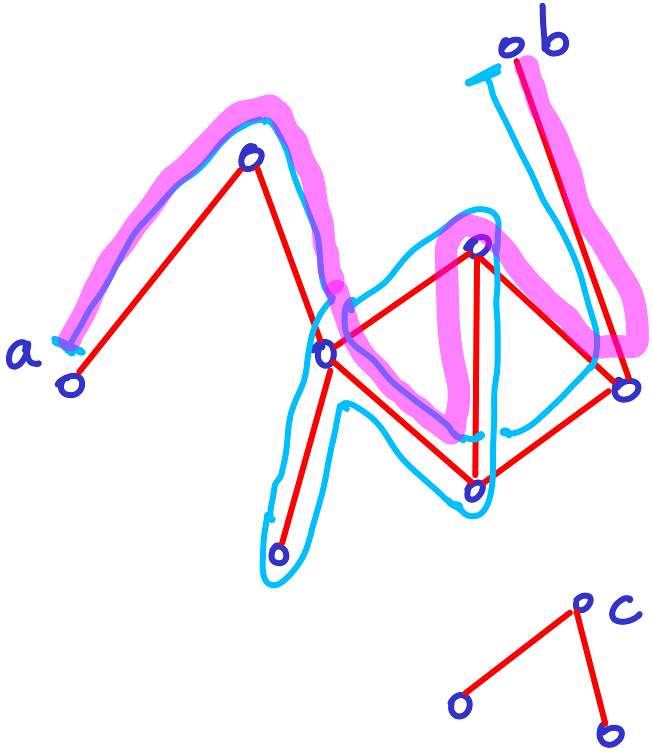
GRAPH CONNECTIVITY



A **walk** is a sequence of vertices $v_i, v_{i+1}, v_{i+2}, \dots, v_k$

s.t. every v_j, v_{j+1} is an edge $(i \le j < k)$

We can walk from a to b, but not from a to c.

A **path** is a walk with distinct vertices

GRAPH CONNECTIVITY

A vertex x is connected to a vertex y (in G)

if there is a path from x to y

could say "walk"

If there is an x-y walk in G then there is an x-y path in G.

↳ Pick the shortest x-y walk that is not a path.

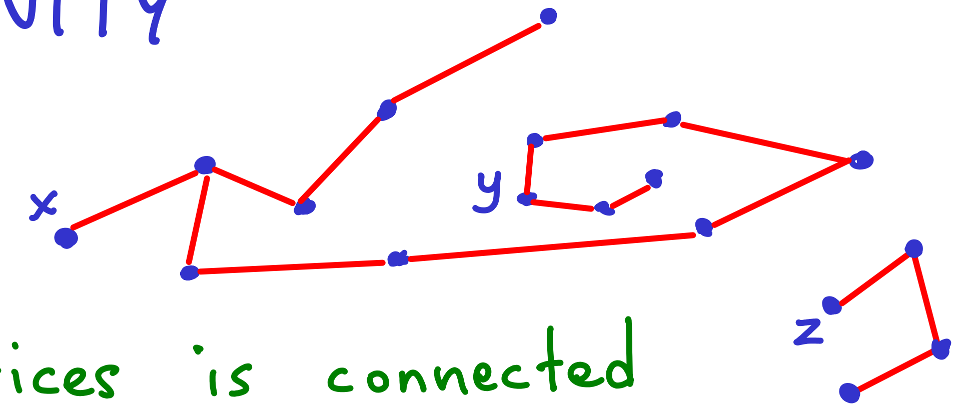
It must have a repeated vertex, u } $x \dots u \underbrace{\dots}_{\text{remove}} u \dots y$

CONTRADICTION

GRAPH CONNECTIVITY

A graph is connected

if every pair of vertices is connected



If vertex x is connected to vertex y

then they are in the same component.

