Prove that the first $n$ odd natural numbers sum to $n^2$. 
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

1  3  5  7  \ldots
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1 \ 2 \ 3 \ 4 \ \cdots \ (n-1) \ n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[
i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n
\]

\[
1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2
\]

Sum: 1 4 9 16 ...

so far so good
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

Sum: \[ 1 \quad 4 \quad 9 \quad 16 \cdots \]

Suppose not. Then \[ \sum_{i=1}^{n} 2i-1 \neq n^2. \] Now what?
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} (2i-1) \neq n^2$. Now what?

We saw the claim is true for small $i$. 
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[
\sum_{i=1}^{n} 2i-1 = n^2
\]

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. Now what?

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If the claim is false, there must be some $j$ for which this happens.
Prove that the first $n$ odd natural numbers sum to $n^2$.

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Suppose not. Then \( \sum_{i=1}^{n} 2i-1 \neq n^2 \). Now what?

We saw the claim is true for small $i$.

If the claim is false, there must be some $j$ for which this happens.

Focus on the smallest such $j$ & on $j-1$
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

?
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false \& \( x-1 \) for which it is true \( \Rightarrow \) in fact for all \( i < x \)
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

if false, then \( \exists x \) for which it is false \& \( x-1 \) for which it is true

\[ 1 + 3 + 5 + \cdots + (2x-3) = (x-1)^2 \]

\[ \sum_{i=x-1}^{n} i = (x-1)^2 \]
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false

\[ 1 + 3 + 5 + \cdots + (2x-3) \]

\[ i = x \]

\[ \Rightarrow (x-1)^2 \]

\[ (2x-3) \]

\[ i = x \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\( \& x-1 \) for which it is true

\( \Rightarrow \) in fact for all \( i < x \)
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]
\[ 1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2 \]

Is this true? If false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true. In fact, for all \( i < x \)

\[
1 + 3 + 5 + \cdots + (2x-3) + (2x-1) = (x-1)^2
\]

\[
(x-1)^2 + 2x-1 \neq x^2
\]
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]
\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

if false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true

\[ 1 + 3 + 5 + \cdots + (2x-3) = (x-1)^2 \]
\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x - 1 \neq x^2 \]
\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]
\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false. \( \Rightarrow \) in fact for all \( i < x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) = (x-1)^2 \]
\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]
\[ (x-1)^2 + 2x - 1 \neq x^2 \]
\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]
\[ i = 1 \quad 2 \quad 3 \quad 4 \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false \& \( x - 1 \) for which it is true

\[ 1 + 3 + 5 + \cdots + (2x-3) \quad i = x-1 \]

\[ = (x-1)^2 \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x-1 \neq x^2 \]

\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]

contradiction
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ? \]

If false, then \( \exists x \) for which it is false \& \( x-1 \) for which it is true, in fact for all \( i < x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) = (x-1)^2 \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x - 1 \neq x^2 \]

\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]

\[ \text{contradiction} \]
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]
\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

if false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true \( \Rightarrow \) in fact for all \( i < x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) \quad \text{for} \quad i = x-1 \]
\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]
\[ (x-1)^2 + 2x-1 \neq x^2 \]
\[ x^2 - 2x + 1 + 2x-1 \neq x^2 \]

contradiction

\[ \leftarrow \text{either this should have been } \neq \right. \]
\[ \leftarrow \text{or this should have been } = \]

{... which contradicts the smallest counterexample assumption.}
\[ i = 1 \enspace 2 \enspace 3 \enspace 4 \enspace \cdots \enspace (n-1) \enspace n \]
\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false & \( x \neq 1 \) for which it is true \( \Rightarrow \) in fact for all \( i < x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) = (x-1)^2 \]
\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]
\[ (x-1)^2 + 2x-1 \neq x^2 \]
\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]

\[ \text{contradiction} \]

\[ \text{either this should have been } \neq \]
\[ \text{or this should have been } = \]

