Graphs
Graphs

Vertices and edges
GRAPHs
vertices and edges
Graphs

Vertices and edges
Graphs

vertices and edges
GRAPHS

vertices and edges
GRAPHS

vertices and edges
Networks

NYC → Boston
NYC → LA
NYC → Miami
NYC → London
London → Paris
London → Madrid
Paris → Madrid
Social networks

- Stan
- Eric
- Kyle
- Kenny

Relationships:
- Stan to Kenny
- Eric to Kenny
- Kyle to Stan
- Kyle to Eric
Scheduling

Directed Graph

Discrete \(\rightarrow\) Algorithms \(\rightarrow\) Advanced Comp. Geom.

Data Structures \(\rightarrow\) Algorithms

Comp. Geom. \(\rightarrow\) Advanced Comp. Geom.

Quantum Mech.
Graphs can be "abstract"

- co-ordinates
- & drawings
- don't matter

Although sometimes it helps to draw & visualize
Graphs can be "abstract" or "geometric"/"embedded"

- co-ordinates & drawings don't matter

although sometimes it helps to draw & visualize

VS

representing physical restrictions
Often it is assumed that there are no loops or multiple edges. Assume this unless specified.
\[ G = (V, E) \]

\[
\downarrow \quad \downarrow
\]

\[ V(G) \quad E(G) \]

\[
(\{a, b, c, d, e\}, \{ad, bc, be, ce, cd\})
\]

\[ b \ & \ c \text{ are adjacent (share an edge)} \]
$G = \{V, E\}$
$G = \{V, E\}$

Adjacency matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<th>4</th>
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Size: ?
$G = \{ V, E \}$

Adjacency matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 1 & 0 & 1 \\
4 & 1 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 \\
7 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]

size: $|V|^2$

(symmetric for undirected)
$G = \{ V, E \}$

Adjacency matrix

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(size: $|V|^2$
(symmetric for undirected)

Adjacency list

1 $\rightarrow$ 2 $\rightarrow$ 4
2 $\rightarrow$ 1 $\rightarrow$ 7
3 $\rightarrow$ 4 $\rightarrow$ 6 $\rightarrow$ 7
4 $\rightarrow$ 1 $\rightarrow$ 3
5
6 $\rightarrow$ 3 $\rightarrow$ 7
7 $\rightarrow$ 2 $\rightarrow$ 3 $\rightarrow$ 6
Representation

\[ G = \{ V, E \} \]

Adjacency matrix

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 1 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 \\
7 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{array}
\]

Adjacency list

(size: \(|V|^2\), symmetric for undirected)

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow 4 \\
2 & \rightarrow 1 \rightarrow 7 \\
3 & \rightarrow 4 \rightarrow 6 \rightarrow 7 \\
4 & \rightarrow 1 \rightarrow 3 \\
5 & \\
6 & \rightarrow 3 \rightarrow 7 \\
7 & \rightarrow 2 \rightarrow 3 \rightarrow 6
\end{align*}
\]
$G = \{V, E\}$

**Adjacency list**

size: $|V| + 2|E|$  
(undirected)

$1 \rightarrow 2 \rightarrow 4$
$2 \rightarrow 1 \rightarrow 7$
$3 \rightarrow 4 \rightarrow 6 \rightarrow 7$
$4 \rightarrow 1 \rightarrow 3$
$5$
$6 \rightarrow 3 \rightarrow 7$
$7 \rightarrow 2 \rightarrow 3 \rightarrow 6$

**Adjacency matrix**

size: $|V|^2$  
(symmetric for undirected)

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Representation

\[ G = \{ V, E \} \]

give one reason for using matrix vs list

Adjacency matrix

size: \(|V|^2\)
(symmetric for undirected)

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Adjacency list

size: \(|V| + 2|E|\)
(undirected)

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow 4 \\
2 & \rightarrow 1 \rightarrow 7 \\
3 & \rightarrow 4 \rightarrow 6 \rightarrow 7 \\
4 & \rightarrow 1 \rightarrow 3 \\
5 & \rightarrow \ \\
6 & \rightarrow 3 \rightarrow 7 \\
7 & \rightarrow 2 \rightarrow 3 \rightarrow 6
\end{align*}
\]
Representation

