Prove that the first n odd natural numbers sum to n^2.

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$i =$	1	2	3	4	$\cdots$	$(n-1)$	n
	1	3	5	7		?	?

Prove that the first n odd natural numbers sum to n^2.

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

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Sum: 1 4 9 16 ...

so far so good

Prove that the first n odd natural numbers sum to n^2.

$$i = \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. Now what?

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{lccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n \\ & 1 + & 3 + & 5 + & 7 + & \cdots + & (2n-3) + & (2n-1) = n^2 \end{array}$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. Now what?

We saw the claim is true for small i.

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{llllllll} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n \end{array}$$

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Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. Now what?

We saw the claim is true for small i.

If the claim is false, there must be some j for which this happens.

SMALLEST COUNTEREXAMPLE

Prove that the first n odd natural numbers sum to n^2.

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. Now what?

We saw the claim is true for small i.

If the claim is false, there must be some j for which this happens.

Focus on the smallest such j & on j-1

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}_{\text{in fact for all } i<x}$ for which it is true

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $\boxed{\underbrace{x-1}\text{ for which it is true}}$

↳ in fact for all $i<x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1+3+5+7+\cdots+(2n-3)+(2n-1) = n^2 \quad ?$$

if false, then $\boxed{\exists x \text{ for which it is false}}$ & $\underbrace{x-1}$ for which it is true

↳ in fact for all $i<x$

$$1+3+5+\cdots+\overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

$$1+3+5+\cdots+(2x-3)+\overbrace{(2x-1)}^{i=x} \neq x^2$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}$ for which it is true

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$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$+ 2x-1 \neq x^2$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

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in fact for all $i<x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$(x-1)^2 + 2x-1 \neq x^2$$

$$x^2 - 2x + 1 + 2x - 1 \neq x^2$$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

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$$(x-1)^2 + 2x - 1 \neq x^2$$

$$x^2 \cancel{-2x} \cancel{+1} \cancel{+2x} \cancel{-1} \neq x^2$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

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$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$(x-1)^2 + 2x-1 \neq x^2$$

$$\underline{x^2} - \cancel{2x} + \cancel{1} + \cancel{2x} - \cancel{1} \; \underline{\neq x^2}$$

contradiction

$$\begin{array}{ccccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n \end{array}$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \;?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}_{\hookrightarrow \text{ in fact for all } i<x}$ for which it is true

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2 \quad \leftarrow \text{either this should have been } \neq$$

$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2 \quad \leftarrow \text{or this should have been } =$$

$$(x-1)^2 + 2x-1 \neq x^2$$

$$\underline{x^2} - \cancel{2x} + \cancel{1} + \cancel{2x} - \cancel{1} \; \underline{\neq x^2}$$

contradiction

$$\begin{array}{lcccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n \end{array}$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}$ for which it is true

↳ in fact for all $i < x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$ ← either this should have been $\neq$

$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)} + \overbrace{(2x-1)}^{i=x} \neq x^2$$ ← or this should have been $=$

$$(x-1)^2 \quad + 2x-1 \neq x^2$$

$$x^2 \cancel{-2x} \cancel{+1} \quad \cancel{+2x} \cancel{-1} \neq x^2$$

contradiction

... which contradicts the smallest counterexample assumption.

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$?

if false, then $\exists x$ for which it is false & $x-1$ for which it is true

↳ in fact for all $i<x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

← either this should have been $\neq$

$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2$$

← or this should have been $=$

$$(x-1)^2 + 2x - 1 \neq x^2$$

$$\underline{x^2} - \cancel{2x} + \cancel{1} + \cancel{2x} - \cancel{1} \; \underline{\neq x^2}$$

contradiction

... which contradicts the smallest counterexample assumption, i.e.,

THERE IS NO (SMALLEST) COUNTEREXAMPLE

↳ CLAIM IS TRUE

SMALLEST COUNTEREXAMPLE

recap

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

(case / example)

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

(case / example)

- prove the claim for smallest instance

(& prove a smallest instance exists)

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim (case / example)
- prove the claim for smallest instance
- assume the claim is false

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

(case / example)

- prove the claim for smallest instance

- assume the claim is false: then there is a smallest example, E_i, for which it is false

(smallest counterexample)

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim (case / example)
- prove the claim for smallest instance
- assume the claim is false: then there is a smallest example, E_i, for which it is false (smallest counterexample)
- this implies the claim is true for the next smallest example, E_{i-1}.

SMALLEST COUNTEREXAMPLE recap

see p. 131 "well-ordering principle"

- be able to "count" & "order" instances of the claim

  (case / example)
- prove the claim for smallest instance
- assume the claim is false: then there is a smallest example, E_i, for which it is false

  (smallest counterexample)
- this implies the claim is true for the next smallest example, E_{i-1}.
- use E_i & E_{i-1} to get a contradiction (to the existence of any counterexample)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

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$$\left(\text{notice} \left\{ \begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right. \right)$$

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice $\left\{ \begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right.$

- use smallest counterexample

↳ which is ... ?

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice $\left\{ \begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right.$

- use smallest counterexample

↳ which is some unknown hypothetical x.

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

$$\left(\text{notice} \begin{cases} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{cases} \right)$$

- use smallest counterexample

(n = 2, 3, 4 are not counterexamples)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice
$$\left(\begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right)$$

- use smallest counterexample

(n = 2, 3, 4 are not counterexamples)

why can we?

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice

n	0	1	2	3	4	5
2^n	1	2	4	8	16	32
n^2	0	1	4	9	16	25

- use smallest counterexample

(n = 2, 3, 4 are not counterexamples)

why can we? → Claim is true for smallest instance (n=5)

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- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. → ?

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice
$$\left\{\begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array}\right.$$

- use smallest counterexample

(n = 2, 3, 4 are not counterexamples)

why can we? → Claim is true for smallest instance (n=5)

- assume $\exists$ smallest counterexample x. $\}$ $\boxed{2^x \leq x^2}$ $(x > 5)$

(what other condition?)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice
$$\left(\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right)$$

- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

  why can we? $\rightarrow$ Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. $\Big\}$ $\boxed{2^x \leq x^2}$ $(x > 5)$

  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice $\left\{ \begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right.$

- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. $\Big\}$ $\boxed{2^x \leq x^2}$ $(x>5)$

  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

- focus on $x-1$:

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice $\left\{ \begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right.$

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- assume $\exists$ smallest counterexample x. $\Big\}$ $\boxed{2^x \leq x^2}$ $(x > 5)$

  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

- focus on $x-1$: $\boxed{2^{x-1} > (x-1)^2}$

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice $\left\{ \begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right.$

- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. } $\boxed{2^x \leq x^2}$ ($x > 5$)

  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

- focus on $x-1$: $\boxed{2^{x-1} > (x-1)^2}$ → combine to get contradiction

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because ?

$\boxed{2^x \leq x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and ?

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and not the smallest case

next?

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$\boxed{2^x \le x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

$\boxed{2^x \leq x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

if $x^2 - 4x + 2 \geqslant 0$

then ?

