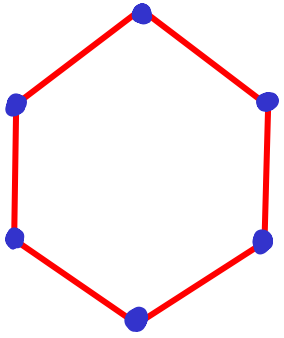

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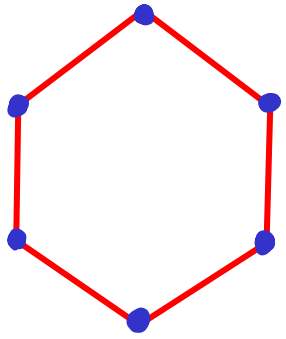

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$\chi = 2$ if V even

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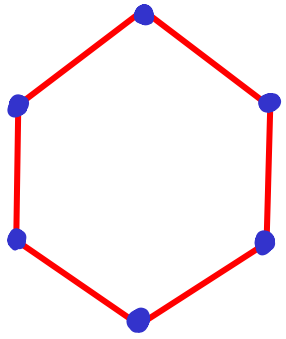

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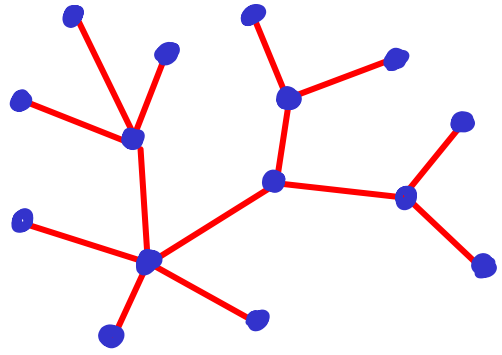

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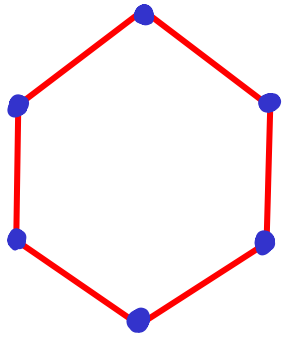

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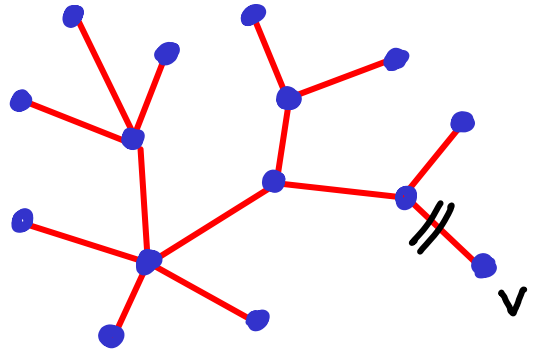

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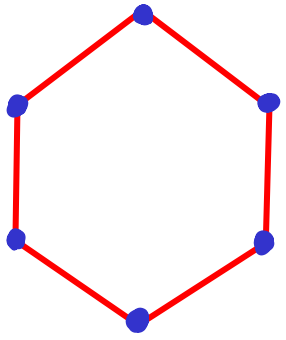

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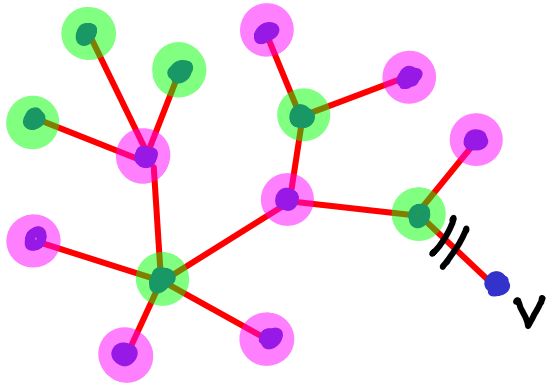

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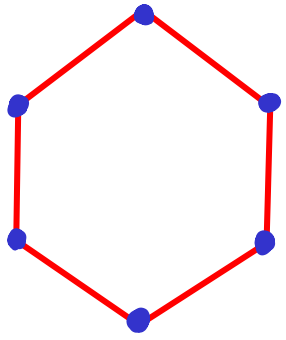

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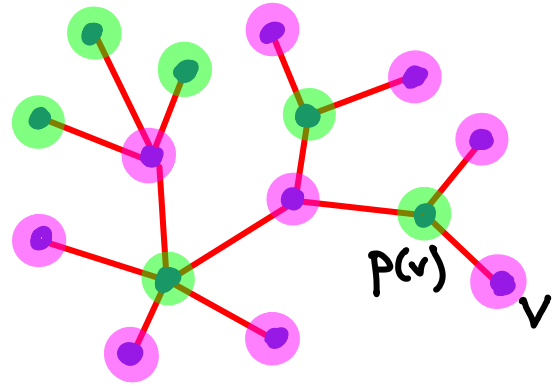

For trees?

