Any graph that can be drawn in the plane without crossings.

\[ G = (V, E) \]

- \( V = 6 \)
- \( E = 8 \)
- \( F = \text{#faces} = 4 \)

Any planar graph can be drawn w/ straight edges.

A planar graph that is “embedded” (drawn) without crossings is a plane graph.
Euler Formula for planar connected graphs: \[ V - E + F = 2 \]

Proof by induction on number of faces:

Base case \( F = 1 \) \( \rightarrow \) \( G \) is a tree \( \rightarrow \) \( V = E + 1 \)

so \( (E+1) - E + 1 = 2 \) \( \checkmark \)

Given \( G = (V, E) \) \( \omega \) \( F > 1 \) faces,
remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).

Either \( f_1 \) or \( f_2 \) is a bounded face,
so \( e \) is on a cycle (\( e \) is not a cut edge)

\( \Downarrow \) \( G - e \) is connected & \( f_1, f_2 \) merge:

\( V - (E - 1) + (F - 1) = 2 \)

\( \Downarrow \) \( V - E + F = 2 \) \( \checkmark \)

Note that this also holds for multigraphs \( \infty \)
For any planar connected graph $G$ with $V > 3$, $E \leq 3V - 6$ ($\& \ F \leq 2V - 4$)

Every edge belongs to 1 or 2 faces \[ \sum_{\text{all faces}} e \leq 2E \]

Every face has $\geq 3$ edges (for $V > 3$) \[ \sum_{\text{all faces}} e \geq 3F \]

\[ V - E + F = 2 \]
\[ V - 2 = E - F \]
\[ V - 2 \geq E - \frac{2E}{3} \]
\[ 3V - 6 \geq E \]
\[ 2V - 4 \geq F \]

Also \[ V - 2 \geq \frac{3F}{2} - F \]

What if $V \leq 3$? \[ \leftrightarrow \text{then } E \leq V \]

$V = 8$  $E = 9$  $F = 3$
$E \leq 3V - 6$

$10 \leq 15 - 6$

Not planar

$E \leq 3V - 6$

$9 \leq 18 - 6$  OK!

Inconclusive

not iff

All planar graphs have $E \leq 3V - 6$

Some non-planar graphs can too
V - E + F = 2  \[\text{What if } G \text{ has no triangles?}\]

Every edge belongs to 1 or 2 faces

Every face has \(\geq 4\) edges (for \(V > 4\))

\[
\begin{align*}
E - F &= V - 2 \\
E - \frac{E}{2} &\leq V - 2 \\
E &\leq 2V - 4
\end{align*}
\]

\[
\sum e \leq 2E \quad \text{all faces} \quad \sum e \geq 4F \quad \text{all faces}
\]

\(E \geq 2F\)
\[ E \leq 3V - 6 \]
\[ 10 \leq 15 - 6 \]

\[ 9 \leq 2 \cdot 6 - 4 \]

**NOT PLANAR**

For triangle free:

\[ E \leq 2V - 4 \]
It turns out that every non-planar graph "contains" one of these two shapes.

*e.g.*

see links
\[ E = 3V - 6 \]

... for triangulations

Every edge belongs to 1 or 2 faces

Every face has 3 edges (for \( V > 3 \))

\[ \sum e \leq 2E \]
\[ \sum e \leq 3F \]
\[ 2E \geq 3F \]

\[ V - E + F = 2 \]
\[ E - F = V - 2 \]
\[ E - \frac{2E}{3} = V - 2 \]
\[ E = 3V - 6 \]
What is the average degree of a triangulation?

\[
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6
\]

Every triangulation has a vertex with degree ≤ 5

Implies the same for any planar graph

(every graph is a spanning subgraph of a triangulation)
COLORING $G \rightarrow$ no adjacent vertices get same color

$G$ is $k$-colorable if we can use $\leq k$ colors

$\chi(G) : \min \# \text{ colors we can use to color } G$

$\chi$ = color

Our map is 4-colorable $\chi \leq 4$

...but not 3-colorable

subgraph $K_4$
so $\chi \geq 4$
EXAM SCHEDULING

students: $S_1$, $S_2$, $S_3$, $S_4$, $S_5$  
classes: $c_1$, $c_2$, $c_3$, $c_4$, $c_5$

Can't schedule exam simultaneously for classes taken by $S_i$
Want to minimize exam slots.

Make $G$: $V =$ classes  $E =$ conflicts

Colors = slots (minimize colors)

If no edge has same color at endpoints,  
then no 2 classes are in same slot
EXAM SCHEDULING

students: $S_1, S_2, S_3, S_4, S_5$

classes: $C_1, C_2, C_3, C_4, C_5$

Can't schedule exam simultaneously for classes taken by $S_i$.

Want to minimize exam slots.

Make $G: V = \text{classes}, E = \text{conflicts}$

Colors = slots (minimize colors)

If no edge has same color at endpoints, then no 2 classes are in same slot.