$\boxed{2^x \leq x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

!

if $x^2 - 4x + 2 \geqslant 0$ we will get a contradiction

$\boxed{2^x \leq x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2-2x+1$$

$$2^{x-1}\cdot 2 > 2x^2-4x+2$$

$$2^x > 2x^2-4x+2$$

$$2^x > x^2+(x^2-4x+2)$$

!

if $x^2-4x+2 \geqslant 0$ we will get a contradiction

$\updownarrow$

$(x-2)\cdot(x-2) \geqslant 2$

true for $x \geqslant 4$

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample and not the smallest case

!

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

if $x^2 - 4x + 2 \geqslant 0$ we will get a contradiction

$\updownarrow$

$(x-2)\cdot(x-2) \geqslant 2$

true for $x \geqslant 4$

DONE

conclusion

For $n \in \mathbb{Z}$, $n \geqslant 5$, $2^n > n^2$

FIBONACCI NUMBERS

FIBONACCI NUMBERS

$F_0 = 1$ $F_1 = 1$

FIBONACCI NUMBERS

for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1 \qquad F_1 = 1$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

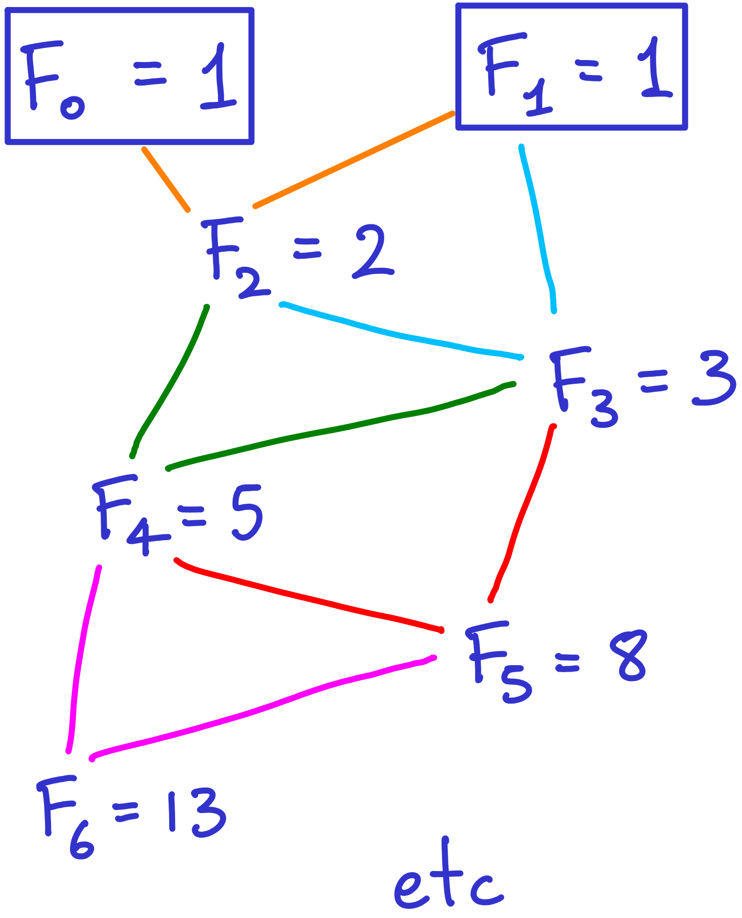
$F_5 = 8$

$F_6 = 13$

etc

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$



Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

etc

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\hookrightarrow F_x > 1.7^x$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

etc

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\hookrightarrow F_x > 1.7^x$

we want a contradiction, so most likely this will involve F_{x-1}

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

etc

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

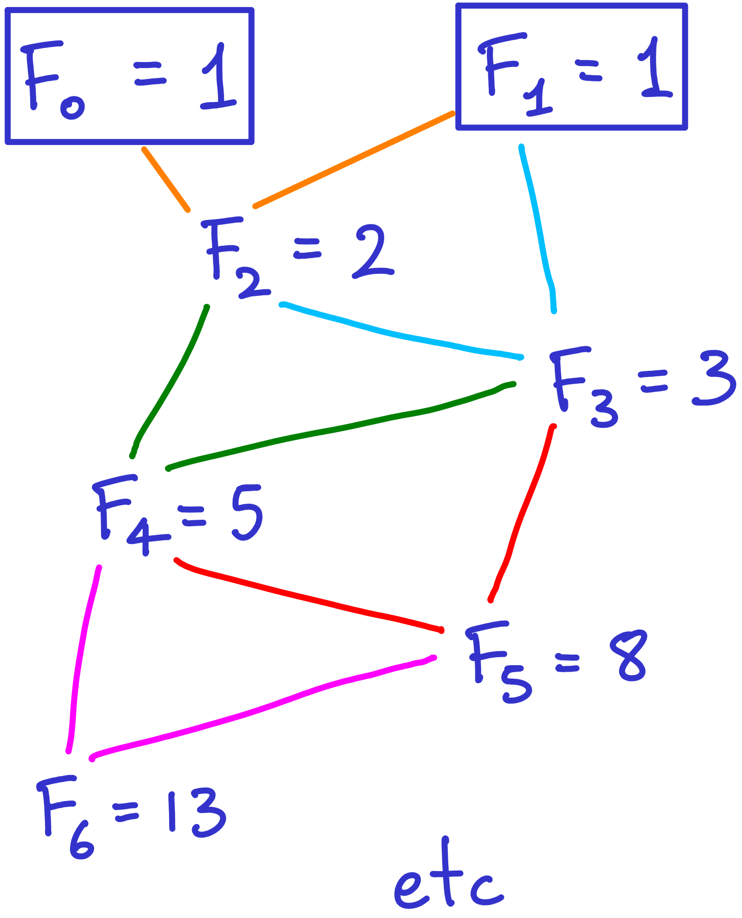
$\hookrightarrow F_x > 1.7^x$

we want a contradiction, so most likely this will involve F_{x-1}

slight hiccup ?

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$



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we want a contradiction, so most likely this will involve F_{x-1}