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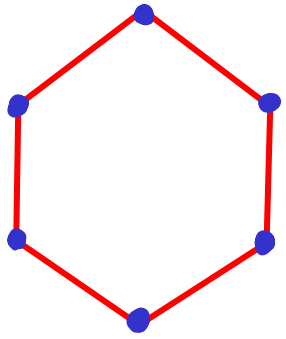
Color v opposite of p(v)

$\chi = 2$



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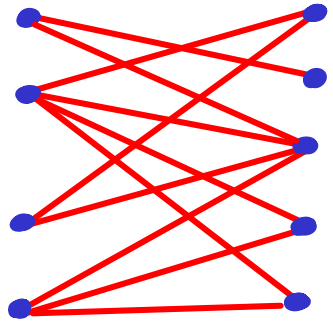
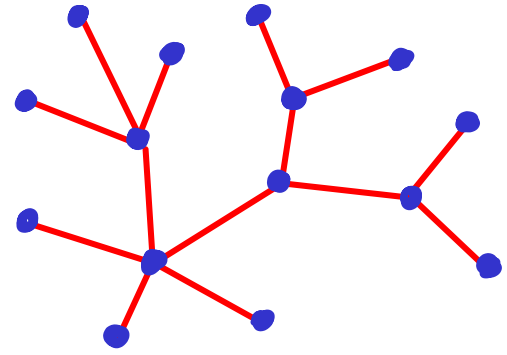


For bipartite graphs?



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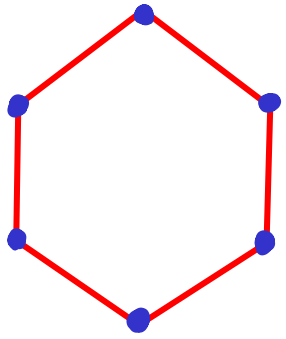

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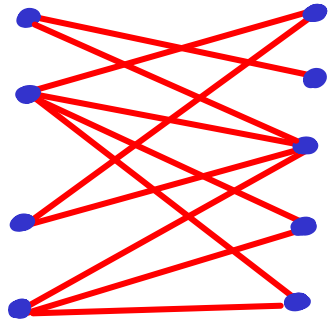

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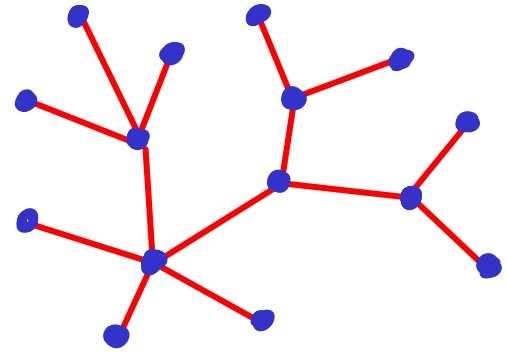

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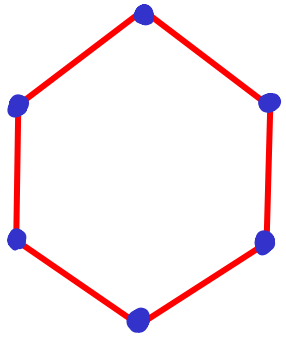

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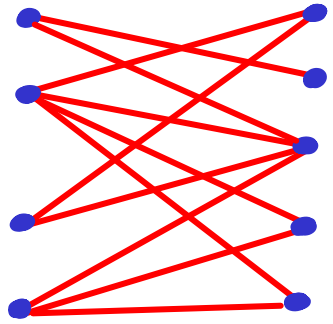

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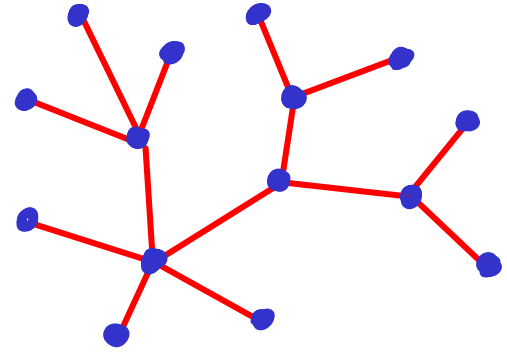
$\chi = 2$

In fact if $\chi(G) = 2$ then G is bipartite by definition



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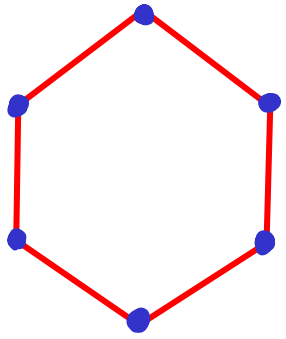
(trees are bipartite)



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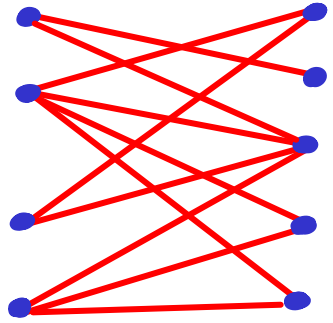

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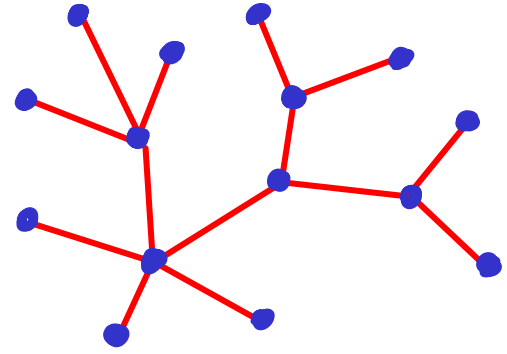