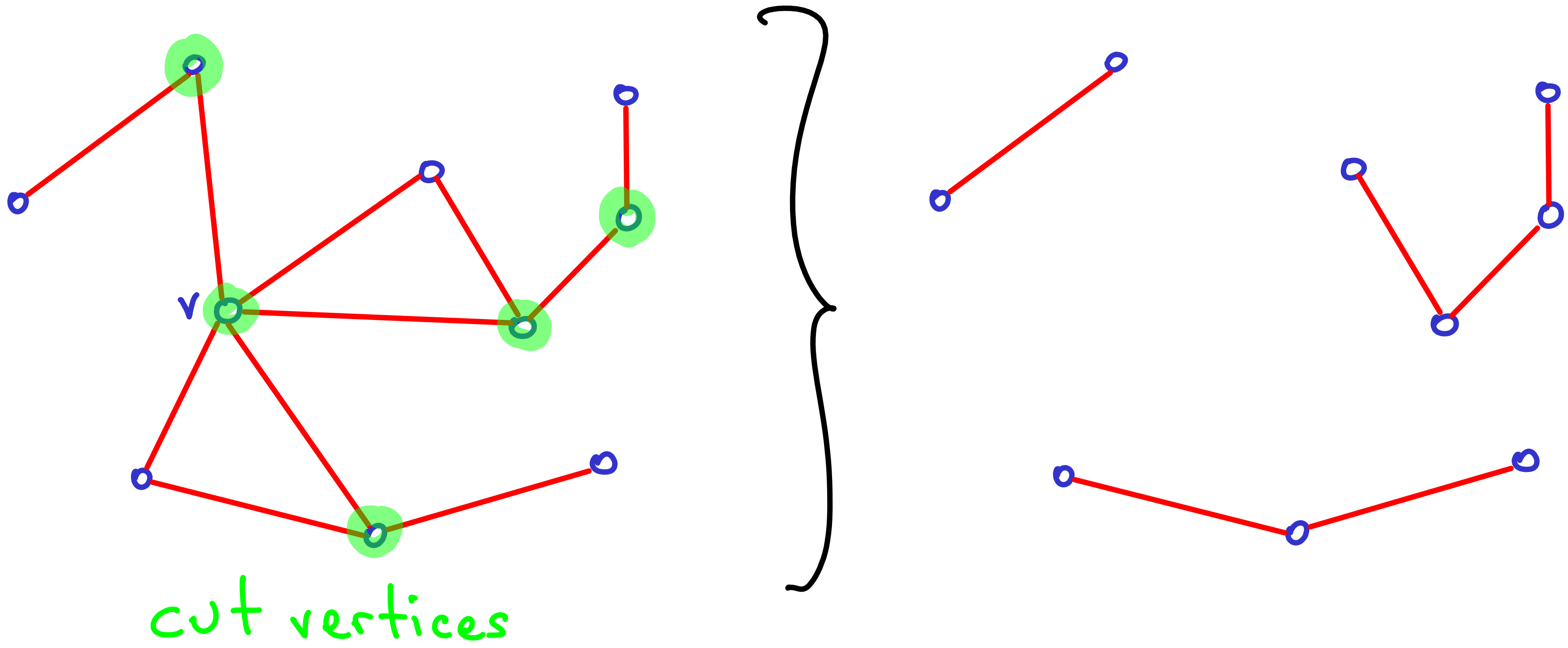
If x & y are in the same component

but x & z are not

then y & z are not.