it will be hard to use only F_x & F_{x-1}

FIBONACCI NUMBERS

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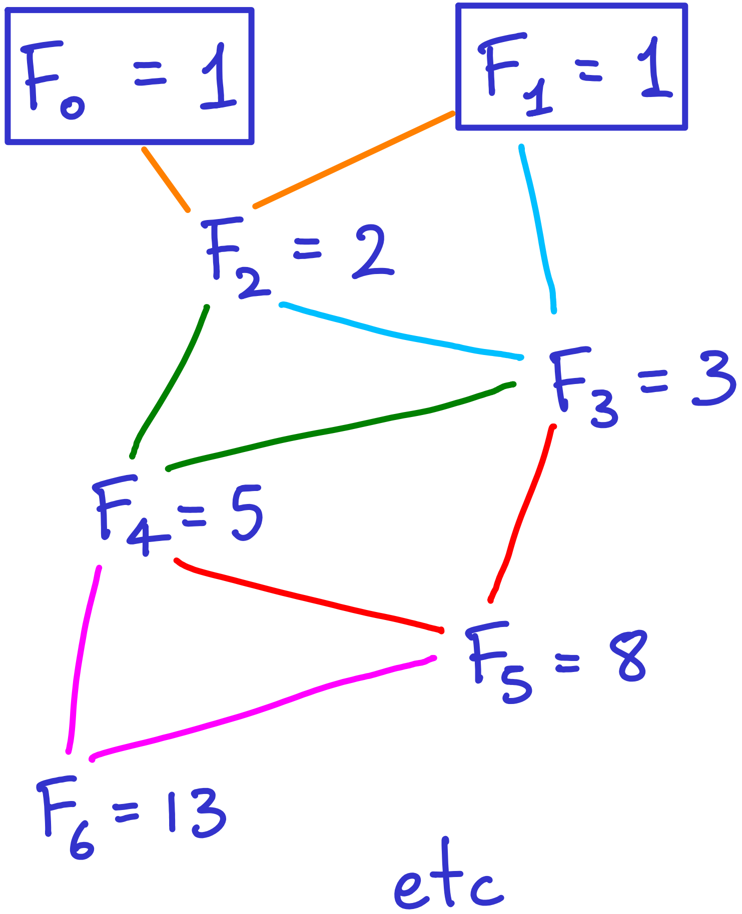

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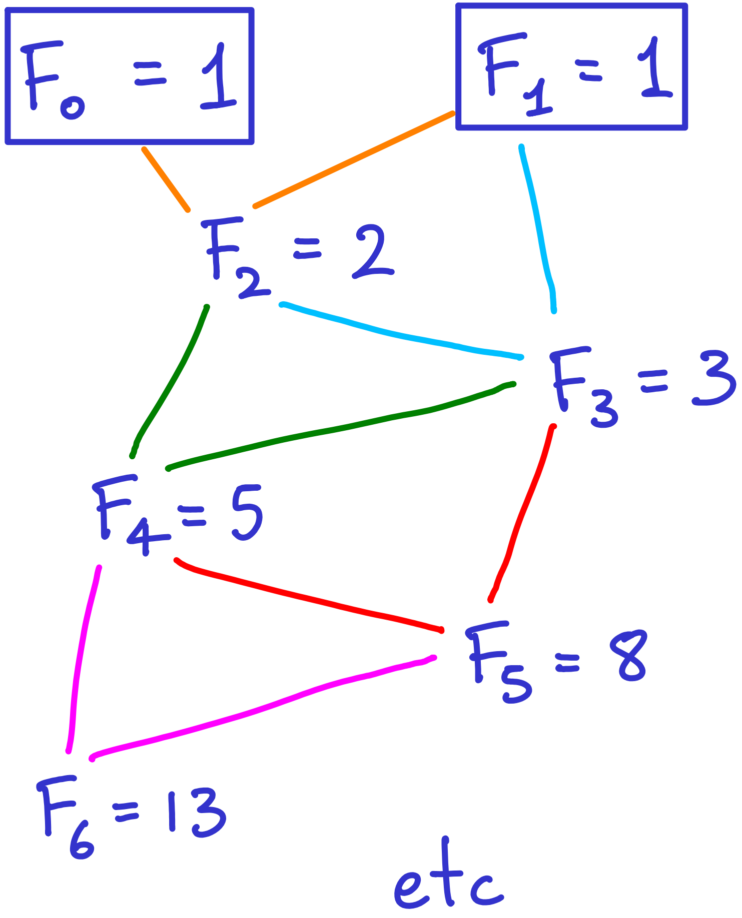
but it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geq 2$

FIBONACCI NUMBERS

for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$



Claim: for $n \in \mathbb{Z}$, $n \geqslant 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\hookrightarrow F_x > 1.7^x$

we want a contradiction, so most likely this will involve F_{x-1}

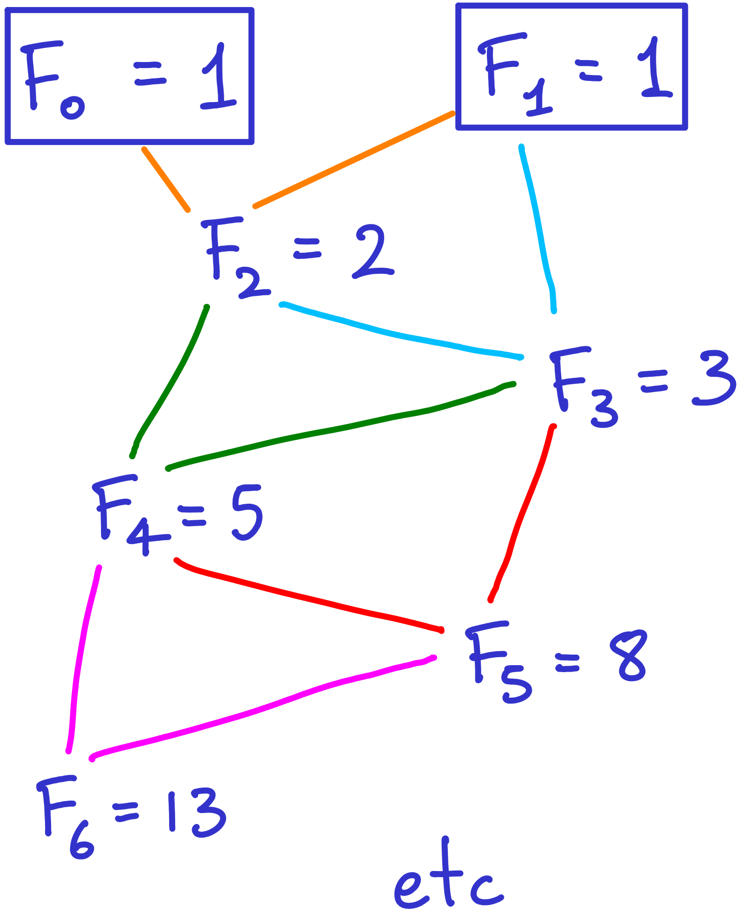
but it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geqslant 2$

$\hookrightarrow$ is $F_0 \leq 1.7^0$?