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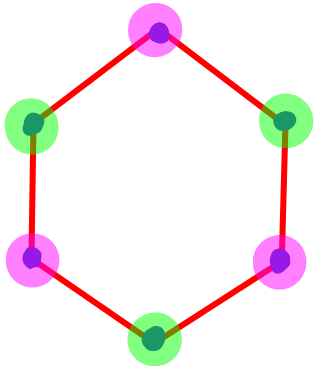

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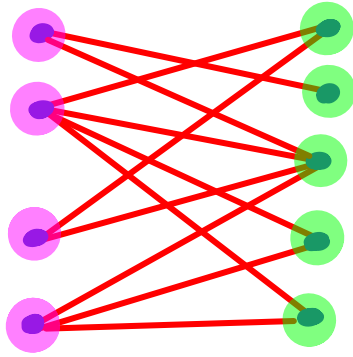

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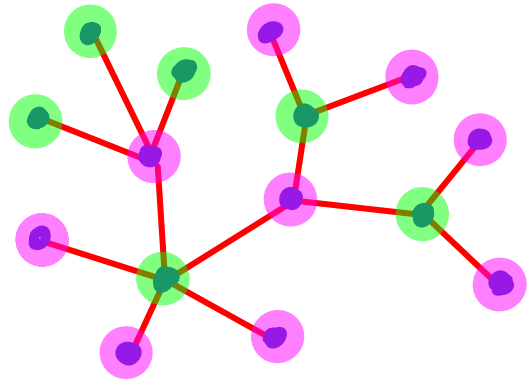

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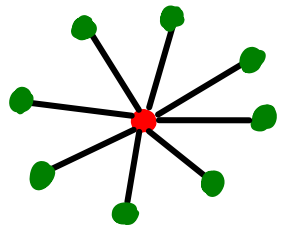
$K_n \rightarrow \Delta = n-1 : \chi = \Delta + 1$



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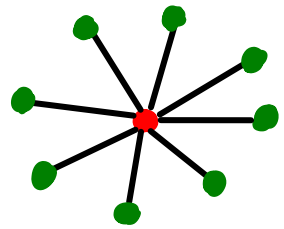

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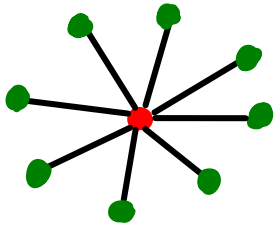

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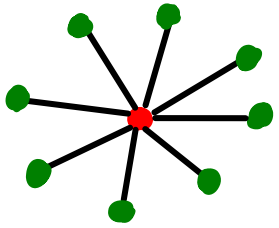

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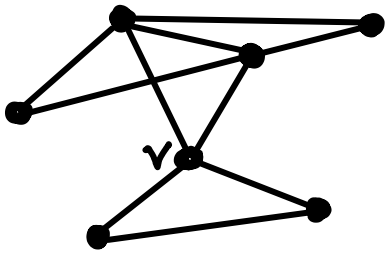

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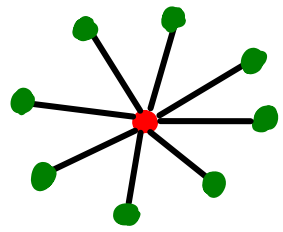

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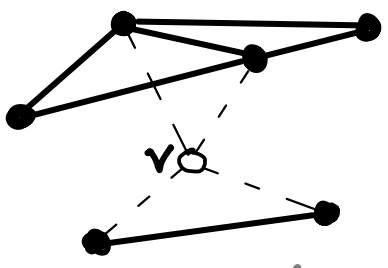

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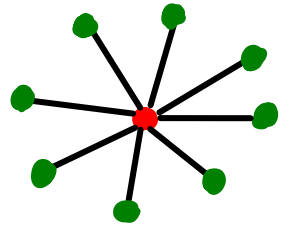

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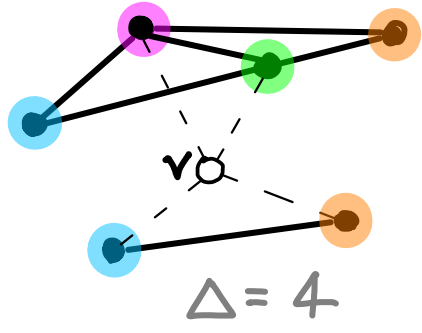

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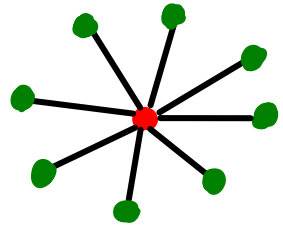

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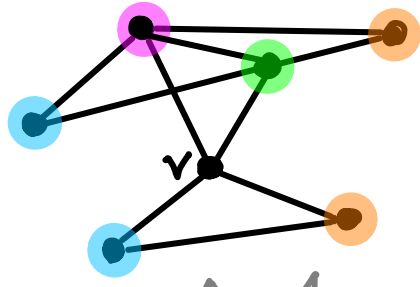

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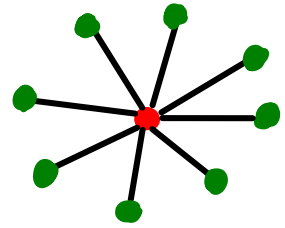

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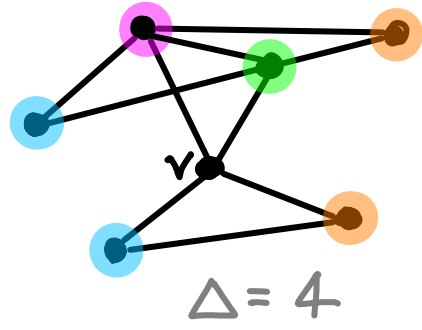

