PLANAR GRAPHS

Any graph that can be drawn in the plane without crossings

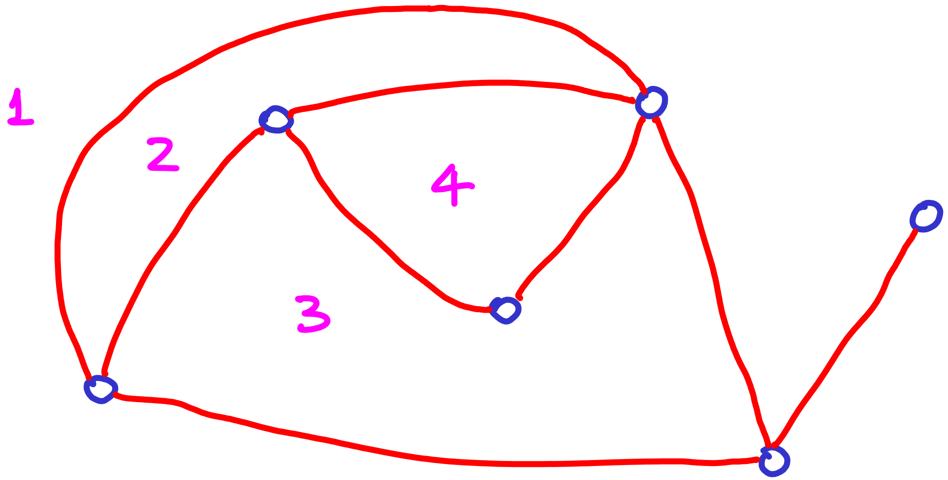
$G=(V,E)$

$V=6$

$E=8$

F = #faces = 4

disjoint regions, one of which is unbounded



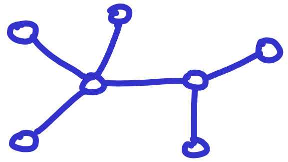
claim:

any planar graph can be drawn w/ straight edges

A planar graph that is "embedded" (drawn) without crossings is a plane graph.

EULER FORMULA for planar connected graphs: $\underline{V - E + F = 2}$

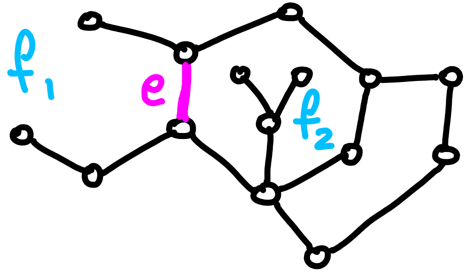
Proof by induction on number of faces:



Base case → $F = 1$ → G is a tree → $V = E + 1$

so $(E+1) - E + 1 = 2$ ✓

Given $G = (V, E)$ w/ $F > 1$ faces,



remove an edge e between 2 faces, f_1 & f_2.

Either f_1 or f_2 is a bounded face,

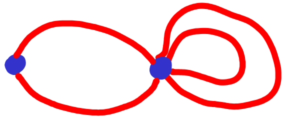
so e is on a cycle (e is not a cut edge)

↳ $G - e$ is connected & f_1, f_2 merge:

$$\overbrace{V - (E-1) + (F-1) = 2}^{\text{hypothesis}}$$

↳ $V - E + F = 2$ ✓

Note that this also holds for multigraphs



For any planar connected graph w/ $V>3$, $E \leq 3V-6$ ($\&$ $F \leq 2V-4$)

Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V>3$) $\sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

$F \leq \frac{2E}{3}$

what if $V \leq 3$?