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$



Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\hookrightarrow F_x > 1.7^x$

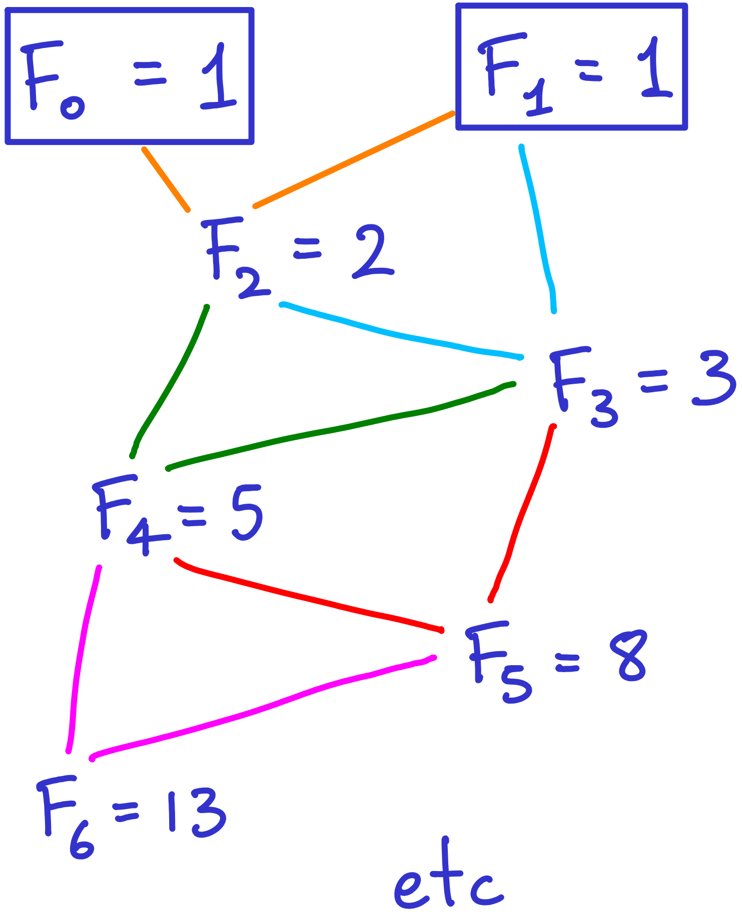
we want a contradiction, so
most likely this will involve F_{x-1}

but it will be hard to use only F_x & F_{x-1}
so why not use F_{x-2} also: assume $x \geq 2$

$\hookrightarrow$ is $F_0 \leq 1.7^0$? yes. is $F_1 \leq 1.7^1$?

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$



Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\hookrightarrow F_x > 1.7^x$

we want a contradiction, so most likely this will involve F_{x-1}

but it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geq 2$

$\hookrightarrow$ is $F_0 \leq 1.7^0$? yes. is $F_1 \leq 1.7^1$? yes. OK!

$F_0 = F_1 = 1$ // for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geqslant 0$, $F_n \leq 1.7^n$

$F_0 = F_1 = 1$ // for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ $(x \geq 2)$

& we can safely assume $F_y \leq 1.7^y$ for $y < x$

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next?

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we can now say: $F_x = F_{x-1} + F_{x-2}$ ...

$F_0 = F_1 = 1$ // for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

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we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

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we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$= 1.7^{x-2} \cdot (1.7 + 1)$

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we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$$= 1.7^{x-2} \cdot (1.7 + 1)$$

$$= 1.7^{x-2} \cdot 2.7$$

$F_0 = F_1 = 1$ // for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$
$(x \geq 2)$

& we can safely assume
$F_y \leq 1.7^y$ for $y < x$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$$= 1.7^{x-2} \cdot (1.7 + 1)$$

$$= 1.7^{x-2} \cdot 2.7$$

$$< 1.7^{x-2} \cdot (1.7)^2$$

$[1.7^2 = 2.89]$

next?

$F_0 = F_1 = 1$ // for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geqslant 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ $(x \geqslant 2)$ & we can safely assume $F_y \leq 1.7^y$ for $y < x$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$$= 1.7^{x-2} \cdot (1.7 + 1)$$

$$= 1.7^{x-2} \cdot 2.7$$

$$< 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89]$$

$$= 1.7^x \quad \text{so?}$$

$F_0 = F_1 = 1$ // for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ $(x \geq 2)$

& we can safely assume $F_y \leq 1.7^y$ for $y < x$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

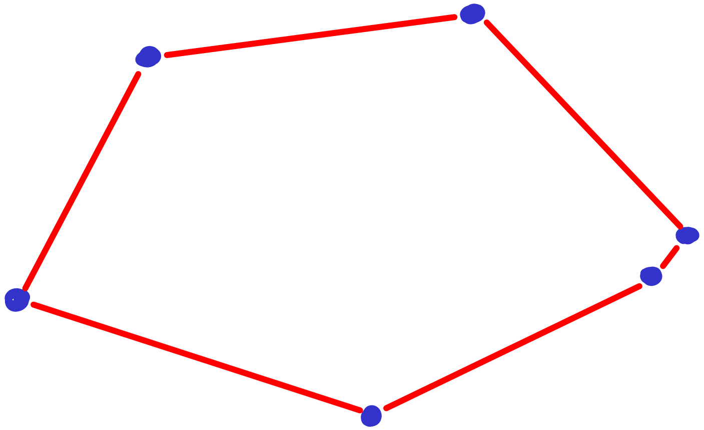
$$= 1.7^{x-2} \cdot (1.7 + 1)$$
$$= 1.7^{x-2} \cdot 2.7$$
$$< 1.7^{x-2} \cdot (1.7)^2$$
$$= 1.7^x$$