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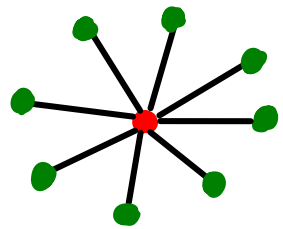

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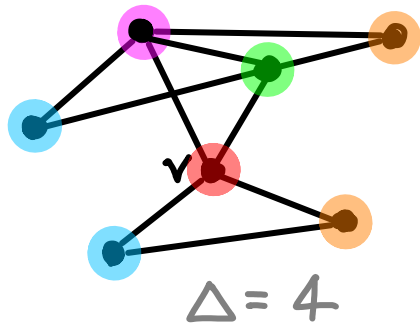

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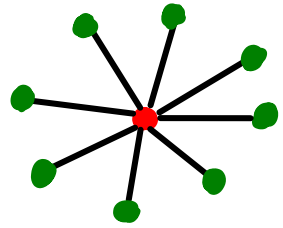

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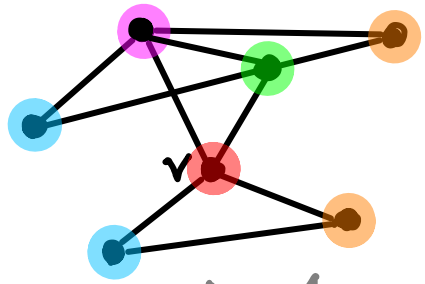
Claim $\chi \le \Delta + 1$

- Incrementally "add" vertices.
- When adding vertex v, look at all neighboring colors.
- Always have ≥ 1 color available.



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COLORING PLANAR GRAPHS (like map duals)

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Claim: $\chi \leq 6$ ... trivial if $V \leq 6$

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We know planar graphs have a vertex w/ degree ≤ 5

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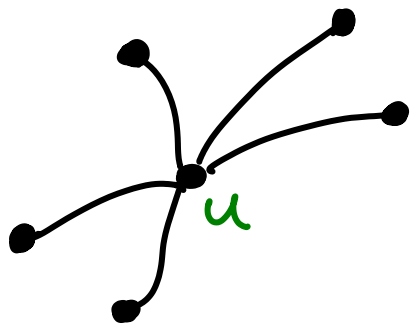

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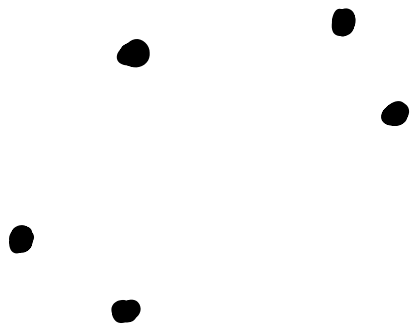

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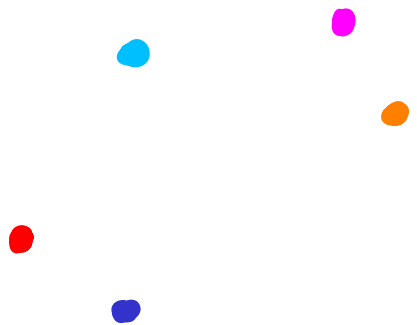

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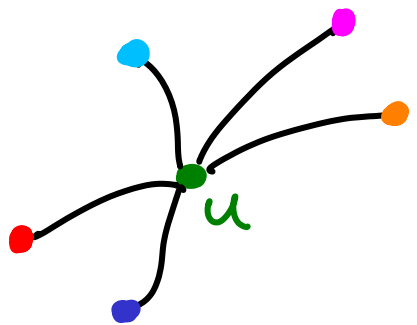

re-insert u: give it a color not used by neighbors

□

Claim: $\chi \leq 5$ ... trivial if ???

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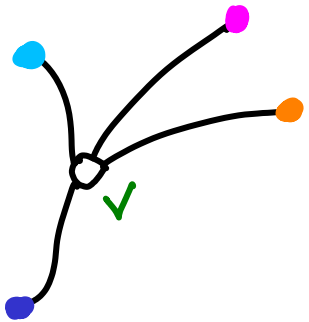

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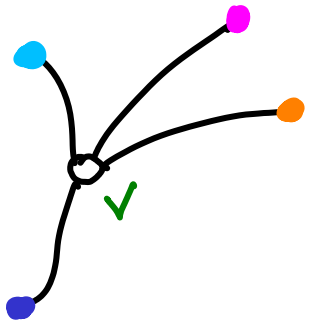
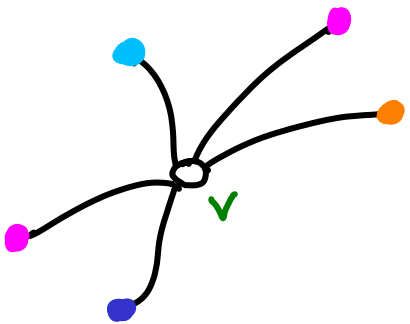


still needs induction

... or if neighbors use < 5 colors



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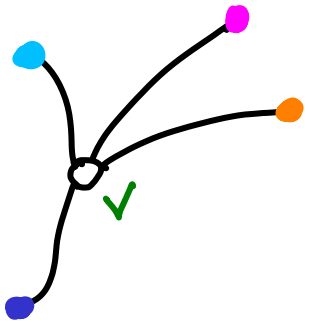