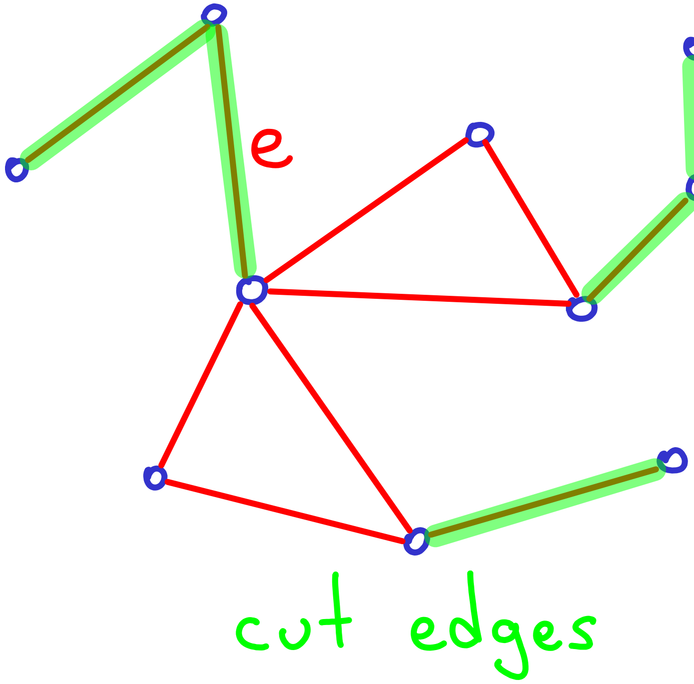
Given G, remove a vertex: $G - v$

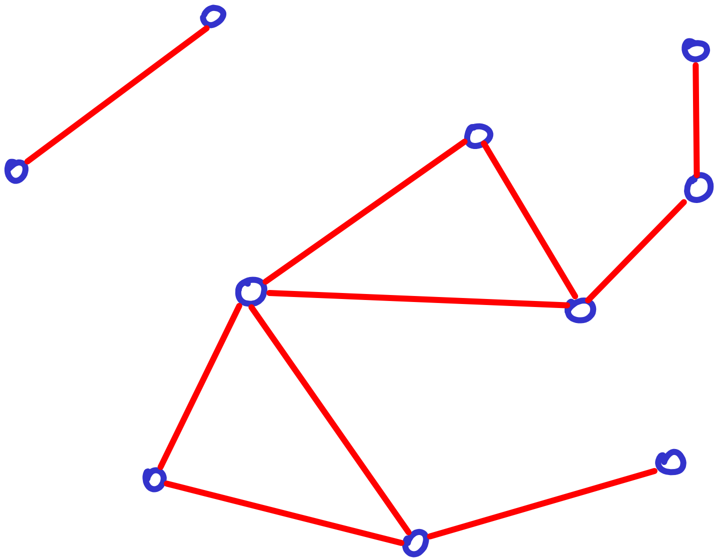
If $G - v$ has more components than G, then v is a cut vertex.



Given G, remove an edge : $G-e$

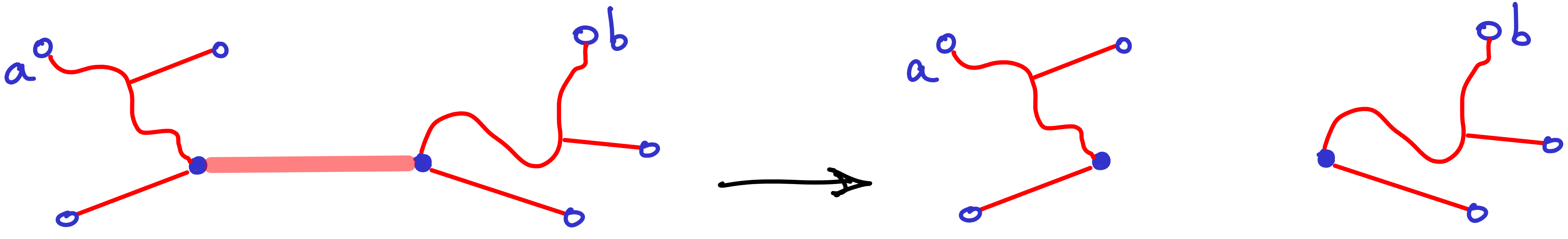
If $G-e$ has more components than G, then e is a cut edge.

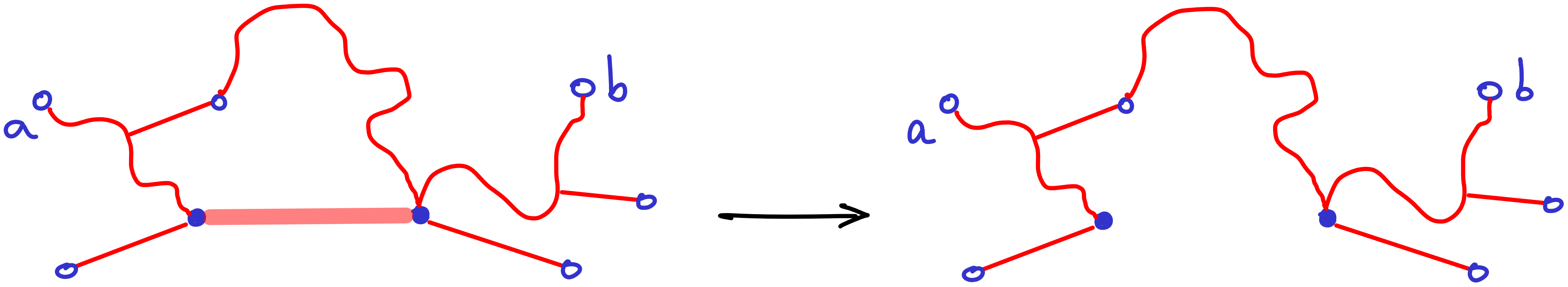




Claim: a cut edge can't be on a cycle.