$[1.7^2 = 2.89]$

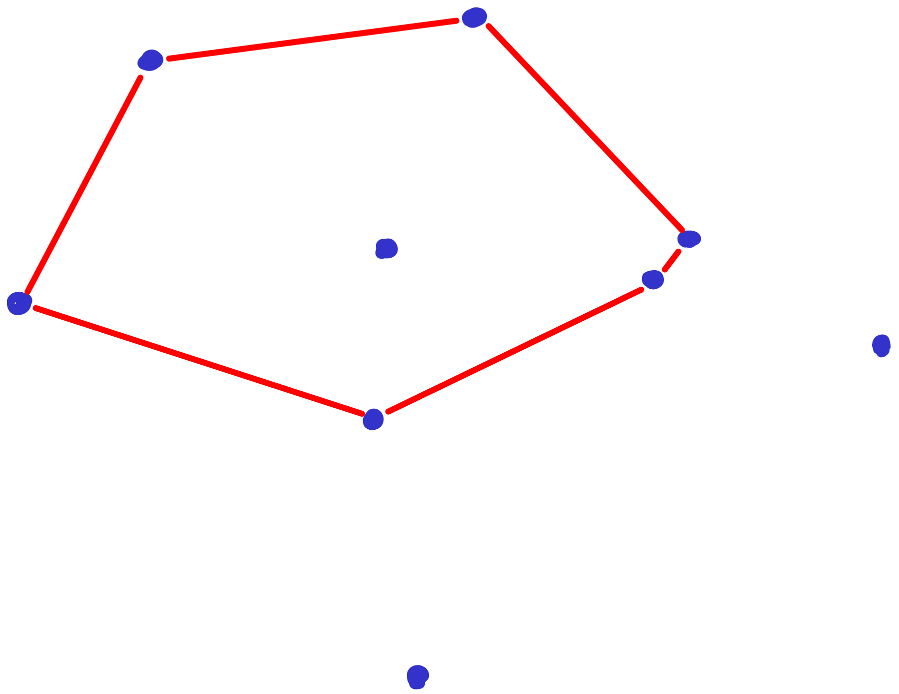
so $F_x < 1.7^x$

CONTRADICTION

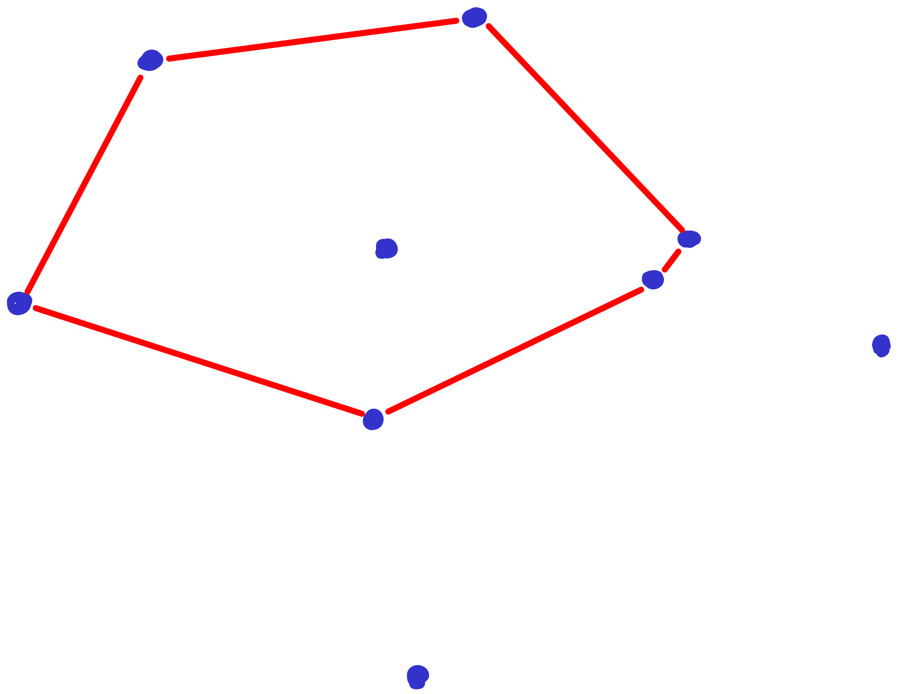
6 points in convex position.



still 6 points in convex position.

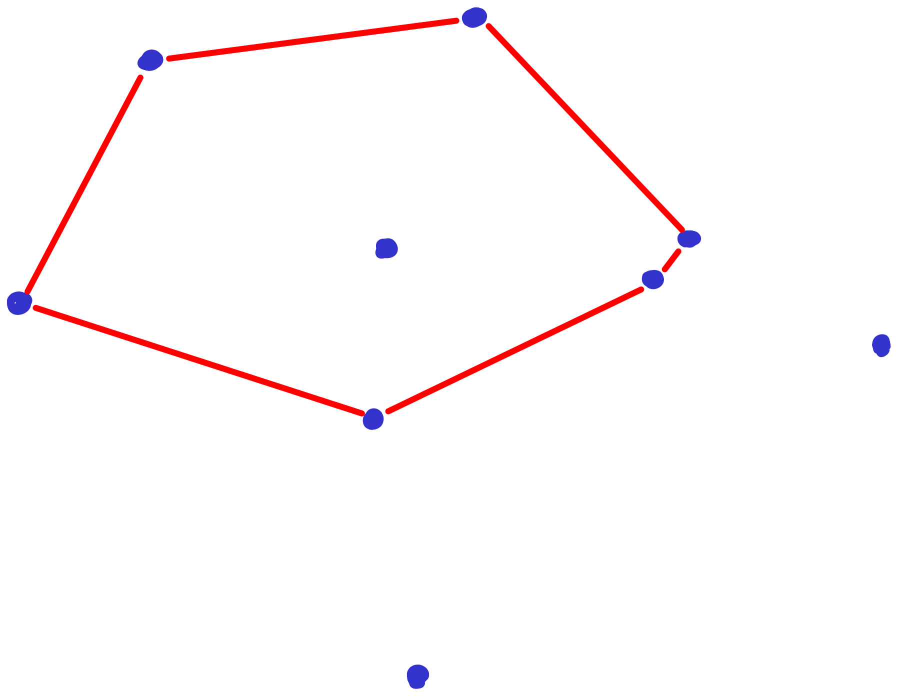


Theorem: in $\mathbb{R}^2$, every set of ≥ 17 points w/ no 3 on a line has 6 points in convex position.



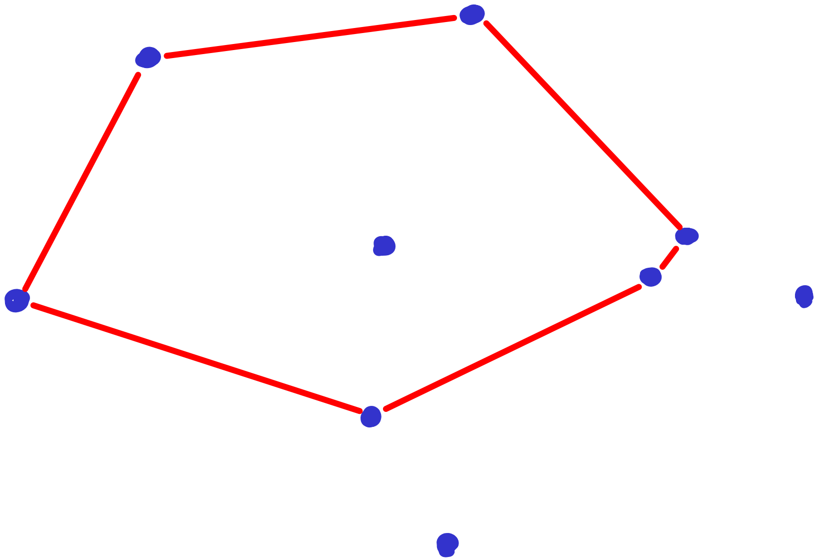
Theorem: in $\mathbb{R}^2$, every set of $\geqslant 17$ points w/ no 3 on a line has 6 points in convex position.

"has a hexagon"
(not necessarily regular)

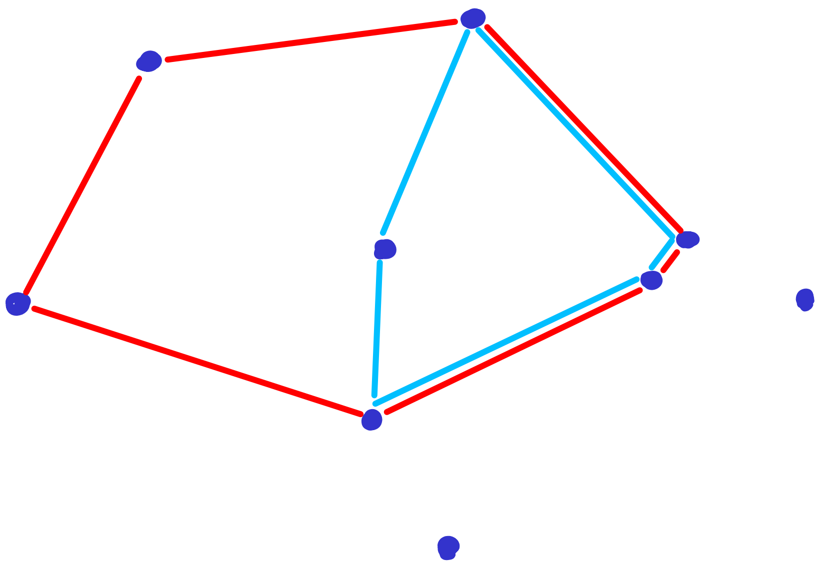


Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,

if P has 6 points forming a hexagon...



Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.



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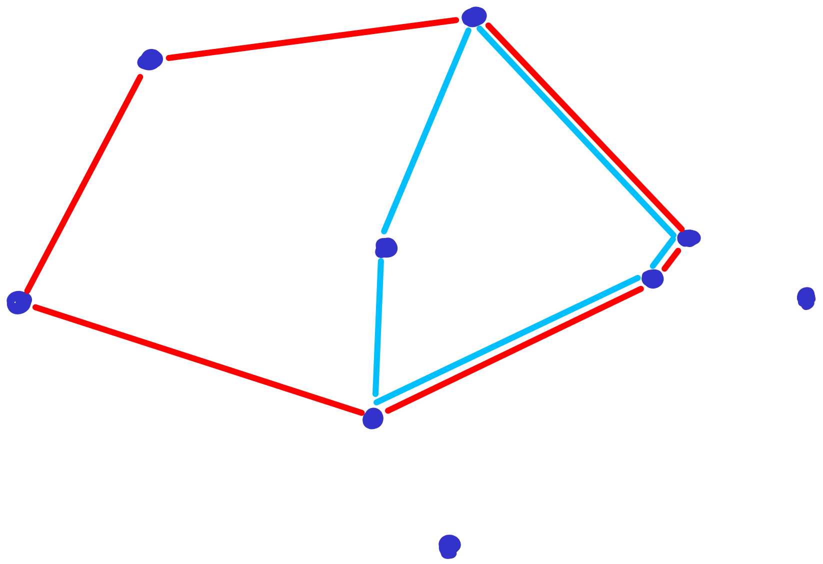
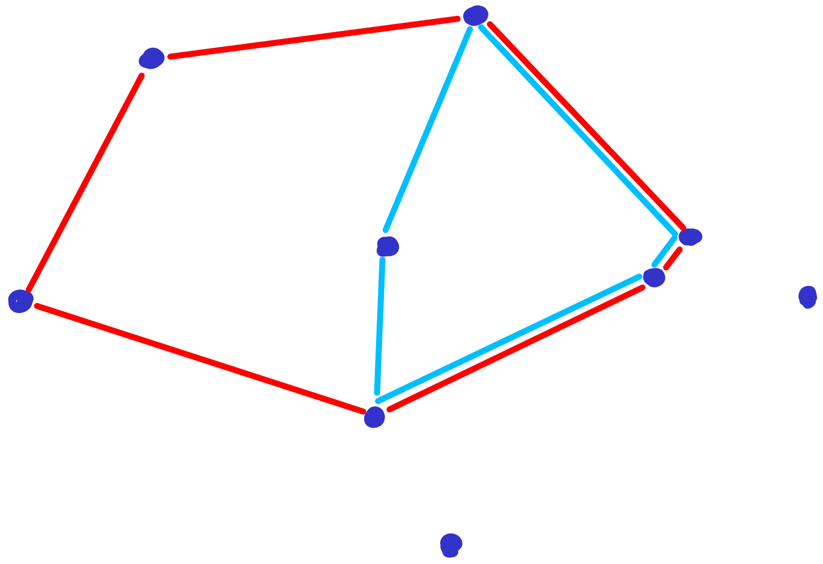


Stronger claim:
the empty pentagon is inside a hexagon

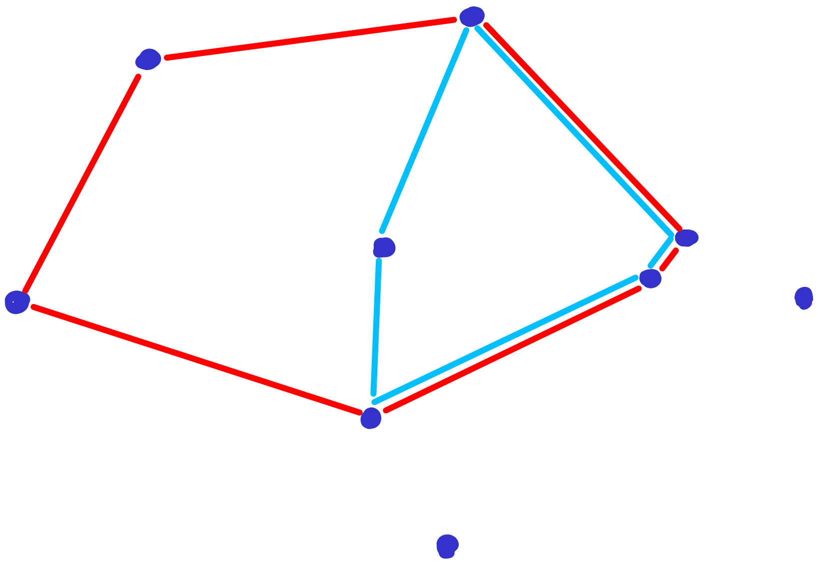
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.



Stronger claim:
the empty pentagon is
inside a hexagon

In fact, inside the hexagon
containing the fewest number of points

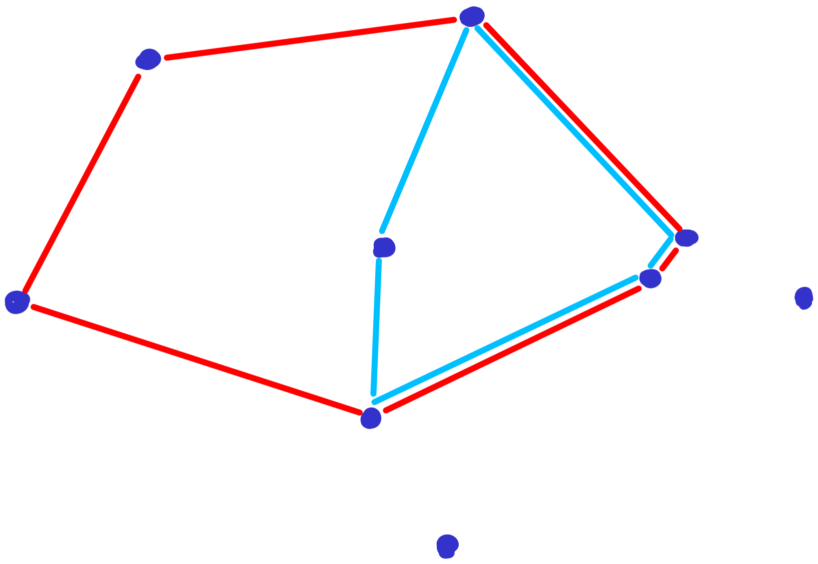
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line, if P has 6 points forming a hexagon then P has 5 points forming an empty pentagon.



PROOF SKETCH:

Choose a hexagon H containing min #pts.

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line, if P has 6 points forming a hexagon then P has 5 points forming an empty pentagon.