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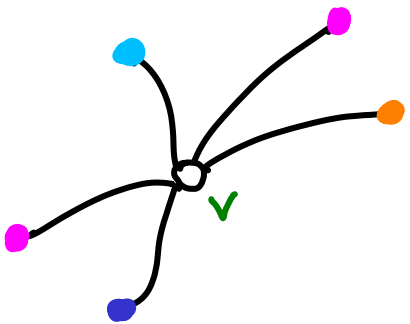
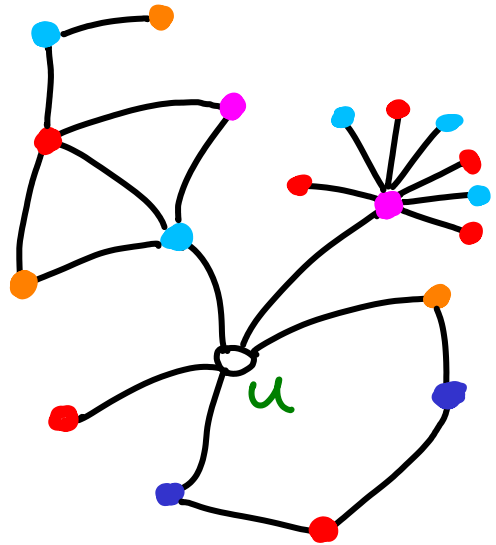


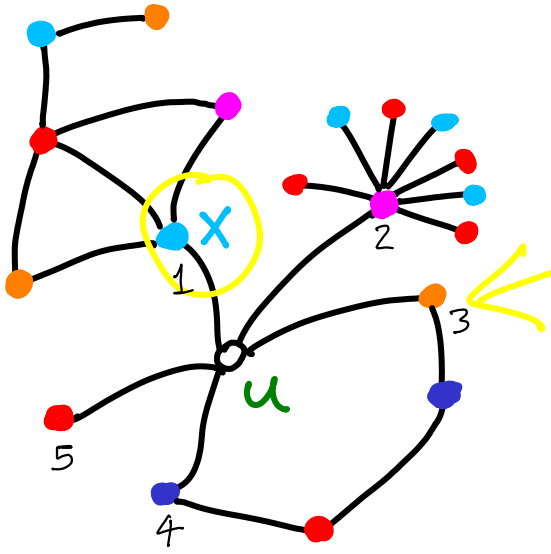
Use induction & $d(u) \leq 5$



= drawing

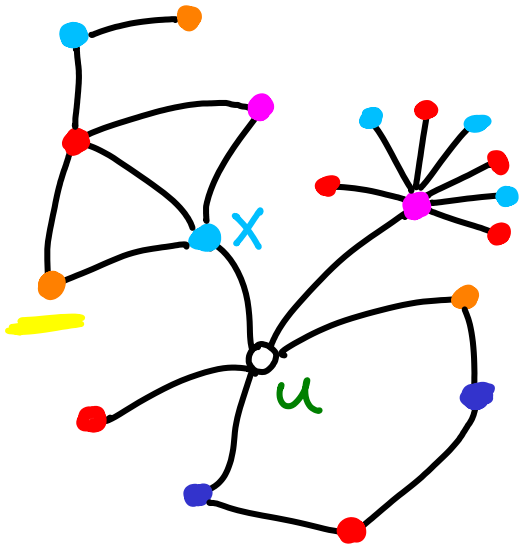
Consider any embedding of G

We need a neighbor of u to change color



Try to change × from ● to ●

[specifically skipping 2 over in adj(u)]



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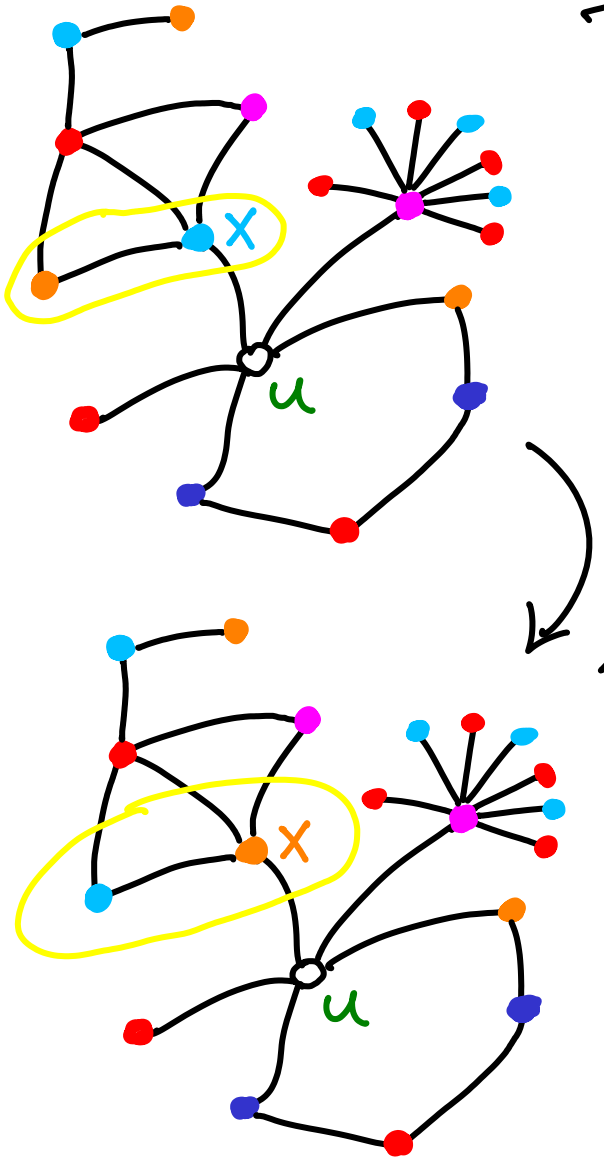
↳ This works if x has no ● neighbors