$G = \{V, E\}$

"is y a neighbor of x"? \[\text{matrix}\]

"report all neighbors of x" \[\text{list}\]

Adjacency matrix

Size: $|V|^2$

(symmetric for undirected)

Adjacency list

Size: $|V| + 2|E|$

(undirected)

1 $\rightarrow$ 2 $\rightarrow$ 4
2 $\rightarrow$ 1 $\rightarrow$ 7
3 $\rightarrow$ 4 $\rightarrow$ 6 $\rightarrow$ 7
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6 $\rightarrow$ 3 $\rightarrow$ 7
7 $\rightarrow$ 2 $\rightarrow$ 3 $\rightarrow$ 6
Vertex degree

$v_3$ has 4 adjacent vertices = 4 neighbors
Vertex degree

$v_3$ has 4 adjacent vertices = 4 neighbors

d($v_3$) = 4
Vertex degree

\[ v_3 \text{ has 4 adjacent vertices } = 4 \text{ neighbors} \]

\[ d(v_3) = 4 \]

\[ \sum_{v \in G} d(v) = ? \]
Vertex degree

$\sum d(v) = 2 + 2 + 4 + 2 + 2 = 12$

$|E| = 6$

$v_3$ has 4 adjacent vertices = 4 neighbors

$d(v_3) = 4$

$\sum_{v \in G} d(v) = 2 \cdot |E|$

(just double counting)
Regular graphs: all vertices have the same degree
Regular graphs: all vertices have the same degree

$d=1$ ?
Regular graphs: all vertices have the same degree

d=1

1-regular
Regular graphs: all vertices have the same degree

\[ d = 1 \ ? \]
\[ d = 2 \ ? \]

1-regular
Regular graphs: all vertices have the same degree

\[ d = 1 \ ? \]

1-regular

\[ d = 2 \ ? \]

2-regular
Regular graphs: all vertices have the same degree

\[ d = 1 \ ? \]

\[ d = 2 \ ? \]

\[ d = 3 \ ? \]
Regular graphs: all vertices have the same degree

$d = 1$ ?  

$d = 2$ ?  

$d = 3$ ?  

1-regular

2-regular
Regular graphs: all vertices have the same degree

$d=1$ ?

1-regular

$d=2$ ?

2-regular

$d=3$ ?
Regular graphs: all vertices have the same degree

$d = 1$ ?

$1$-regular

$d = 2$ ?

$2$-regular

$d = 3$ ?
Regular graphs: all vertices have the same degree

$d = 1$ ?

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$d = 2$ ?

2-regular

$d = 3$ ?
Regular graphs: all vertices have the same degree

$d=1$ ?

1-regular

$d=2$ ?

2-regular

$d=3$ ?

3-regular
$n = 5$ 4-regular
$n=5$

4-regular

$(n-1$ regular) $\rightarrow$ complete graph
\( n = 5 \)

4-regular

\((n-1)\text{ regular}\) \(\rightarrow\) complete graph

\#edges?
n=5

4-regular

(n-1 regular) → complete graph

#edges? \[ n-1 + n-2 + n-3 + \ldots + 3 + 2 + 1 \]
4-regular
(n-1 regular) $\rightarrow$ complete graph

$\#$ edges?

$n - 1 + n - 2 + n - 3 + \ldots + 3 + 2 + 1 = \binom{n}{2}$

$= \sum_{i=1}^{n-1} i$
A 4-regular graph $(n-1)$-regular $\rightarrow$ complete graph