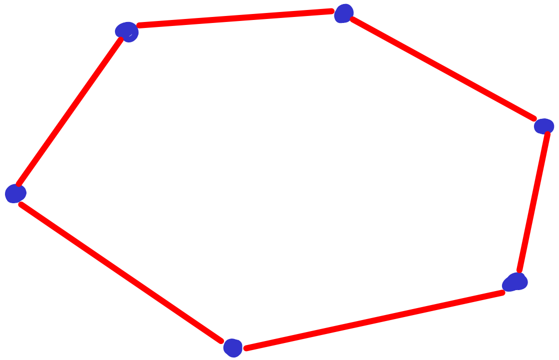


PROOF SKETCH:

Choose a hexagon H containing min # pts.

- if H is empty, DONE.

basis case (or base case)



Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
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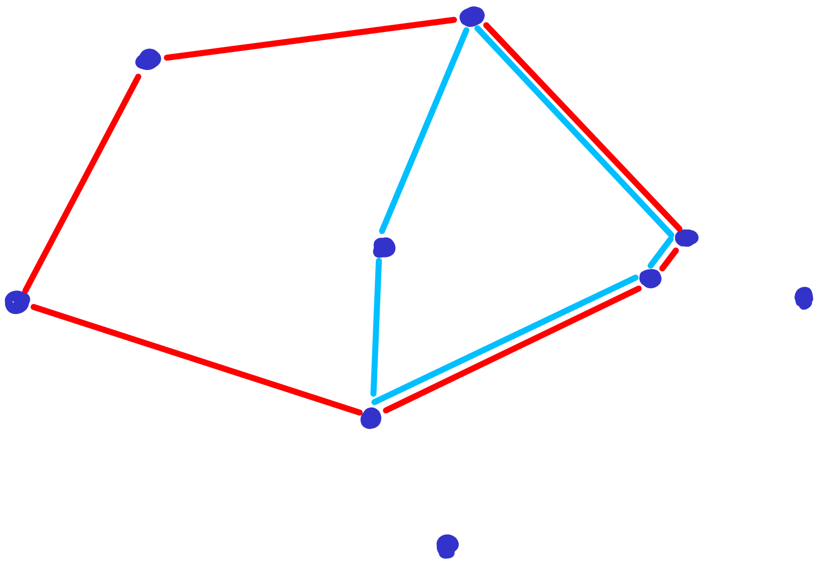
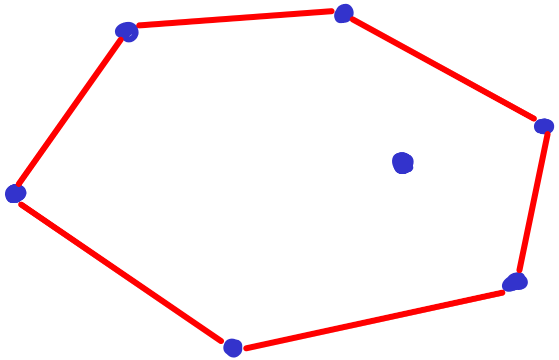
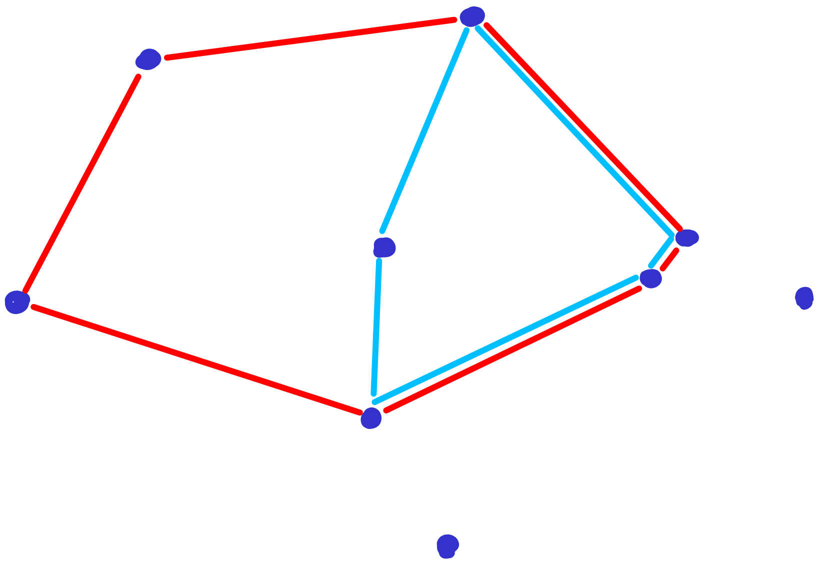


PROOF SKETCH:

Choose a hexagon H containing min #pts.
- if H is empty, DONE.
- if H contains exactly 1 point,



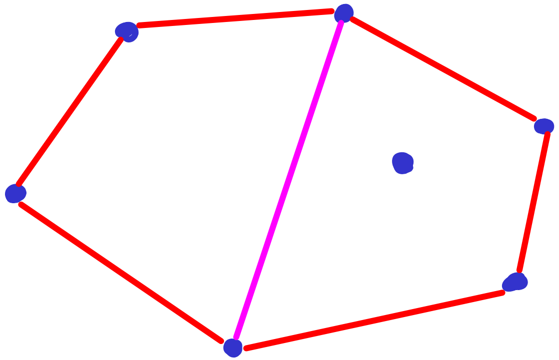
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line, if P has 6 points forming a hexagon then P has 5 points forming an empty pentagon.



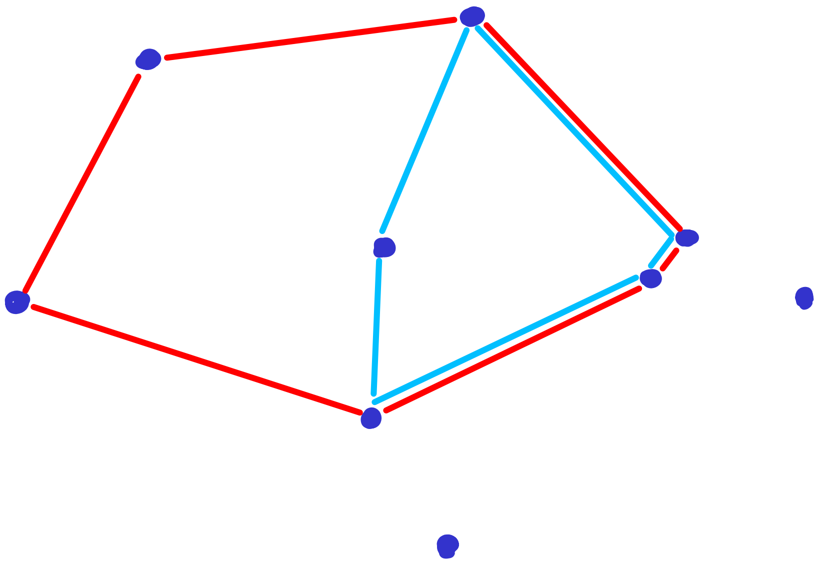
PROOF SKETCH:

Choose a hexagon H containing min #pts.