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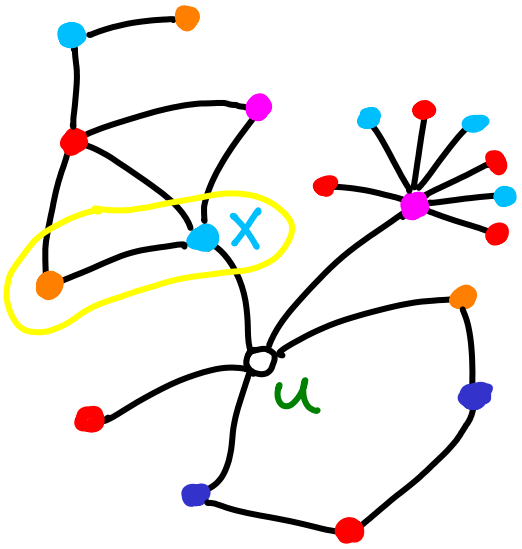

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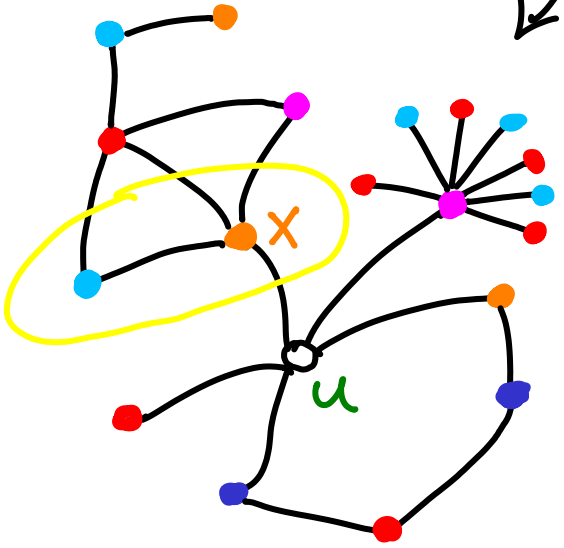