... which contradicts the smallest counterexample assumption, i.e.,

THERE IS NO
(SMALLEST) COUNTEREXAMPLE
\[ \Rightarrow \text{CLAIM IS TRUE} \]
SMALLEST COUNTEREXAMPLE

recap
SMALLEST COUNTEREXAMPLE

- be able to "count" & "order" instances of the claim

(case/example)
SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim
- prove the claim for smallest instance (case/example)

(& prove a smallest instance exists)
Smallest Counterexample: Recap

- be able to "count" & "order" instances of the claim
- prove the claim for smallest instance (case/example)
- assume the claim is false
SMALLEST COUNTEREXAMPLE

- be able to "count" & "order" instances of the claim
- prove the claim for smallest instance (case/example)
- assume the claim is false: then there is a smallest example, $E_i$, for which it is false (smallest counterexample)
SMALLEST COUNTEREXAMPLE recap

• be able to "count" & "order" instances of the claim

• prove the claim for smallest instance (case/example)

• assume the claim is false: then there is a smallest example, $E_i$, for which it is false (smallest counterexample)

• this implies the claim is true for the next smallest example, $E_{i-1}$.
SMALLEST COUNTEREXAMPLE

- be able to "count" & "order" instances of the claim
- prove the claim for smallest instance (case/example)
- assume the claim is false: then there is a smallest example, $E_i$, for which it is false (smallest counterexample)
- this implies the claim is true for the next smallest example, $E_{i-1}$.

- use $E_i$ & $E_{i-1}$ to get a contradiction (to the existence of any counterexample)
Claim: For \( n \in \mathbb{Z}, n \geq 5, 2^n > n^2 \)
Claim: For \( n \in \mathbb{Z}, n > 5 \), \( 2^n > n^2 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^n )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>( n^2 )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>
Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$.

- Use smallest counterexample

$\Rightarrow$ which is ... ?
Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

- use smallest counterexample

$\Rightarrow$ which is some unknown hypothetical $x$. 
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

- use smallest counterexample

$(n=2, 3, 4$ are not counterexamples)
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

- use smallest counterexample
  
  $(n = 2, 3, 4 \text{ are not counterexamples})$

why can we?
Claim: For \( n \in \mathbb{Z}, \ n \geq 5, \ 2^n > n^2 \) (notice \( \begin{array}{c|c|c|c|c|c|c|}
 n & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
2^n & 1 & 2 & 4 & 8 & 16 & 32 \\
\hline
n^2 & 0 & 1 & 4 & 9 & 16 & 25 \\
\end{array} \))

- use smallest counterexample
  
  \((n = 2, 3, 4 \text{ are not counterexamples})\)

→ why can we? Claim is true for smallest instance \((n = 5)\)
Claim: For \( n \in \mathbb{Z}, n \geq 5, \ 2^n > n^2 \) (notice \( 2^n \)): \[
\begin{array}{cccccc}
\text{n} & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
n^2 & 0 & 1 & 4 & 9 & 16 & 25 \\
2^n & 1 & 2 & 4 & 8 & 16 & 32 \\
\end{array}
\]

- use smallest counterexample 
  
  \[ (n=2, 3, 4 \text{ are not counterexamples}) \]

why can we? \( \rightarrow \) Claim is true for smallest instance \( (n=5) \)

- assume \( \exists \) smallest counterexample \( x \). \( \rightarrow ? \)
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

- use smallest counterexample
  
  ($n = 2, 3, 4$ are not counterexamples)

why can we? $\rightarrow$ Claim is true for smallest instance ($n = 5$)

- assume $\exists$ smallest counterexample $x$. $\exists 2^x \leq x^2 \quad (x > 5)$

(what other condition?)
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

- Use smallest counterexample
  
  $(n = 2, 3, 4$ are not counterexamples $)$

  - Why can we? $\rightarrow$ Claim is true for smallest instance $(n = 5)$

- Assume $\exists$ smallest counterexample $x$. $\exists \quad 2^x \leq x^2$ $(x > 5)$