↳ then $E \leq V$

$V-E+F=2$

$V-2 = E-F$

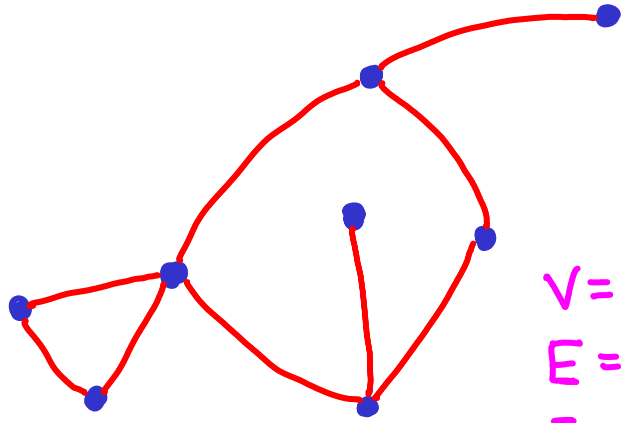
$V-2 \geq E - \frac{2E}{3}$

$3V-6 \geq E$

Also

$V-2 \geq \frac{3F}{2} - F$

$2V-4 \geq F$



$V=8$

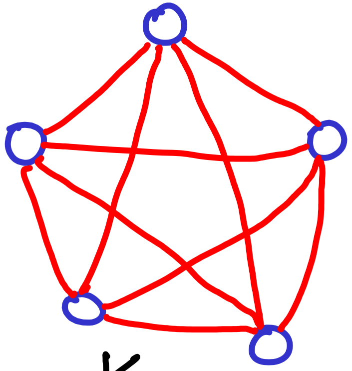
$E=9$

$F=3$

$E \leq 3V-6$

$10 \leq 15-6$

!!!



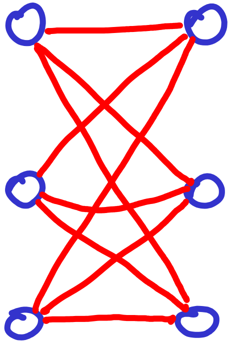
K_5

Not planar

$E \leq 3V-6$

$9 \leq 18-6$ ok!

Inconclusive



$K_{3,3}$

<u>not iff</u>

All planar graphs have $E \leq 3V-6$

Some non-planar graphs can too

$V - E + F = 2$

What if G has no triangles?

Every edge belongs to 1 or 2 faces

$$\sum_{\text{all faces}} e \leq 2E$$

Every face has $\cancel{\geq 3}$ ≥ 4 edges (for $V > 4$)

$$\sum_{\text{all faces}} e \geq 4F$$

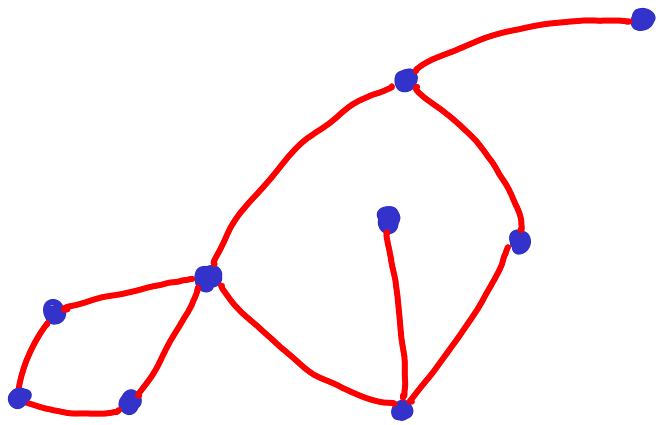
$E \geq 2F$

$E - F = V - 2$

$E - \frac{E}{2} \leq V - 2$

$E \leq 2V - 4$

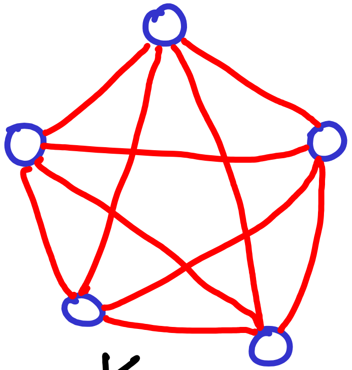
instead of $\leq 3V - 6$



$E \leq 3V-6$

$10 \leq 15-6$

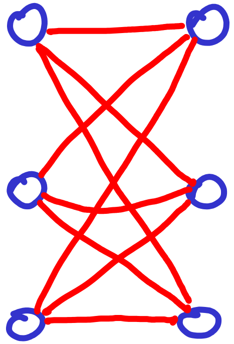
!!!