(a cycle is a path w/ start = end)



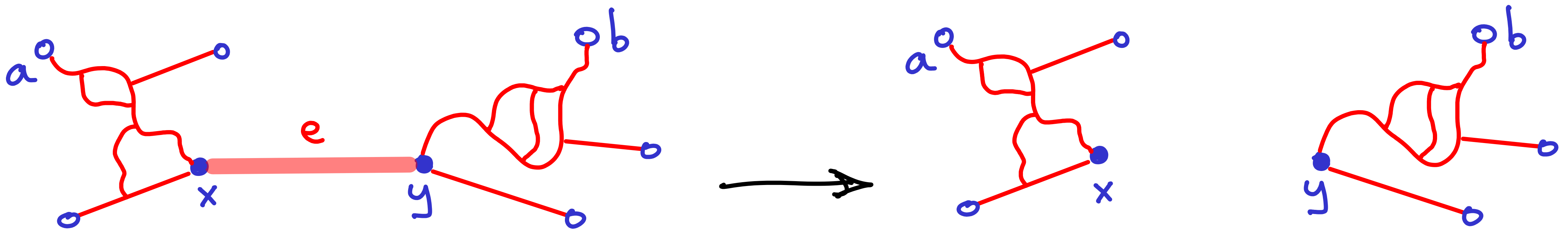


...cont'd: if e is a cut edge then $\exists\ a, b$ s.t.

a is not connected to b in $G - e$.

So all paths from a to b in G use e,

i.e traverse $a \cdots xy \cdots b$ where $e = \overline{xy}$.



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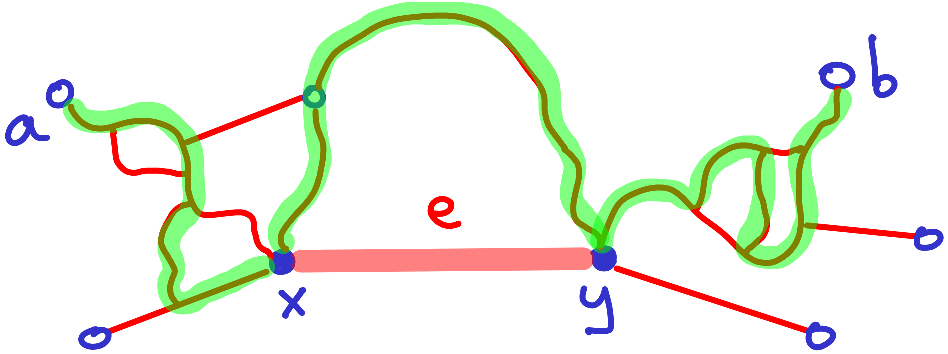


But if e is on a cycle then we can walk from a to x, walk from x to y, without using e, and walk from y to b.

CONTRADICTION

Claim: Removing a cut edge $e=(x,y)$ increases the number of components by 1.

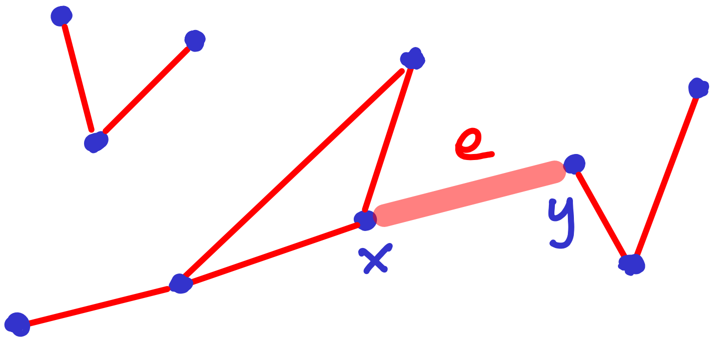
Types of vertex pairs (a,b) in G:

0) no path exists between a & b

1) not all paths between a & b use e

2) all paths use e

(path exists)



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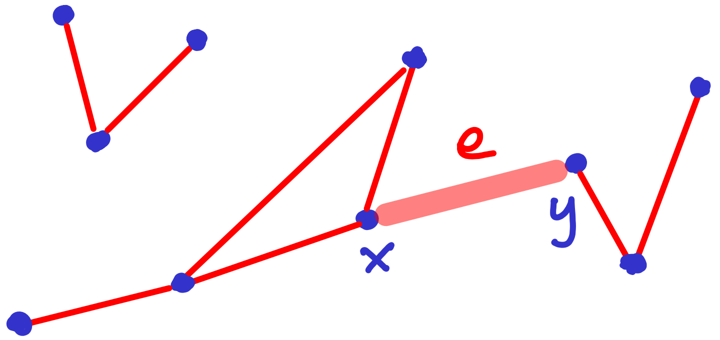


