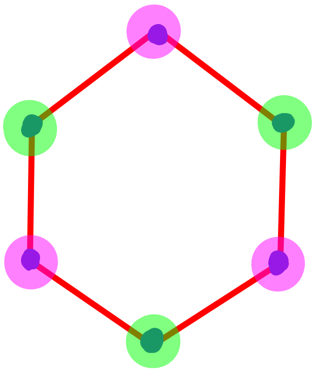


What is χ for cycles?

$\chi = 2$ if V even

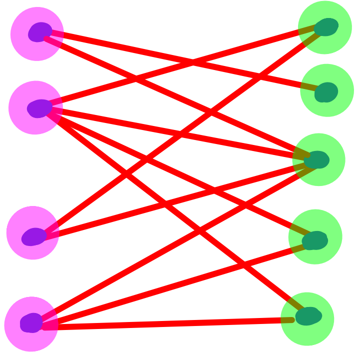
$= 3$ if V odd

Claim:

G is bipartite

$\Longleftrightarrow$

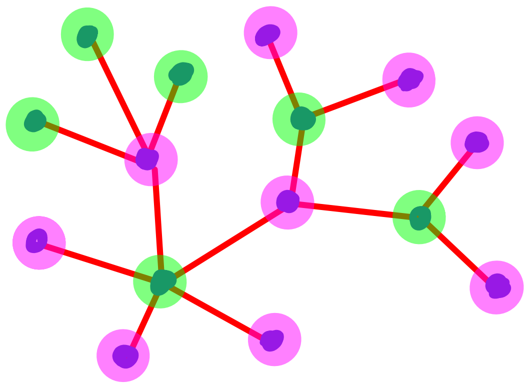
G contains no odd cycle



For bipartite graphs?

$\chi = 2$

In fact if $\chi(G) = 2$ then G is bipartite by definition



For trees?

Remove a leaf, v.

2-color the rest.

Color v opposite of p(v)

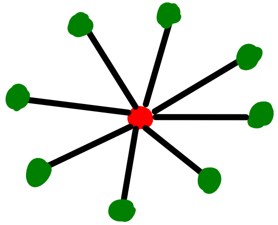
$\chi = 2$

(trees are bipartite)

If G has $n > 1$ vertices, trivial bounds : $2 \le \chi \le n$ (K_n)

What can χ be if max degree of $G = \Delta$?

$K_n \to \Delta = n-1 \;:\; \chi = \Delta + 1$

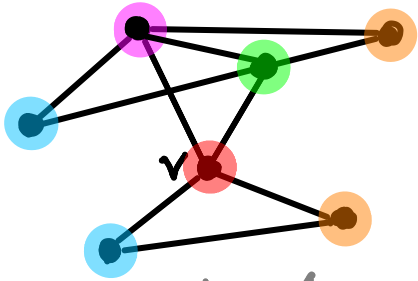


$\Delta = n-1 \;:\; \chi = 2$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)

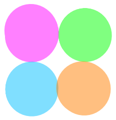
Claim $\chi \le \Delta + 1$

- Incrementally "add" vertices.
- When adding vertex v, look at all neighboring colors.
- Always have ≥ 1 color available.



$\Delta = 4$

- Remove any vertex v.
- Color $G - v$ by induction.
- Re-insert v.
- v has $\le \Delta$ neighbors.
- Use color $\Delta+1$ for v.



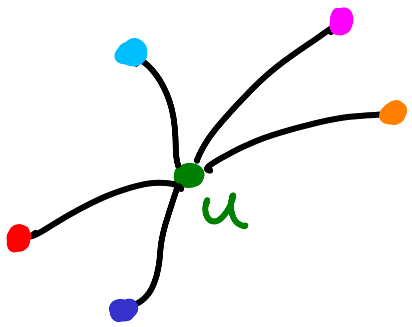
COLORING PLANAR GRAPHS (like map duals)