K_5

NOT PLANAR

$V=6,\ E=9$

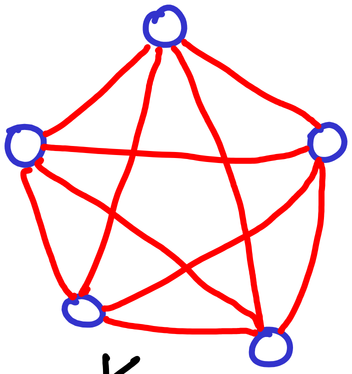


$K_{3,3}$

for triangle free: $E \leq 2V-4$

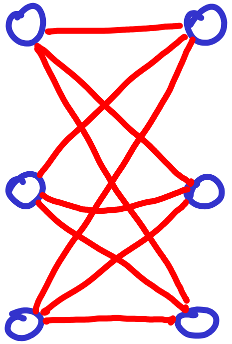
$9 \leq 2 \cdot 6 - 4$

!!!



K_5

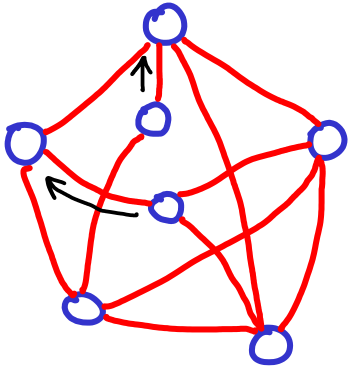
⟵ non-planar ⟶



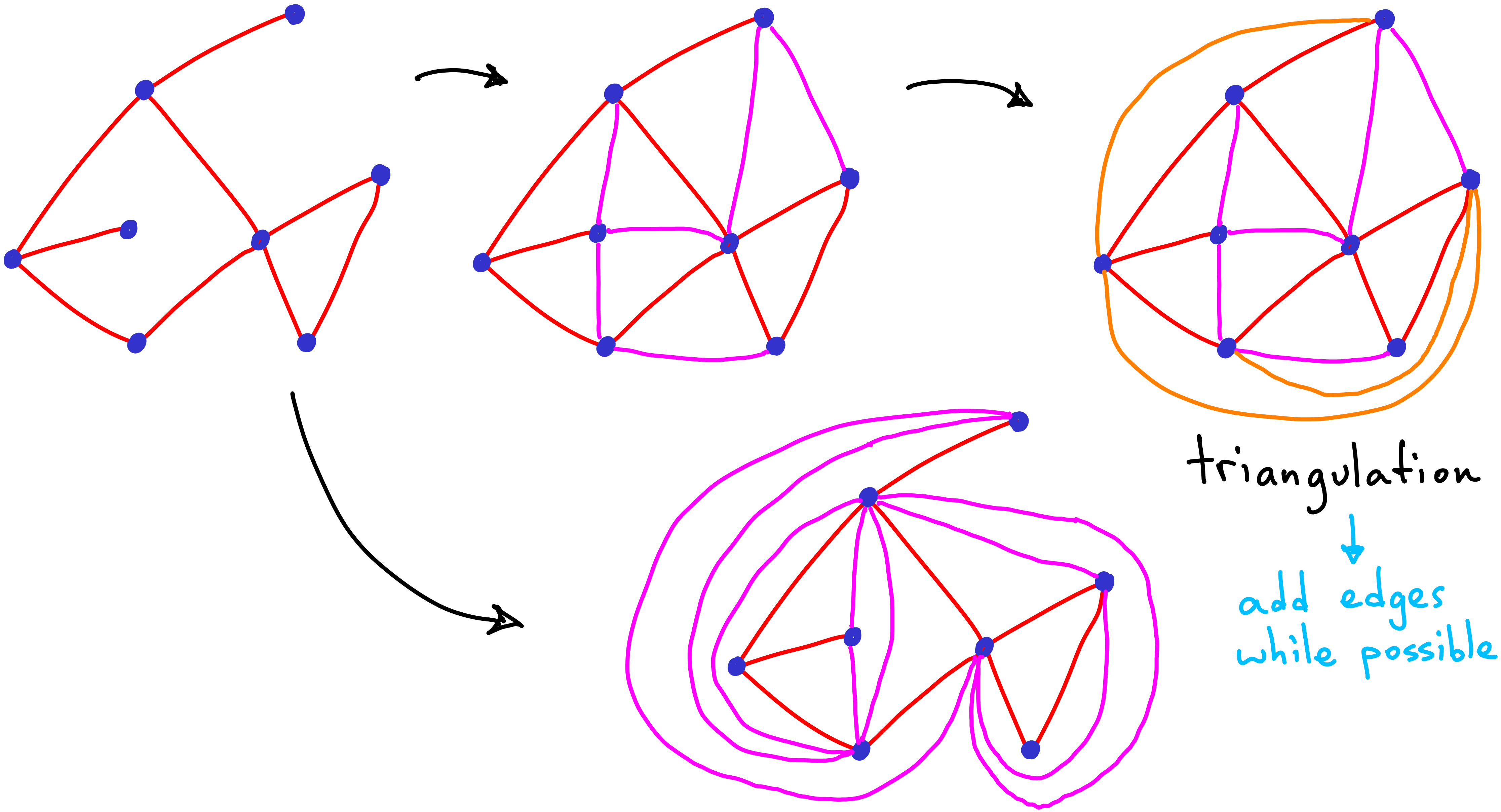
$K_{3,3}$

It turns out that every non-planar graph "contains" one of these two shapes.

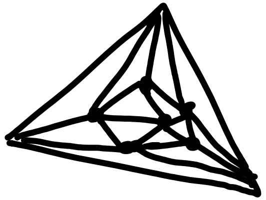
e.g:



see links


triangulation
add edges
while possible

$E = 3V-6$



... for triangulations