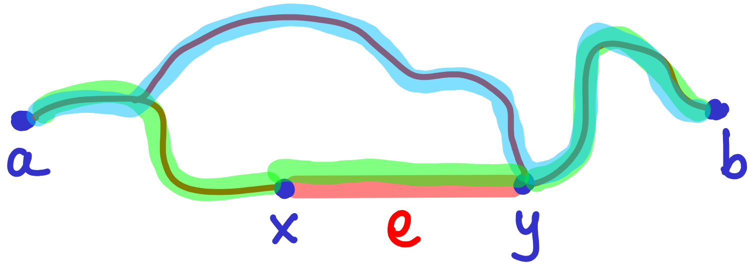
Types of vertex pairs (a,b) in G:

0) no path exists between a & b
1) no paths between a & b use e
2) all paths use e

CONTRADICTION

If path P uses e
& path Q doesn't
then e is on a cycle

but cut edges don't exist on cycles



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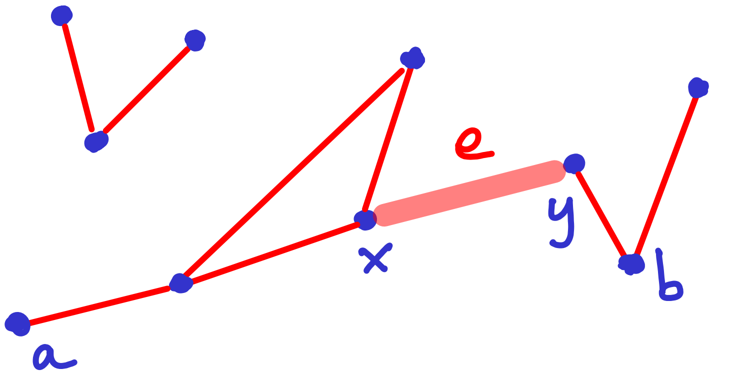


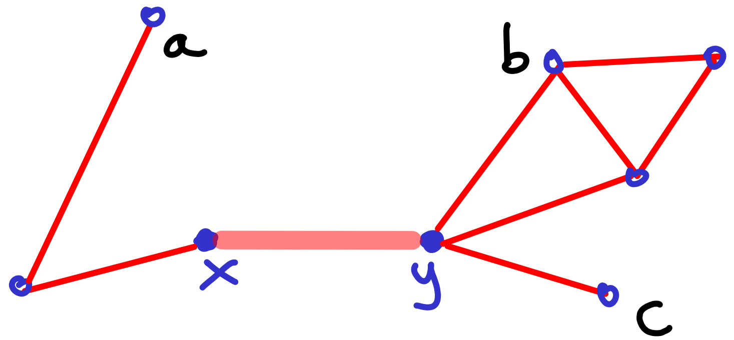
0) no path exists between a & b
1) no paths between a & b use e
2) all paths use e

Type 0 or 1: not affected by removal of e

Type 2: implies a connects to x but not to y
& b connects to y but not to x } in $G-e$

↳ Type 2 partitions one component into two.

Claim: Removing a cut edge $e=(x,y)$ increases the number of components by 1.



* e can only affect the component it's in. So focus on connected graphs.

Proof by contradiction.

Suppose $G-e$ has ≥ 3 components. $\exists a,b,c$ in different components.