$\#\text{edges?} = n-1 + n-2 + n-3 + \ldots + 3 + 2 + 1 = \binom{n}{2}$

$= \sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$
$n=5$

4-regular

$(n-1$ regular) $\rightarrow$ complete graph

$\# \text{edges?} \quad n-1 + n-2 + n-3 + \ldots + 3 + 2 + 1 = \binom{n}{2}$

$= \sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$

$\# \text{possible graphs on } n \text{ vertices?}$
$n=5$

4-regular

$(n-1$ regular) $\rightarrow$ complete graph

$\# \text{edges?} = n-1 + n-2 + n-3 + \ldots + 3 + 2 + 1 = \binom{n}{2}$

$= \sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$

$\# \text{possible graphs on n vertices?} = 2^\binom{n}{2}$
$n = 5$

4-regular

$(n-1$ regular) $\rightarrow$ complete graph

# edges?

$n-1 + n-2 + n-3 + \ldots + 3 + 2 + 1 = \binom{n}{2}$

$= \sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$

# possible graphs on $n$ vertices?

$2^{\binom{n}{2}}$

... but some of them are similar in shape
isomorphism
isomorphism

\[ a : d' \]
\[ b : a' \]
\[ c : c' \]
\[ d : b' \]
\[ e : e' \]
isomorphism \rightarrow \text{map vertices of graph } G \text{ to vertices of graph } H

\begin{align*}
a & : d' \\
b & : a' \\
c & : c' \\
d & : b' \\
e & : e'
\end{align*}
isomorphism → map vertices of graph $G$ to vertices of graph $H$
if vertices $x, y \in G$ share an edge in $G$ then
vertices $m(x), m(y) \in H$ must share an edge in $H$. 

$\begin{align*}
a & : d' \\
b & : a' \\
c & : c' \\
d & : b' \\
e & : e'
\end{align*}$
Isomorphism → map vertices of graph $G$ to vertices of graph $H$

If vertices $a, b \in G$ share an edge in $G$, then vertices $m(a), m(b) \in H$ must share an edge in $H$.
isomorphism → map vertices of graph $G$ to vertices of graph $H$
if vertices $a, b \in G$ share an edge in $G$ then
vertices $m(a), m(b) \in H$ must share an edge in $H$. 

\begin{align*}
a & : d' = w \\
b & : a' = v \\
c & : c' = x \\
d & : b' = z \\
e & : e' = y
\end{align*}
isomorphism  →  map vertices of graph $G$ to vertices of graph $H$ if vertices $a, b \in G$ share an edge in $G$ then vertices $m(a), m(b) \in H$ must share an edge in $H$. 

\begin{align*}
\text{a} : \text{d'} : \text{w} : \text{?} \\
\text{b} : \text{a'} : \text{v} : \text{?} \\
\text{c} : \text{c'} : \text{x} : \text{?} \\
\text{d} : \text{b'} : \text{z} : \text{?} \\
\text{e} : \text{e'} : \text{y} : \text{?}
\end{align*}
isomorphism $\rightarrow$ map vertices of graph $G$ to vertices of graph $H$

if vertices $a, b \in G$ share an edge in $G$ then

vertices $m(a), m(b) \in H$ must share an edge in $H$. 

$a : d' : w : r$
$b : a' : v : t$
$c : c' : x : u$
$d : b' : z : q$
$e : e' : y : s$
Determining if two graphs are isomorphic (without a mapping) is difficult (time complexity as function of \( n \)) (not just because drawings look complicated)
Determining if two graphs are isomorphic (without a mapping) is difficult (time complexity as function of $n$) (not just because drawings look complicated)

Counting # possible graphs without "doublecounting" isomorphs is complicated
G is a subgraph of H if \[ \begin{cases} V(G) \subseteq V(H) \\ E(G) \subseteq E(H) \end{cases} \]
Subgraphs

$G$ is a subgraph of $H$ if\[ \begin{cases} V(G) \subseteq V(H) \\ E(G) \subseteq E(H) \end{cases} \] if equal then it’s a "spanning" subgraph
Subgraphs

$G$ is a subgraph of $H$ if

\[
\begin{cases} 
V(G) \subseteq V(H) \\
E(G) \subseteq E(H)
\end{cases}
\]

if equal then it's a "spanning" subgraph

If you only remove edges as a result of removing vertices then $G$ is an "induced" subgraph.
Subgraphs

$G$ is a subgraph of $H$ if

\[
\begin{cases}
V(G) \subseteq V(H) \\
E(G) \subseteq E(H)
\end{cases}
\]

\text{if equal then it's a "spanning" subgraph}

If you only remove edges as a result of removing vertices then $G$ is an "induced" subgraph.

\[\text{unlike}\]