Every edge belongs to ~~1 or~~ 2 faces $\quad \sum_{\text{all faces}} e \overset{=}{\cancel{\le}} 2E$

Every face has $\overset{=}{\cancel{\ge}}$ 3 edges ~~(for $V>3$)~~ $\quad \sum_{\text{all faces}} e \overset{=}{\cancel{\ge}} 3F$

$$2E \overset{=}{\cancel{\ge}} 3F$$

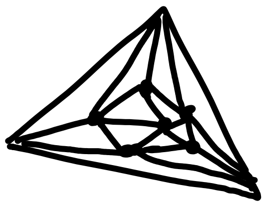
$$V - E + F = 2$$

$$E - F = V - 2$$

$$E - \frac{2E}{3} = V - 2$$

$$E = 3V - 6$$

$E = 3V - 6$



What is the average degree of a triangulation?

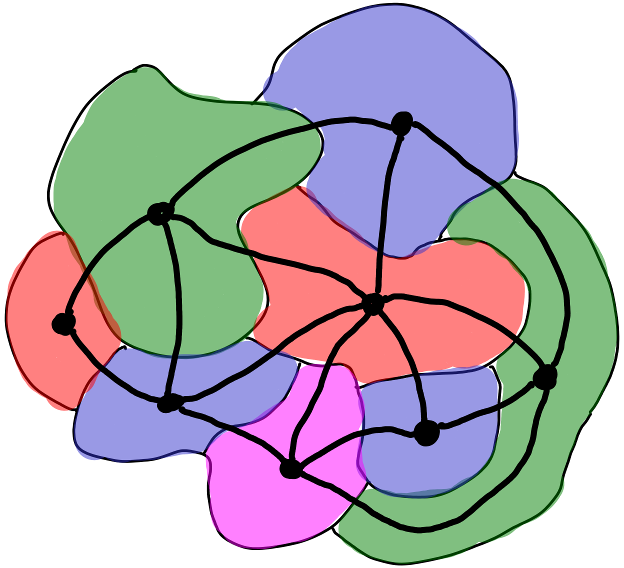
$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