In G, all paths $a\to b$ use e } wlog $a\to x\to y\to b$

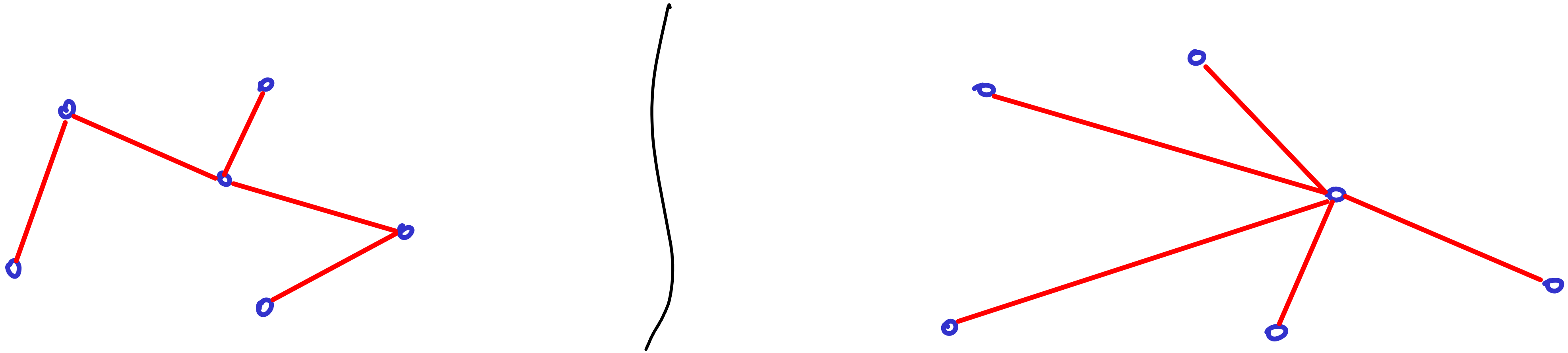
all paths $a\to c$ use e { if $a\to x\to y\to c$

⇒ b & c in same component

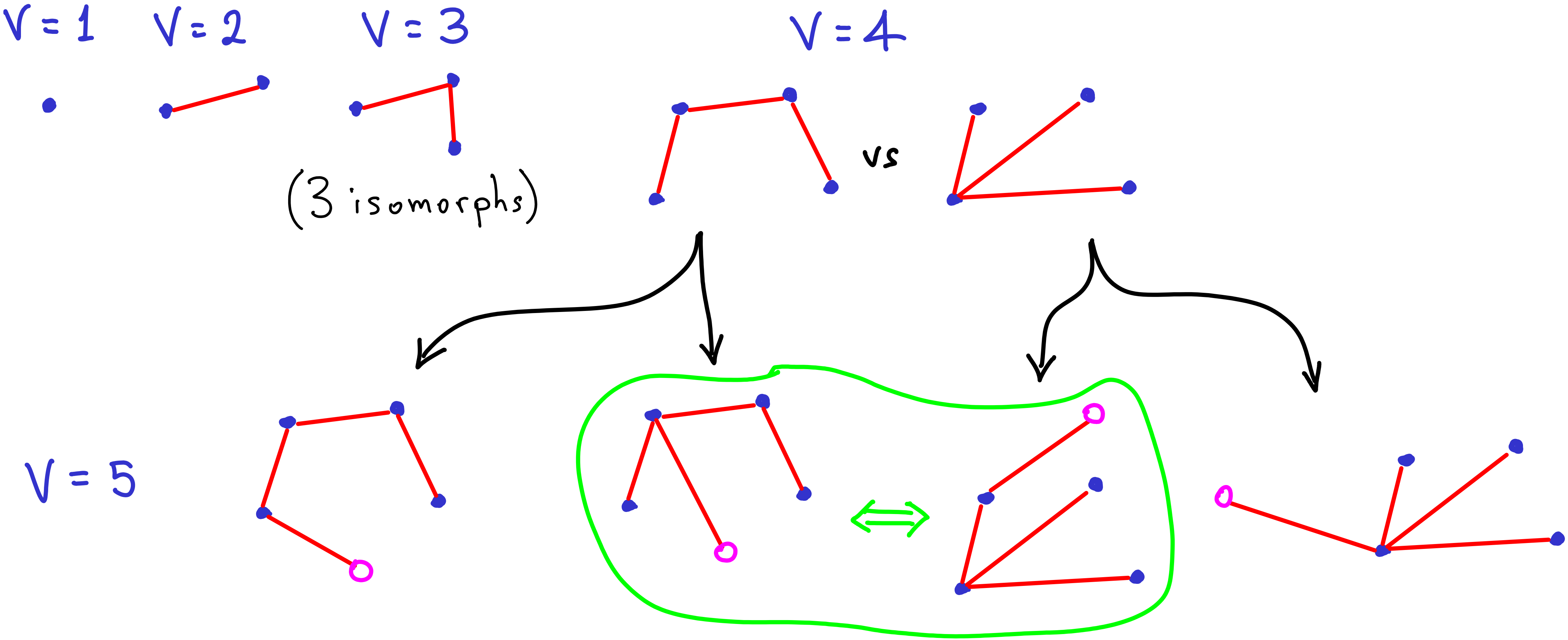
if $a\to y\to x\to c$

⇒ e not a cut edge

TREES : CONNECTED ACYCLIC GRAPHS



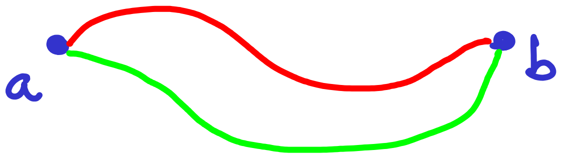
FORESTS : ACYCLIC GRAPHS (collections of trees)