  $(\& \; for \; y \geq 5, \; if \; y < x \; then \; 2^y > y^2)$
Claim: For \( n \in \mathbb{Z}, n > 5 \), \( 2^n > n^2 \) 

- Use smallest counterexample
  
  \((n=2, 3, 4 \text{ are not counterexamples})\)

- Why can we? \(\Rightarrow\) Claim is true for smallest instance \( (n=5) \)

- Assume \( \exists \) smallest counterexample \( x \). \( \implies \) \( 2^x \leq x^2 \) \((x > 5)\)
  
  \((& \text{ for } y > 5, \text{ if } y < x \text{ then } 2^y > y^2)\)

- Focus on \( x-1 \):
Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

- Use smallest counterexample
  
  ($n=2, 3, 4$ are not counterexamples)

- Why can we? → Claim is true for smallest instance ($n=5$)

- Assume $\exists$ smallest counterexample $x$. $\exists \left( 2^x \leq x^2 \right)$ ($x>5$)
  
  (& for $y>5$, if $y<x$ then $2^y > y^2$)

- Focus on $x-1$: $2^{x-1} > (x-1)^2$
Claim: For \( n \in \mathbb{Z}, \ n \geq 5, \ 2^n > n^2 \) (notice \( 2^0 = 1 \ 2^2 = 4 \ 2^4 = 16 \ 2^5 = 32 \) \( n^2 = 0 \ 1 \ 4 \ 9 \ 16 \ 25 \))

- Use smallest counterexample
  
  \( (n=2, 3, 4 \text{ are not counterexamples}) \)

  why can we? \( \rightarrow \) Claim is true for smallest instance \( (n=5) \)

- Assume \( \exists \) smallest counterexample \( x. \) \( 2^x \leq x^2 \) \( (x>5) \)
  
  \( \& \text{ for } y>5, \text{ if } y<x \text{ then } 2^y>y^2 \)

- Focus on \( x-1 : \ 2^{x-1} > (x-1)^2 \)

  combine to get contradiction
$2^x \leq x^2$

because $x$ is a counterexample
\[2^x \leq x^2\]

because \(x\) is a counterexample

\[2^{x-1} > (x-1)^2\]

because ?
$2^x \leq x^2$ because $x$ is a counterexample

$2^{x-1} > (x-1)^2$ because $x$ is the smallest counterexample and $?$
\[ 2^x \leq x^2 \] because \( x \) is a counterexample

\[ 2^{x-1} > (x-1)^2 \] because \( x \) is the smallest counterexample and not the smallest case

next?
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$
\[ 2^x \leq x^2 \]

because \( x \) is a counterexample

\[ 2^{x-1} > (x-1)^2 \]

because \( x \) is the smallest counterexample and not the smallest case

\[ 2^{x-1} > x^2 - 2x + 1 \]

\[ 2^{x-1} \cdot 2 > 2x^2 - 4x + 2 \]
$2^x \leq x^2$ because $x$ is a counterexample

$2^{x-1} > (x-1)^2$ because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1}.2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} \geq (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$
because $x$ is a counterexample and not the smallest case

$2^{x-1} > (x-1)^2$

if $x^2 - 4x + 2 > 0$ then?

because $x$ is the smallest counterexample

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1}.2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1}.2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$

if $x^2 - 4x + 2 > 0$ we will get a contradiction
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$

if $x^2 - 4x + 2 > 0$ we will get a contradiction

$(x-2)(x-2) \geq 2$

true for $x \geq 4$
\[ 2^x \leq x^2 \]  
because \( x \) is a counterexample

\[ 2^{x-1} > (x-1)^2 \]
because \( x \) is the smallest counterexample and not the smallest case

\[ 2^{x-1} > x^2 - 2x + 1 \]
\[ 2^{x-1} \cdot 2 > 2x^2 - 4x + 2 \]
\[ 2^x > 2x^2 - 4x + 2 \]
\[ 2^x > x^2 + (x^2 - 4x + 2) \]