Every triangulation has a vertex with degree ≤ 5

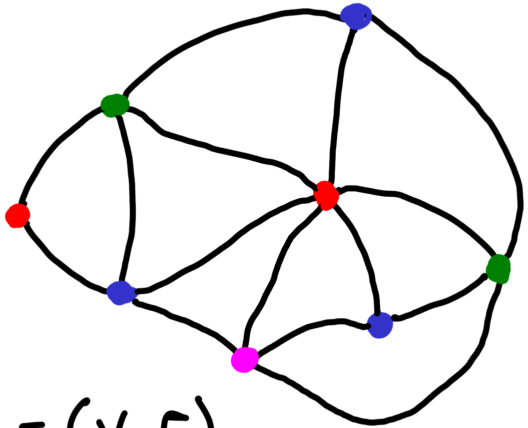
Implies the same for any planar graph

(every graph is a spanning subgraph of a triangulation)

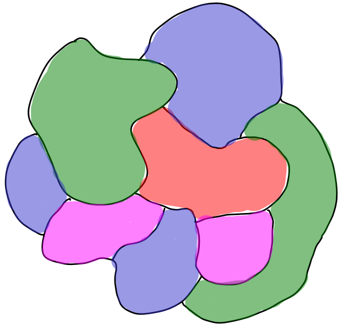
MAP COLORING

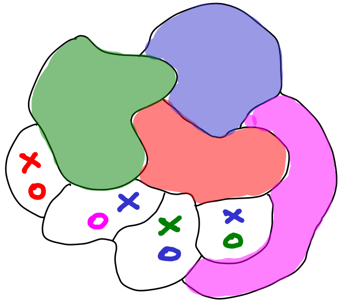


$\Rightarrow$



$G=(V,E)$





etc

COLORING G → no adjacent vertices get same color

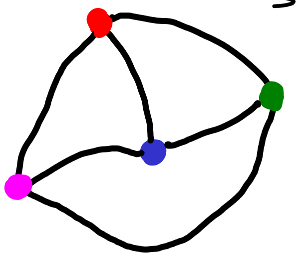
G is k-colorable if we can use $\leq k$ colors

