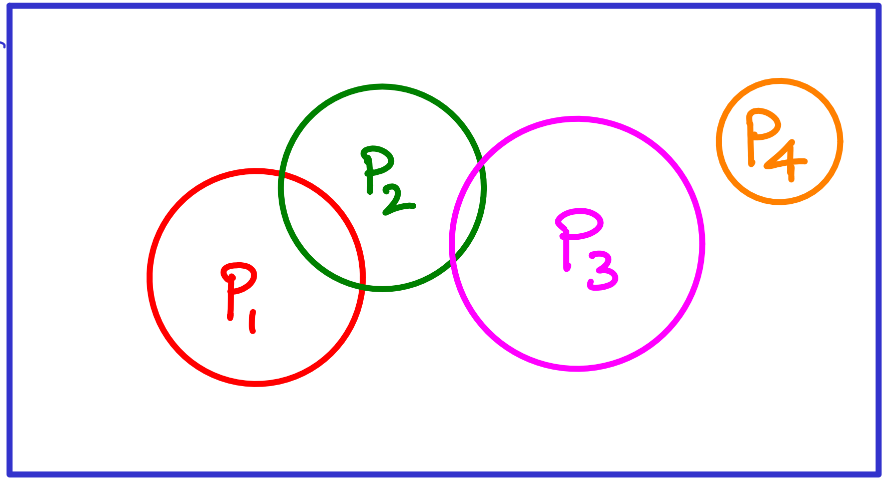
80 students $\sim 5\%$

$$P_i + \bar{P}_i = 1$$

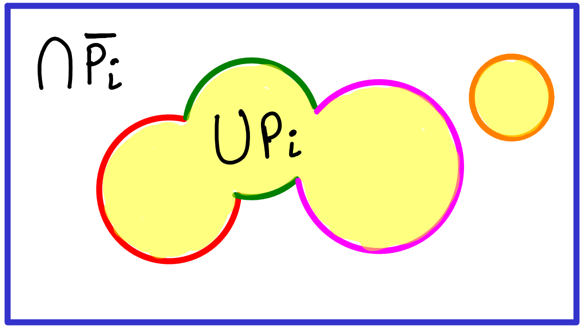
area in circle i
+ area outside circle i $= 1$



area=1



$$\underbrace{\bigcup P_i}_{\text{inside any circle}} = 1 - \underbrace{\bigcap \bar{P}_i}_{\text{outside every circle}}$$



$P(\geq 2$ people in a group of k have same birthday$)$ no Feb. 29 allowed

$\hookrightarrow P[(1,2) \cup (1,3) \cup (1,4) \ldots \cup \underbrace{(1,k)}_{1/365} \cup (2,3) \cup (2,4) \ldots \cup (2,k) \ldots \ldots \cup (k-1,k)]$

awful

$\boxed{\text{if } k > 365 \text{ use pigeonhole}}$

$= 1 - P(\text{all } k \text{ have distinct birthdays})$

$\hookrightarrow P(\text{2nd person has different bday than 1st}) = \dfrac{364}{365} = P(A)$

$P(\text{3rd} \ldots \ldots \ldots \underbrace{\text{1st \& 2nd}}_{\text{assuming 1st \& 2nd differ}}) = \dfrac{363}{365} = P(B)$

this is actually "conditional probability" which will be covered next time.

$P(\geq 2$ people in a group of k have same birthday$)$ no Feb. 29 allowed

$\hookrightarrow P[(1,2) \cup (1,3) \cup (1,4) \ldots \cup \underbrace{(1,k)}_{1/365} \cup (2,3) \cup (2,4) \ldots \cup (2,k) \ldots \ldots \cup (k-1,k)]$

awful

$\boxed{\text{if } k > 365 \text{ use pigeonhole}}$

$= 1 - P(\text{all } k \text{ have distinct birthdays})$

$\hookrightarrow P(\text{2nd person has different bday than 1st}) = \frac{364}{365} = P(A)$

$P(\text{3rd} \ldots \ldots \ldots \text{1st \& 2nd}) = \frac{363}{365} = P(B)$

$P(\text{4th} \ldots \ldots \ldots (1-3)) = \frac{362}{365} = P(C)$

etc

$= 1 - [P(A) \cap P(B) \cap P(C) \ldots] = 1 - \frac{\frac{365!}{(365-k)!}}{365^k} = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (363-k+1)}{365^k}$

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

$$= 1 - \frac{\frac{365!}{(365-k)!}}{365^k} = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (363-k+1)}{365^k} = 1 - \frac{(365)_k}{365^k}$$

$k = 2 \rightarrow P \sim 0{,}27\%$ $\left(\frac{1}{365}\right)$

$k = 4 \rightarrow P \sim 1.64\%$

$k = 23 \rightarrow P \sim 50.73\%$

$k = 30 \rightarrow P \sim 70.6\%$

$k = 70 \rightarrow P \sim 99.9\%$

$k \sim 116 \rightarrow P \sim 1 - \frac{1}{10^9}$

$k = 300 \rightarrow P \sim 1 - \frac{1}{10^{80}}$

($10^{80} \sim$ # atoms in universe)

$(k > 365 \rightarrow P = 1)$

$P(\geqslant 2 \text{ people in a group of } k \text{ have same birthday})$

$$1 - \frac{365 \cdot 364 \cdot 363 \cdots (363-k+1)}{365^k}$$

Didn't cover this slide in class.
It explains bet #2.

For the bet involving k people born in a month w/ 30 days

substitute $365 \to 30$

$(k=10)$ $$1 - \frac{30 \cdot 29 \cdot 28 \cdot \cdots \cdot 23 \cdot 22 \cdot 21}{30^{10}} \sim 0.815$$