Clique

Given $G$, a subset $S$ of $V(G)$ is a clique if every $s_i, s_j \in S$ share an edge in $G$.

The induced subgraph obtained by removing all but $S$ from $V(G)$ is a complete graph $(K_s)$.

How many cliques?
What is the largest clique?
Independent Sets

Given $G$, a subset $S$ of $V(G)$ is an independent set if no $s_i, s_j \in S$ share an edge in $G$.

The induced subgraph obtained by removing all but $S$ from $V(G)$ is an edgeless graph.

i.e. its complement is a complete graph

Largest independent set?
cliques vs. independent sets

max clique

complement

max ind. set

max ind. set

max clique
cliques vs. independent sets

max clique

max ind. set

complement

max ind. set

max clique
Claim: Every graph with $|V| \geq 6$ contains a triangle (clique of size 3) OR an independent set of size 3

Rephrase: (for $|V| \geq 6$) a graph contains a triangle or its complement does

Proof: pick any vertex $v$.

If $d(v) \geq 3$ we have $v \xrightarrow{y} x \xrightarrow{z} y$.

If $xy$ or $xz$ or $yz$: we find a clique $\Delta$.

Otherwise, $x, y, z$ are an independent set.

If $d(v) \leq 2$, there are $\geq 3$ vertices not neighboring $v$. $\rightarrow v \xrightarrow{a} x \xrightarrow{b} c$.

If $ab, bc, ac$ are edges, they are a clique $\Delta$.

Otherwise one edge is missing (w.l.o.g. $ab$) ... so $vab$ is an ind. set. \[\square\]
(for $n \geq 6$) a graph contains a triangle or its complement does.

Equivalent statement:
If we color each edge of $K_n$ red or black, then we must get a red triangle or a black triangle.
Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as $|V| \geq 6$

$\Rightarrow R(3,3) = 6$
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$\Rightarrow R(3,3) = 6$

$R(4,4) \quad R(5,5) \quad R(n,n) \sim \text{exponential}$

$18 \quad \text{we don't know!} \quad [43...49]$

even for small values we will probably never know the exact answer

For more, see Ramsey's Thm.
\(R(x, y)\) : smallest number \(N\) such that any graph with \(N\) vertices has a clique of size \(x\) or an independent set of size \(y\).

\[R(4, 2) = 4\]
$R(4,3) \leq 10$

Suppose $|V| \geq 10$

Pick any vertex, $A$. $\geq 9$ vertices remain.

Form 2 groups:

- $S$ & $T$
- $S$: edges to $A$
- $T$: no edges to $A$

If $|S| \geq 6$, use $R(3,3) = 6$: $S$ has 3 independent vertices (done), or $S$ has a 3-clique, so with $A$ we get a 4-clique.

If $|T| \geq 4$, if $T$ is a clique: done.
Otherwise $\exists a,b \in T$ w/ no edge. Combine w/ $A$. □
$R(4,3) \leq 10$

$R(4,3) > 8$

... turns out $R(4,3) = 9$

⇒ not terribly hard

⇒ notice $R(x,y) = R(y,x)$
This is where class ended,
but you should be able to follow the previous examples
and work through the rest as well.
Suppose $|V| \geq 18$

Pick any vertex, A. \geq 17 vertices remain.

Form 2 groups: $S$ \& $T$

\[ R(4,4) \leq 18 \]

If $|S| \geq 9$, use $R(3,4) = 9$ : $S$ has 4 independent vertices (done)

\text{or } S \text{ has a 3-clique, so with A we get a 4-clique.}

If $|T| \geq 9$, use $R(4,3) = 9$ : $T$ has a 4-clique, (done)\text{ or }

\text{T has 3 independent vertices, so with A we have 4.}
\[ R(4,4) \leq 18 \]

Suppose \(|V| \geq 18\)

Pick any vertex, A. \(\geq 17\) vertices remain.

Form 2 groups: 

\[ S \text{ } \& \text{ } T \]

edges to A

no edges to A

\[ \begin{cases} |S| \geq 9 & \text{either} \\ |T| \geq 9 & \text{or} \end{cases} \]

If \(|S| \geq 9\), use \(R(3,4) = 9\): S has 4 independent vertices (done) or S has a 3-clique, so with A we get a 4-clique.

If \(|T| \geq 9\), use \(R(3,4) = 9\) on the complement graph. (SYMMETRIC)
Notes:

(1) if we only knew that $R(4, 3) \leq 10$ (instead of $= 9$)

we could have used $|V| \geq 20$ for $R(4, 4)$

(2) there is a graph w/ 17 vertices with no clique or independent set of size 4

$\Rightarrow R(4, 4) = 18$