$\chi(G)$: min # colors we can use to color G

chromatic number

χρῶμα = color

Our map is 4-colorable $\Big\}\ \chi \leq 4$

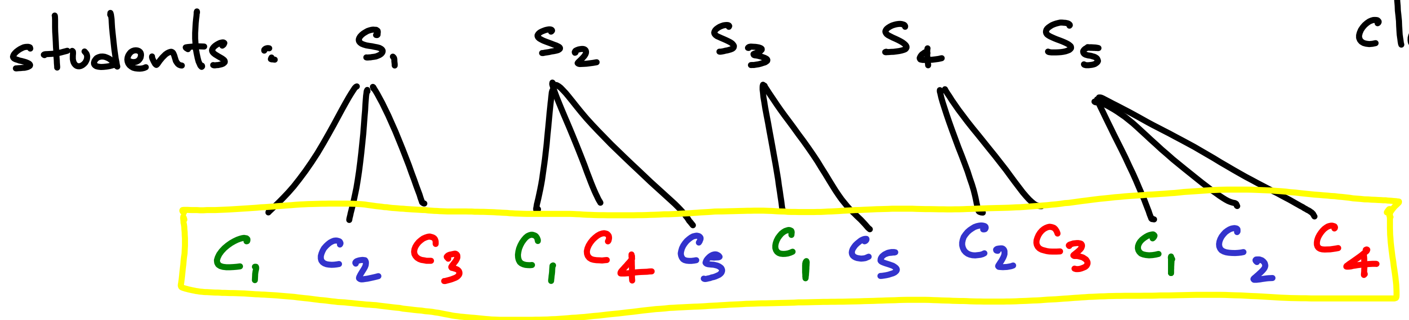


subgraph K_4

so $\chi \geq 4$

...but not 3-colorable

EXAM SCHEDULING

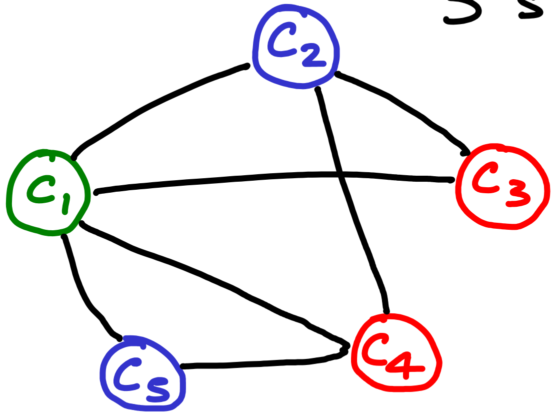


classes c_1 c_2 c_3 c_4 c_5

Can't schedule exam simultaneously for classes taken by s_i

Want to minimize exam slots.

3 slots



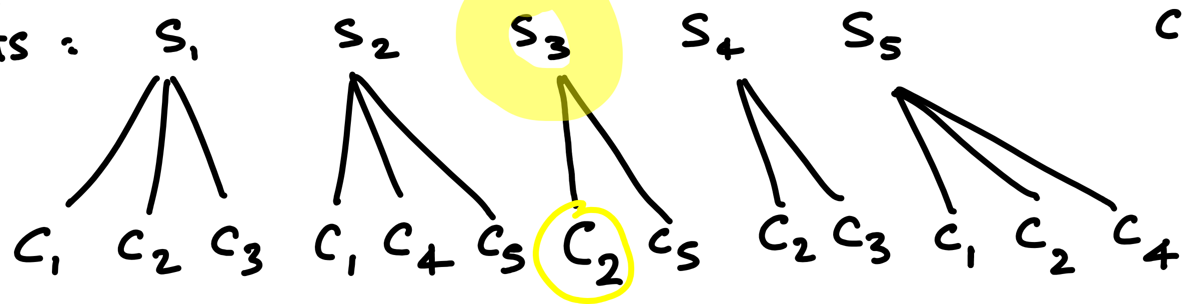
Make G: V = classes E = conflicts

Colors = slots (minimize colors)

If no edge has same color at endpoints,
then no 2 classes are in same slot

EXAM SCHEDULING

students :



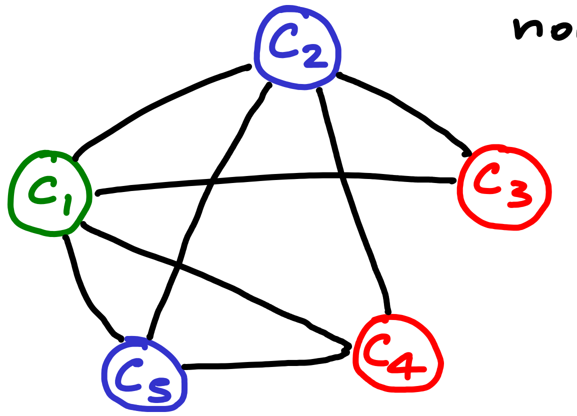
classes c_1 c_2 c_3 c_4 c_5

Can't schedule exam simultaneously for classes taken by s_i

Want to minimize exam slots.

Make G: V = classes E = conflicts

Need 4 now



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