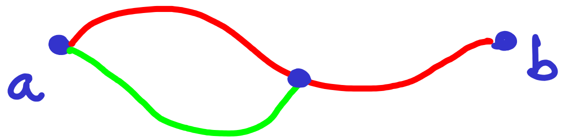

V=1
V=2
V=3
(3 isomorphs)
V=4
vs
V=5

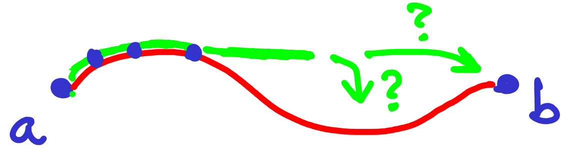
tree $\iff$ there is a unique path between every pair of vertices

$\Rightarrow$ • for any vertices a, b : a path exists (trees are connected)

• suppose ≥ 2 paths.







cycle : contradiction of tree: acyclic

$\Leftarrow$ • if for every 2 vertices a path exists, then graph is connected

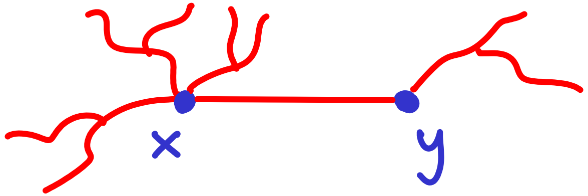
• if any 2 vertices are on a cycle, then they are on ≥ 2 paths but we assume unique paths, so no 2 vertices are on a cycle.

acyclic

$\square$

For any connected graph,

tree $\Longleftrightarrow$ every edge is a cut edge

$\Rightarrow$ 

$\left.\right\}$ for any x, y $\left.\right\}$ tree $\Rightarrow$ unique path from x to y

$\Rightarrow$ $\overline{xy}$: cut edge

$\Leftarrow$ suppose graph $\neq$ tree.

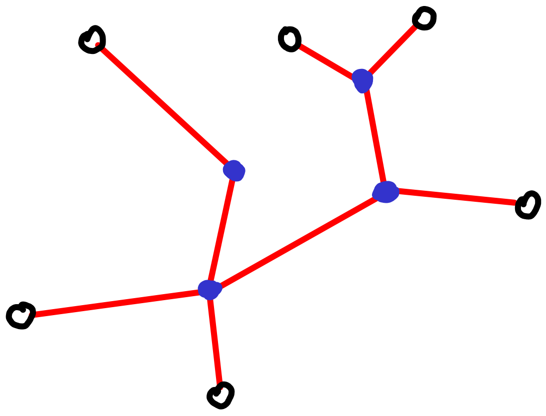
Then it has a cycle.

We have proved: A cut edge can't be on a cycle.

not every edge is a cut edge. (CONTRADICTION)

LEAVES : vertices of degree 1

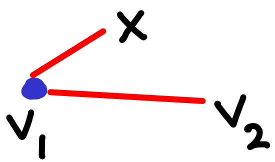
If $V \geqslant 2$, then T has $\geqslant 2$ leaves



Consider longest path in T. $v_1 \dots v_k$

$(k \geqslant 2)$

If $v_1 \neq$ leaf, then



$x \neq v_i$ (not on path)

($x = v_i$ would create cycle)

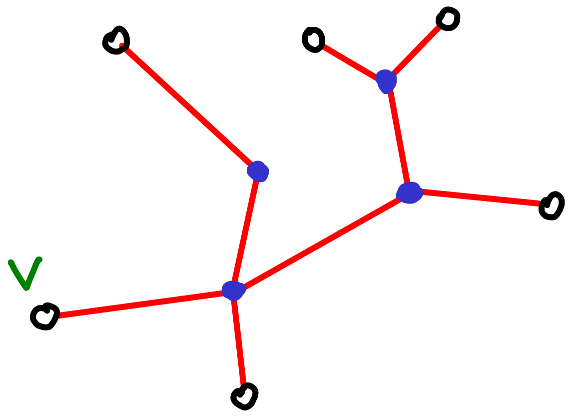
Then $x v_1 \dots v_k$: longer path : CONTRADICTION

So v_1 & v_k : leaves

If v is a leaf in tree T, then $T-v$ is a tree

- removing v doesn't create cycles.