Conclusion:

For \( n \in \mathbb{Z}, n \geq 5 \), \( 2^n > n^2 \)
FIBONACCI NUMBERS
Fibonacci numbers

$F_0 = 1 \quad F_1 = 1$
Fibonacci Numbers

\[ F_0 = 1 \quad F_1 = 1 \]

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)
Fibonacci Numbers

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1$
$F_1 = 1$
$F_2 = 2$
FIBONACCI NUMBERS

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)
**Fibonacci Numbers**

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)
Fibonacci Numbers

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]
\[ F_5 = 8 \]

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)
FIBONACCI NUMBERS

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]
\[ F_5 = 8 \]
\[ F_6 = 13 \]

**etc**

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)
Fibonacci Numbers

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]
\[ F_5 = 8 \]
\[ F_6 = 13 \]
\[ \text{etc} \]

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, \ n \geq 0 \), \( F_n \leq 1.7^n \)
Fibonacci Numbers

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]
\[ F_5 = 8 \]
\[ F_6 = 13 \]

etc

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, n > 0 \), \( F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n = x \)

\[ F_x > 1.7^x \]
FIBONACCI NUMBERS

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]
\[ F_5 = 8 \]
\[ F_6 = 13 \]
\[ \text{etc} \]

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, n \geq 0 \), \( F_n \leq 1.7^n \)

suppose smallest counterexample is \( n = x \)

\( F_x > 1.7^x \)

we want a contradiction, so most likely this will involve \( F_{x-1} \)
For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, n \geq 0 \), \( F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n = x \)

\( \iff F_x > 1.7^x \)

We want a contradiction, so most likely this will involve \( F_{x-1} \)

Slight hiccup?
Fibonacci Numbers

\[ F_0 = 1 \quad F_1 = 1 \]

\[ F_2 = 2 \quad F_3 = 3 \]

\[ F_4 = 5 \quad F_5 = 8 \]

\[ F_6 = 13 \quad \text{etc} \]

\[ F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 \]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \ F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n = x \)

\[ F_x > 1.7^x \]

We want a contradiction, so most likely this will involve \( F_{x-1} \)

It will be hard to use only \( F_x \) & \( F_{x-1} \).
FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\Rightarrow F_x > 1.7^x$

we want a contradiction, so most likely this will involve $F_{x-1}$

but it will be hard to use only $F_x$ & $F_{x-1}$

so why not use $F_{x-2}$ also: assume $x \geq 2$
Fibonacci Numbers

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n > 0$, $F_n \leq 1.7^n$

Suppose smallest counterexample is $n = x$

Then $F_x > 1.7^x$

We want a contradiction, so most likely this will involve $F_{x-1}$

But it will be hard to use only $F_x$ & $F_{x-1}$

So why not use $F_{x-2}$ also: assume $x \geq 2$

Then is $F_0 \leq 1.7^0$?
Fibonacci Numbers

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, n \geq 0 \), \( F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n = x \)

\[ F_x > 1.7^x \]

We want a contradiction, so most likely this will involve \( F_{x-1} \)

But it will be hard to use only \( F_x \) & \( F_{x-1} \), so why not use \( F_{x-2} \) also: assume \( x \geq 2 \)

\[ F_0 \leq 1.7^0? \text{ yes. } F_1 \leq 1.7^1? \]

\( F_0 = 1 \)

\( F_1 = 1 \)

\( F_2 = 2 \)

\( F_3 = 3 \)

\( F_4 = 5 \)

\( F_5 = 8 \)

\( F_6 = 13 \)

etc
FIBONACCI NUMBERS

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: For \( n \in \mathbb{Z} \), \( n > 0 \), \( F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n = x \)

\[ F_x > 1.7^x \]

We want a contradiction, so most likely this will involve \( F_{x-1} \)