Claim: $\chi \le 6$ ... trivial if $V \le 6$

We know planar graphs have a vertex w/ degree ≤ 5

Given planar G s.t. $V > 6$ $\exists u \in G$, $d(u) \le 5$: look at $G-u$

Assume by induction that $G-u$ is 6-colorable

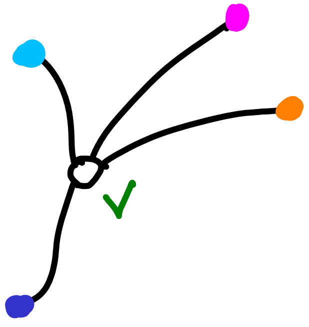


re-insert u: give it a color not used by neighbors $\square$

Claim: $\chi \leq 5$ ... trivial if $V \leq 5$

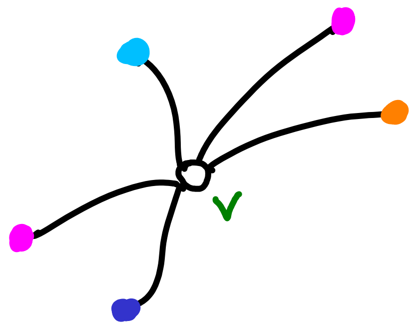
Use induction & $d(u) \leq 5$

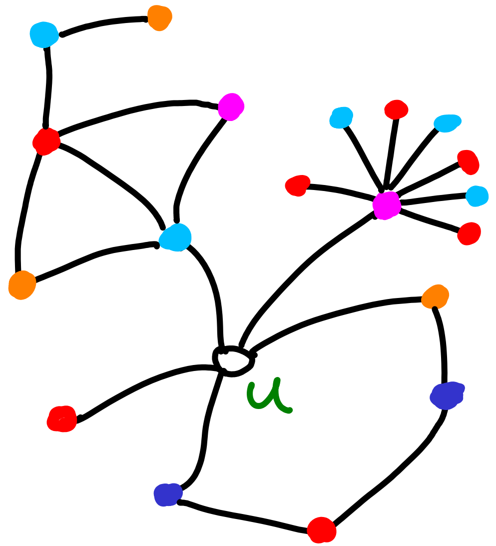
... or if $\exists v : d(v) \leq 4$



still needs induction

... or if neighbors use < 5 colors

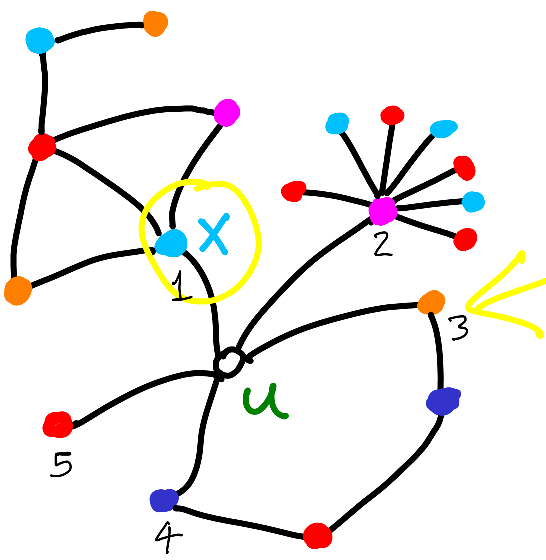




= drawing

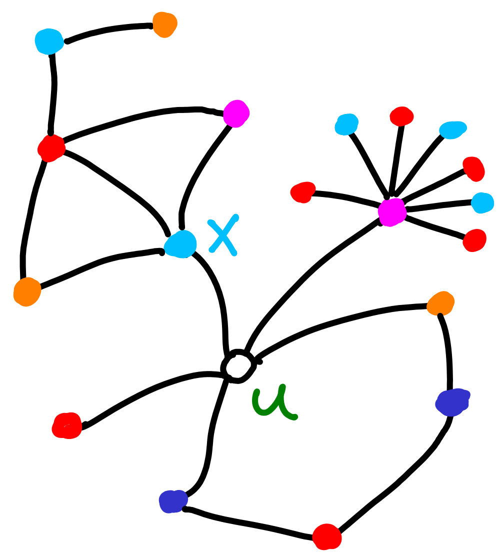
Consider any embedding of G

We need a neighbor of u to change color



Try to change x from to

[specifically skipping 2 over in adj(u)]