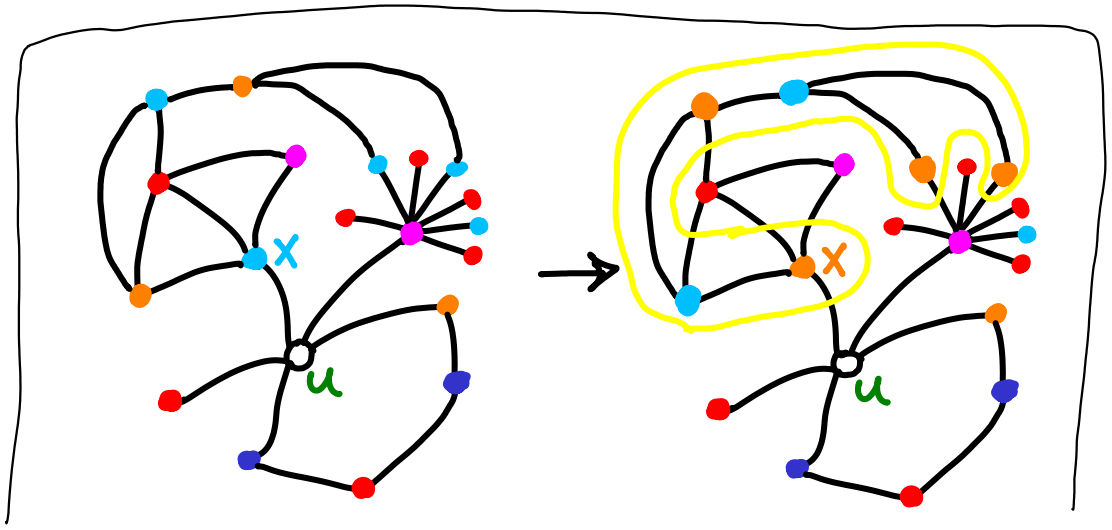

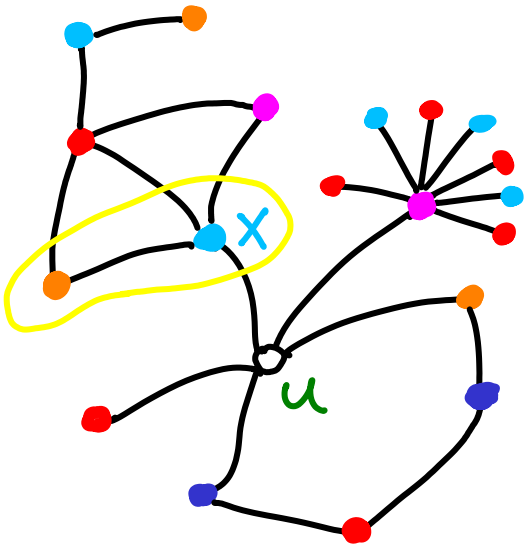

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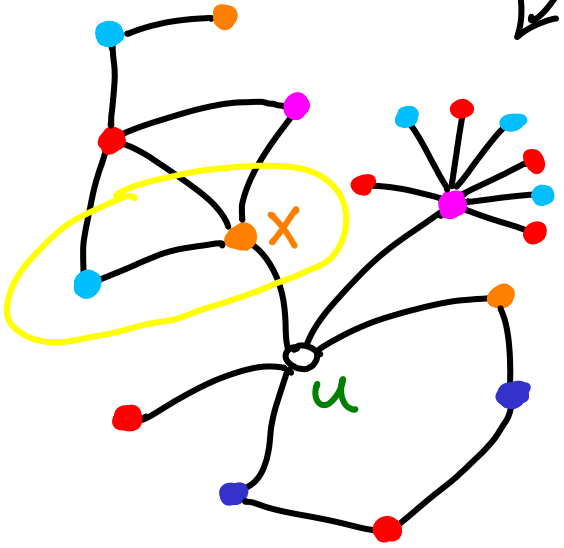
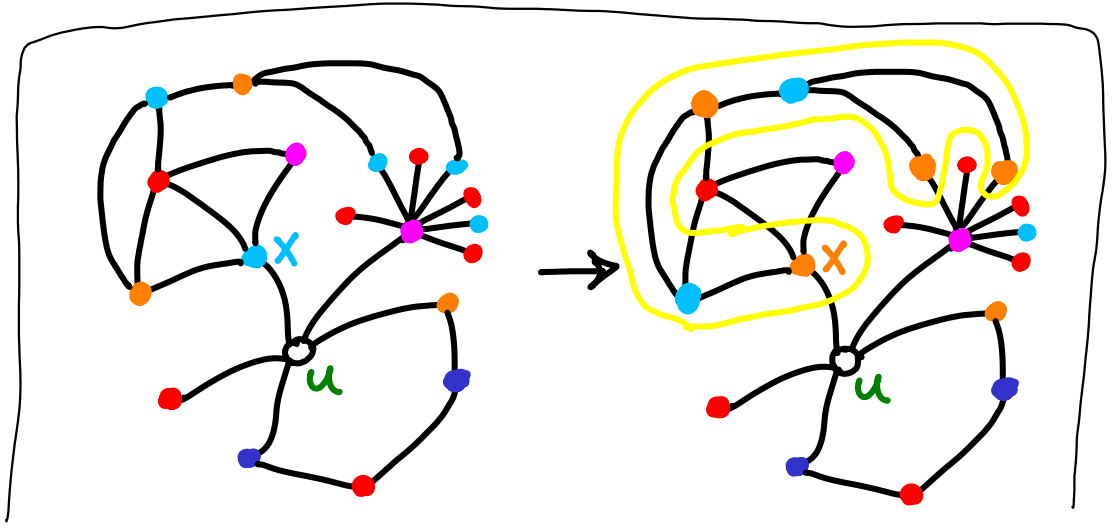
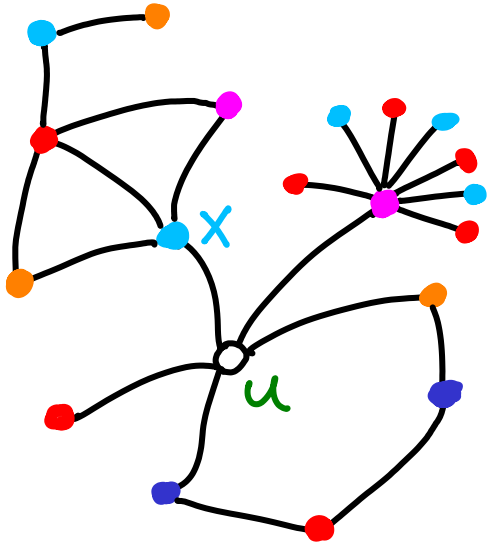
ONE PROBLEM

?







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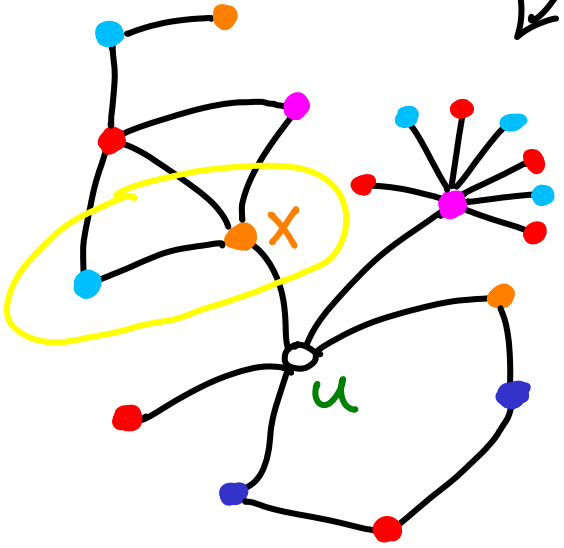