- removing v doesn't disconnect.
  ($v \neq$ cut vertex ; $T-v$ is connected)



↳ if v were a cut vertex, then $\exists\, a, b$ ($a \neq v$, $b \neq v$) s.t.
any path $a \to b$ must use v.
But v is a dead end:
can't be part of such a path

in fact,
"the only path"
(T : unique paths)

If v is a leaf in tree T, then $T-v$ is a tree

This allows us to use induction

ex: if $|V(T)| = n \geqslant 2$ then $|E(T)| = n-1$

pf: Base case: $n=2$  trivial

Hypothesis: for $2 \leq k < n$, statement holds.

Suppose T has n vertices. Find a leaf v & delete.

- v had degree 1, so we delete 1 edge.
- $T-v$ is a tree, w/ $n-1$ vertices $\rightarrow$ $n-2$ edges.
- Replace v: total edges $= n-2+1 = n-1$

Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

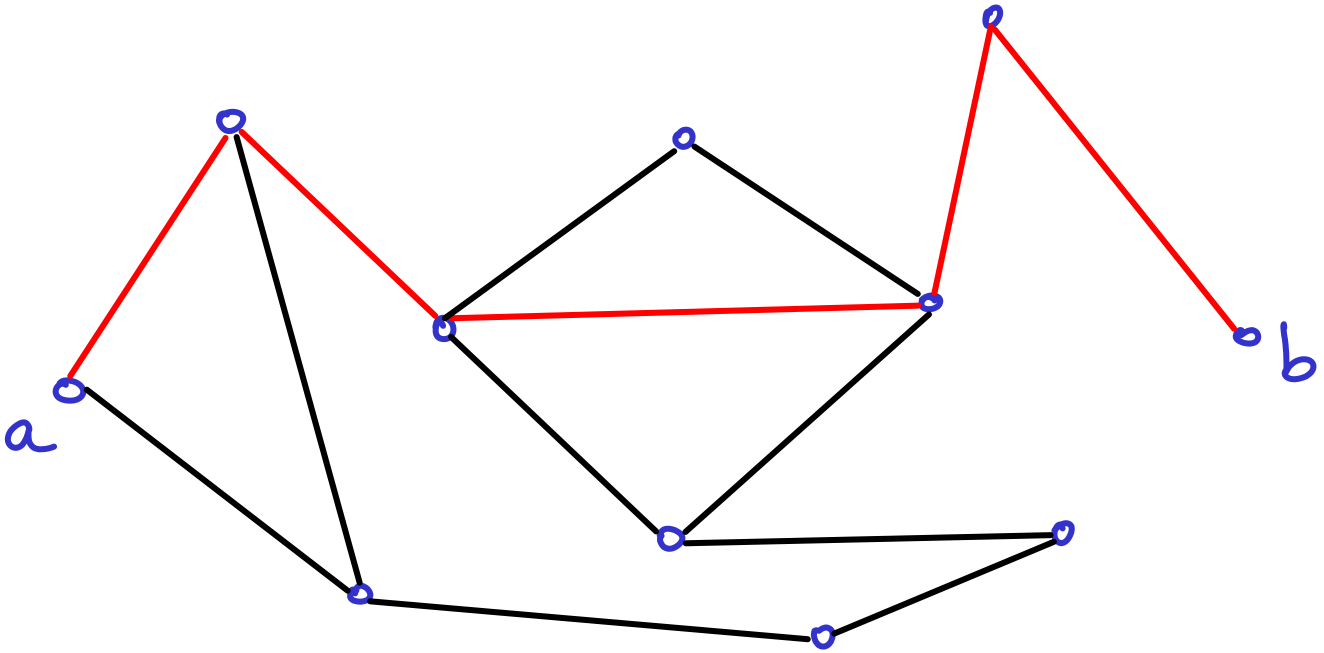
Also true: for connected G with $n \geq 1$ vertices,

if $|E(G)| = n-1$ then G is a tree

See p. 354

Also defines spanning trees → comp 160

DISTANCE IN GRAPHS



$d(a,b) = 5$

$d(a,b)$ = length of shortest path between a & b

more in COMP-160