But it will be hard to use only \( F_x \) & \( F_{x-1} \), so why not use \( F_{x-2} \) also:

Assume \( x \geq 2 \)

\[ \Rightarrow \text{is } F_0 \leq 1.7^0? \quad \text{yes.} \quad \text{is } F_1 \leq 1.7^1? \quad \text{yes.} \quad \text{OK!} \]
$F_0 = F_1 = 1$ \quad // \quad \text{for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$
F_0 = F_1 = 1 \quad \forall \ n \geq 2, \ F_n = F_{n-1} + F_{n-2}

Claim: for n \in \mathbb{Z}, n \geq 0, \ F_n \leq 1.7^n

smallest counterexample: F_x > 1.7^x \quad \& \text{ we can safely assume } F_y \leq 1.7^y \text{ for } y < x
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \; n \geq 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) \( (x \geq 2) \) & we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

next?
\( F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \quad \& \quad \text{we can safely assume} \quad F_y \leq 1.7^y \quad \text{for } y < x \)

\((x \geq 2)\)

we can now say: \( F_x = F_{x-1} + F_{x-2} \quad \ldots \)
$F_0 = F_1 = 1 \quad // \quad \text{for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}, \ n \geq 0,$ \quad $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ \quad & we can safely assume \quad $F_y \leq 1.7^y$ \quad for \quad $y \leq x$

$(x \geq 2)$

we can now say: \quad $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$
Claim: for \( n \in \mathbb{Z}, n > 0 \), \( F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) & we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)
\[ = 1.7^{x-2} \cdot (1.7 + 1) \]
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \quad \& \text{we can safely assume} \quad F_y \leq 1.7^y \text{ for } y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)
\[ = 1.7^{x-2} \cdot (1.7 + 1) \]
\[ = 1.7^{x-2} \cdot 2.7 \]
$F_0 = F_1 = 1 \quad \text{// for } n \gg 2, \quad F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}, \ n \gg 0, \ F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ & we can safely assume $F_y \leq 1.7^y$ for $y < x$

$x \gg 2$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$= 1.7^{x-2} \cdot (1.7 + 1)$

$= 1.7^{x-2} \cdot 2.7$

$< 1.7^{x-2} \cdot (1.7)^2$  

$[1.7^2 = 2.89]$

next?
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n \]

smallest counterexample: \( F_x > 1.7^x \quad \text{(x \geq 2)} \)

& we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)

\[ = 1.7^{x-2} \cdot (1.7 + 1) \]

\[ = 1.7^{x-2} \cdot 2.7 \]

\[ < 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89] \]

\[ = 1.7^x \quad \text{so?} \]
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, n \geq 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) \quad (x \geq 2)

& we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)

\[ = 1.7^{x-2} \cdot (1.7 + 1) \]

\[ = 1.7^{x-2} \cdot 2.7 \]

\[ < 1.7^{x-2} \cdot (1.7)^2 \]

\[ = 1.7^x \]

\[ 1.7^2 = 2.89 \]

\( \text{CONTRADICTION} \)
6 points in **convex** position.
still 6 points in convex position.
Theorem: in $\mathbb{R}^2$, every set of $\geq 17$ points with no 3 on a line has 6 points in convex position.
Theorem: in $\mathbb{R}^2$, every set of $\geq 17$ points with no 3 on a line has 6 points in convex position.

"has a hexagon" (not necessarily regular)
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon...
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an *empty* pentagon.
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon, then $P$ has 5 points forming an empty pentagon.

Stronger claim: the empty pentagon is inside a hexagon.
Claim: in $\mathbb{R}^2$, given a set of points $P$ w/ no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Stronger claim: the empty pentagon is inside a hexagon.

In fact, inside the hexagon containing the fewest number of points...
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

**Proof Sketch:**

Choose a hexagon $H$ containing min #pts.
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an \underline{empty} pentagon.