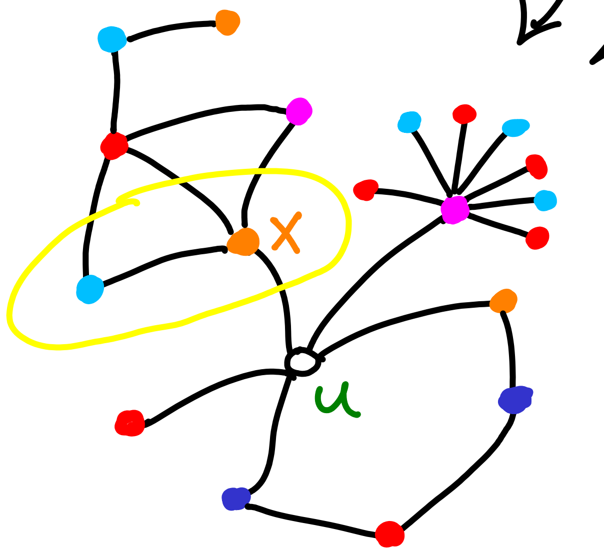
Try to change x from to

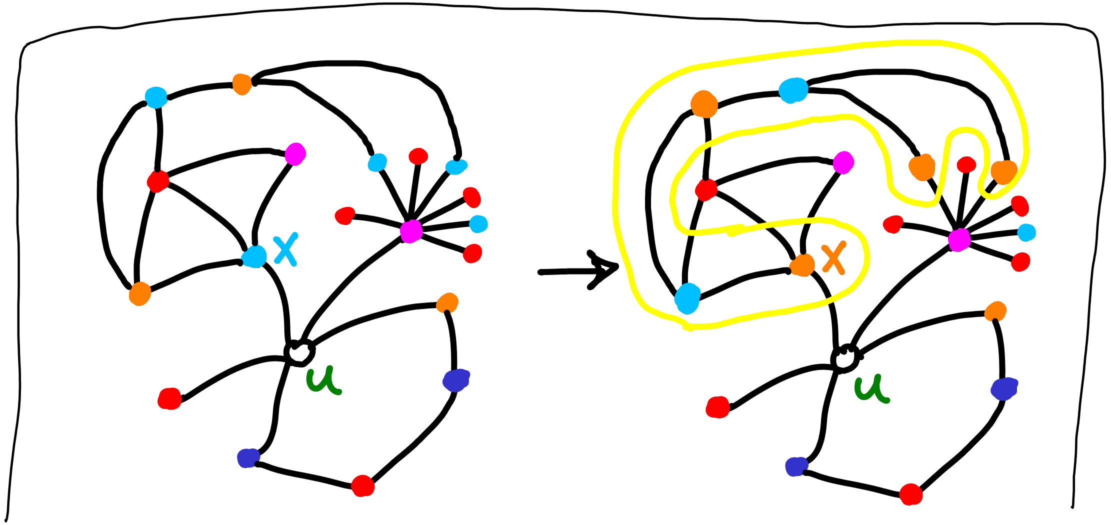
[specifically skipping 2 over in adj(u)]

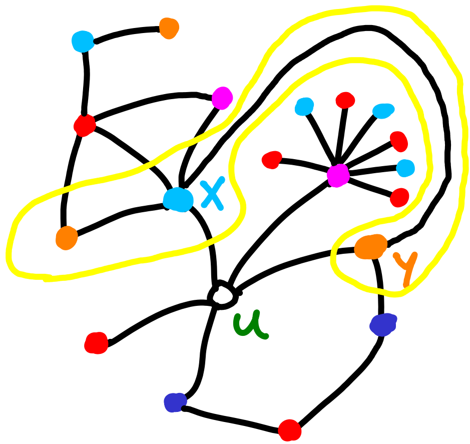
↳ This works if x has no neighbors

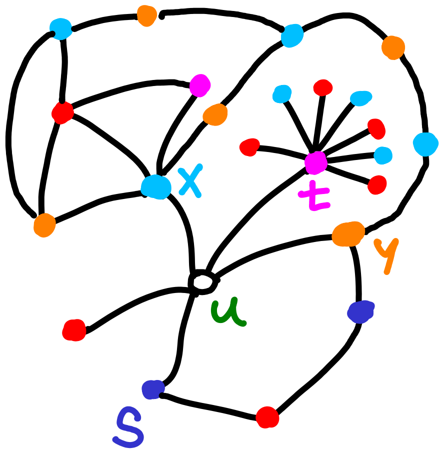
↳ else, swap colors on the connected component of the subgraph of G that contains only colors and x.

ONE PROBLEM









The only bad case involves a path from x to y that alternates x ● ● ● ●… y

Together with u the path forms a cycle surrounding the ● neighbor of u.

Restart the entire procedure using s & t instead of x & y.

The only way to fail is if there is a path s ● ● ● ● t but this would have to cross x ● ● ● ●… y

Impossible: This is a plane drawing

$\square$

Planar graphs:

6-coloring: ~ trivial

5-coloring: short elegant proof

4-coloring:
- unsolved from ≤ 1850 until 1977
- proof involved ~2000 cases solved by computer

3-coloring:
- clearly not always possible: K_4
- if triangle-free then 3-colorable

(in fact if ≤ 3 triangles)