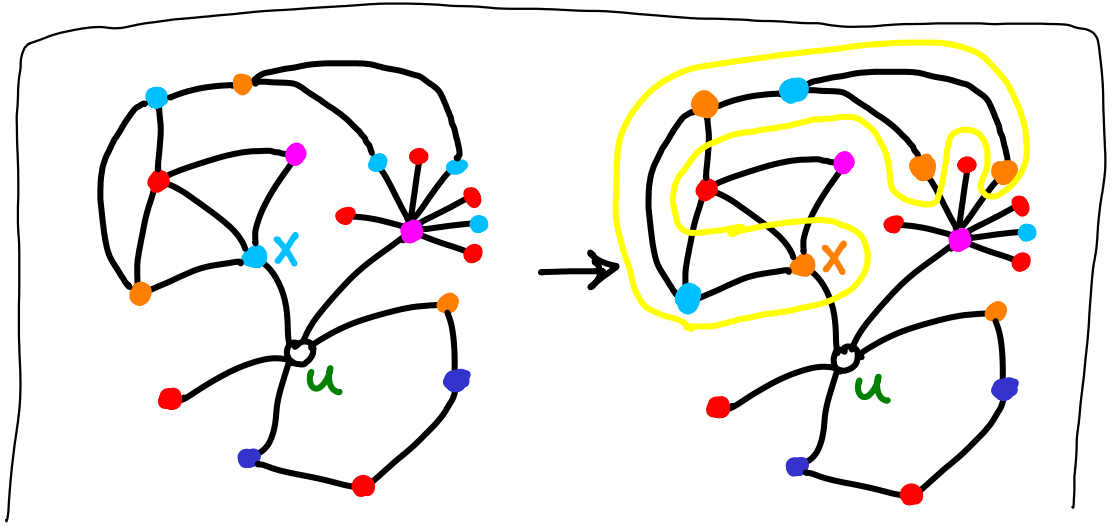

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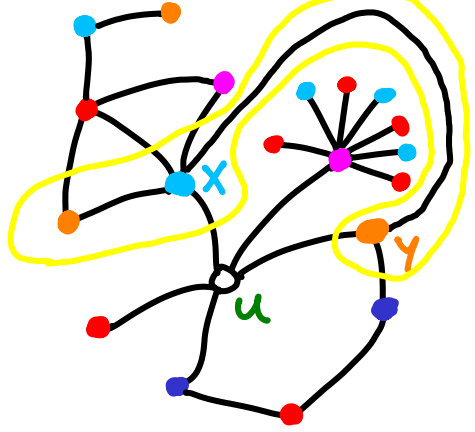
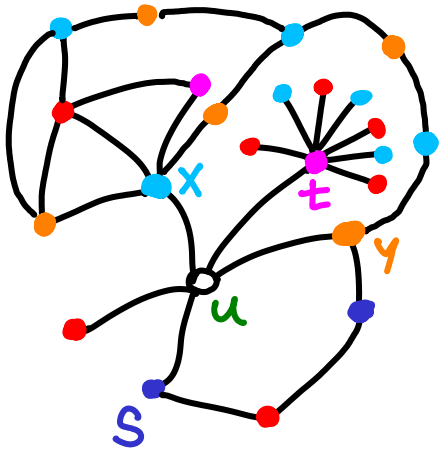
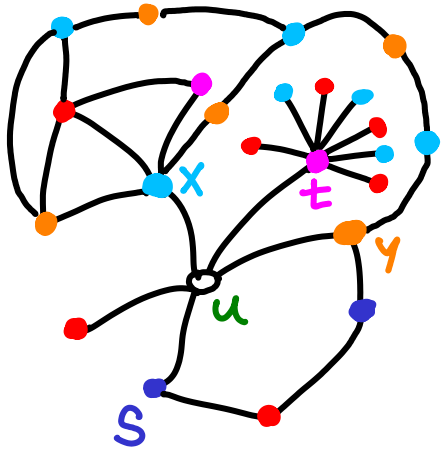




The only bad case involves a path from x to y that alternates x●■●■... y



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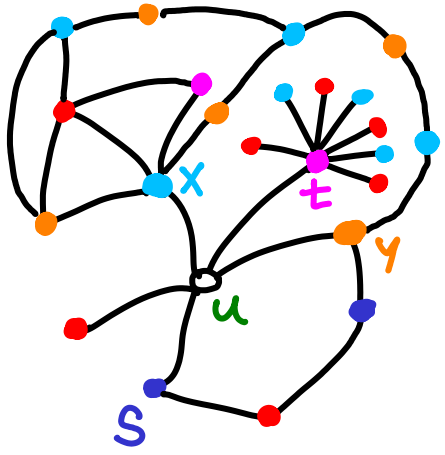
Together with (u) the path forms a cycle surrounding the ● neighbor of u.

So ... ?



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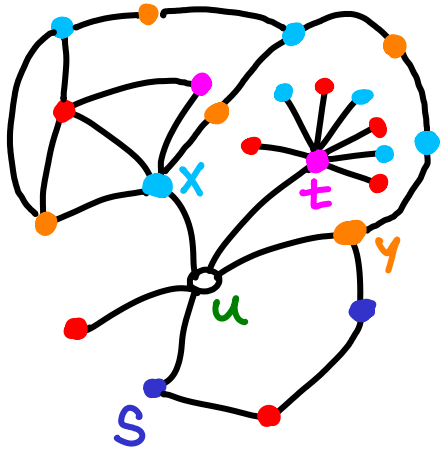
Together with u the path forms a cycle surrounding the ● neighbor of u.

Restart the entire procedure using s & t instead of x & y.



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The only way to fail is if there is a path s●●●●t