**Proof Sketch:**
Choose a hexagon $H$ containing min # pts.
- if $H$ is empty, DONE.

\[basis\ \text{case (or base case)}\]
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

**Proof Sketch:**

Choose a hexagon $H$ containing min #pts. 
- If $H$ is empty, DONE.
- If $H$ contains exactly 1 point,
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon, then $P$ has 5 points forming an empty pentagon.

Proof sketch:

Choose a hexagon $H$ containing min #pts.
- If $H$ is empty, DONE.
- If $H$ contains exactly 1 point, "split" $H$
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

**Proof Sketch:**

Choose a hexagon $H$ containing min #pts.
- if $H$ is empty, DONE.
- if $H$ contains exactly 1 point, "split" $H$ and then we are done.

(cont'd ...)

(so if there is a counterexample to our claim, then $H$ must contain at least 2 points)
Claim: in $\mathbb{R}^2$, given a set of points $P$ w/ no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min #pts.

Shown: if $H$ contains $\leq 1$ points, DONE $\Rightarrow$ so assume $\geq 2$ pts inside.

Note: we will next assume that there is some counterexample, for the sake of contradiction. A counterexample will consist of a point set where $H$ does not contain an empty pentagon.

We order all possible occurrences of $H$, according to the number of points, $n$, in $H$. We've seen that if $n=0$ or 1, there is no counterexample. So we have a base case. Next we assume that there is some smallest counterexample, where $H$ contains $x>1$ points. So we will draw a hexagon $H$, with the constraint that it contains at least 2 points, and that it is the hexagon with fewest number of points inside.
Claim: in $\mathbb{R}^2$, given a set of points $P$ w/ no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min #pts.
- Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of $H$...

As explained in class:
There are at least 2 points strictly inside the hexagon $H$.
Form a tight "fence" around those points. (f.y.i: this is actually called the "convex hull").
For every "extreme segment" on that fence (shown in yellow here), extend into a line (green). That line has all of the interior points on one side, by its definition.
On the other side of the line are some number of points from $H$.
I say that those points are "isolated". In the picture here, 3 points are isolated.

(actually, if $H$ contains exactly 2 points, then there are no other interior points to be "on one side" of the green line. In this case we must isolate at least 3 points on one side of the green line)
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min #pts.
- Shown: if $H$ contains $\leq 1$ points, DONE $\Rightarrow$ so assume $\geq 2$ pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.

... because we isolated 3 points using an extreme segment, we find an empty pentagon.

This contradicts any attempt to claim that $H$ is a (smallest) counterexample. So if $H$ has any hope of being a counterexample, then no extreme segment can isolate $\geq 2$ points of $H$. So we will proceed by assuming this.
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min #pts.
- Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.
- If any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.
  $\Rightarrow$ so every such segment isolates 1 or 2 points.

(note that I have drawn a picture where every extreme segment isolates 2 points from $H$.
If only 1 point is isolated, we'll get the same conclusion. Convince yourself that every extreme segment of the interior points isolates at least one point of $H$.)
Claim: in $\mathbb{R}^2$, given a set of points $P$ w/ no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min # pts.

- Shown: if $H$ contains $\leq 1$ points, DONE $\implies$ so assume $\geq 2$ pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.
  $\implies$ so every such segment isolates 1 or 2 points.

$\implies$ use one segment & form a hexagon

(why?)
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min #pts.

- Shown: if $H$ contains $\leq 1$ points, DONE $\Rightarrow$ so assume $\geq 2$ pts inside.
- If any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.
  - so every such segment isolates 1 or 2 points.

$\Rightarrow$ use one segment
& form a hexagon containing fewer points than $H$.
(Not?)
Claim: in $\mathbb{R}^2$, given a set of points $P$ w/ no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Choose a hexagon $H$ containing min # pts.
- Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.
  $\implies$ so every such segment isolates 1 or 2 points.

$\implies$ use one segment & form a hexagon containing fewer points than $H$.
contradicts [ $H$ = "smallest" ]