- if H is empty, DONE.
- if H contains exactly 1 point, "split" H



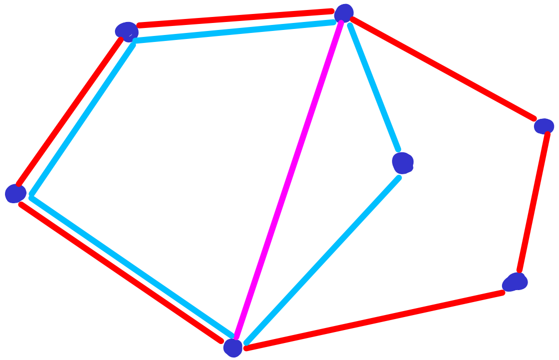
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.



PROOF SKETCH:

Choose a hexagon H containing min #pts.

- if H is empty, DONE.
- if H contains exactly 1 point,
  "split" H and then we are DONE.



cont'd ...

(so if there is a counterexample to our claim, then H must contain at least 2 points)

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.
-shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.

Note: we will next assume that there is some counterexample, for the sake of contradiction.
A counterexample will consist of a point set where H does not contain an empty pentagon.

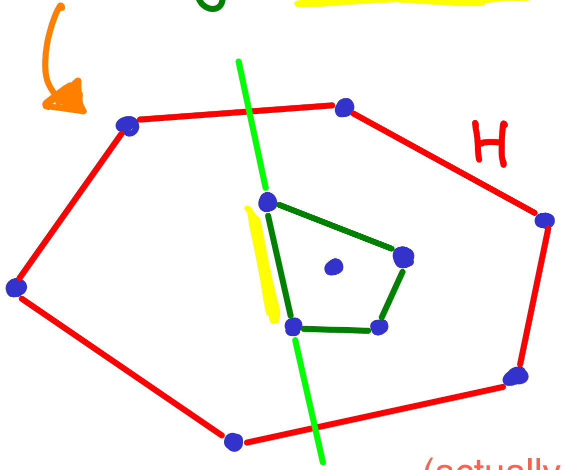
We order all possible occurrences of H, according to the number of points, n, in H.
We've seen that if n=0 or 1, there is no counterexample. So we have a base case.
Next we assume that there is some smallest counterexample, where H contains x>1 points.
So we will draw a hexagon H, with the constraint that it contains at least 2 points, and that it is the hexagon with fewest number of points inside.

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.

- if any "extreme segment" of interior points "isolates" 3 points of H ...



As explained in class:
There are at least 2 points strictly inside the hexagon H.
Form a tight "fence" around those points. (fyi: this is actually called the "convex hull").
For every "extreme segment" on that fence (shown in yellow here), extend into a line (green). That line has all of the interior points on one side, by its definition.
On the other side of the line are some number of points from H.
I say that those points are "isolated". In the picture here, 3 points are isolated.

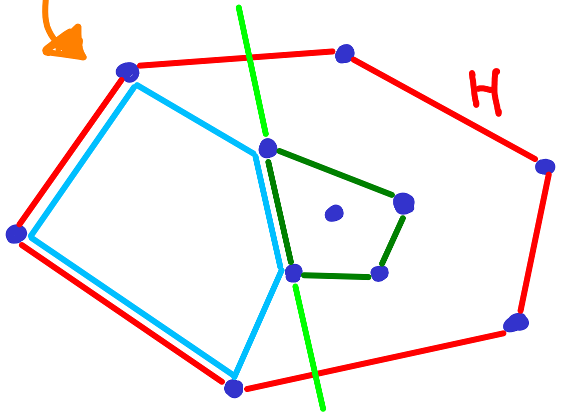
(actually, if H contains exactly 2 points, then there are no other interior points to be "on one side" of the green line. In this case we must isolate at least 3 points on one side of the green line)

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min #pts.

- shown: if H contains ≤ 1 points, DONE → so assume ≥ 2 pts inside.

- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.



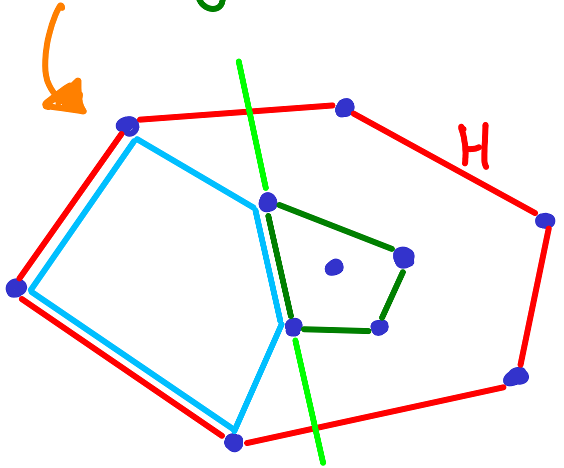
… because we isolated 3 points using an extreme segment,
we find an empty pentagon.

This contradicts any attempt to claim that H is a (smallest) counterexample.
So if H has any hope of being a counterexample, then no extreme segment can isolate >2 points of H. So we will proceed by assuming this.

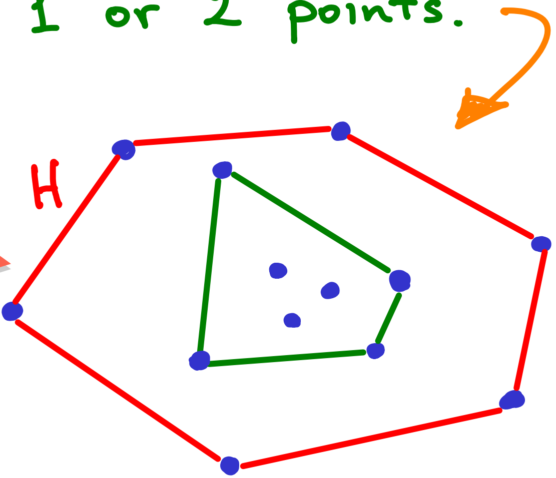
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.
  ↳ so every such segment isolates 1 or 2 points.



(note that I have drawn a picture where every extreme segment isolates 2 points from H.
If only 1 point is isolated, we'll get the same conclusion. Convince yourself that every extreme segment of the interior points isolates at least one point of H.)

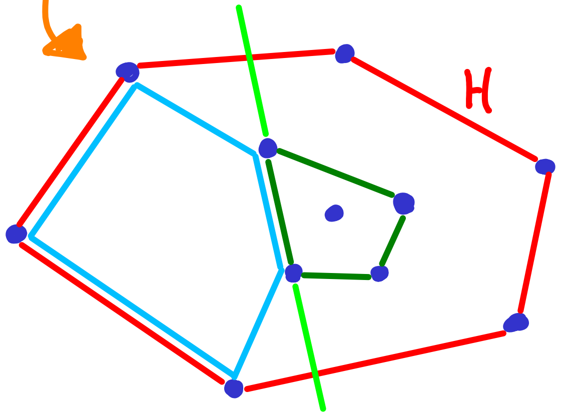


Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

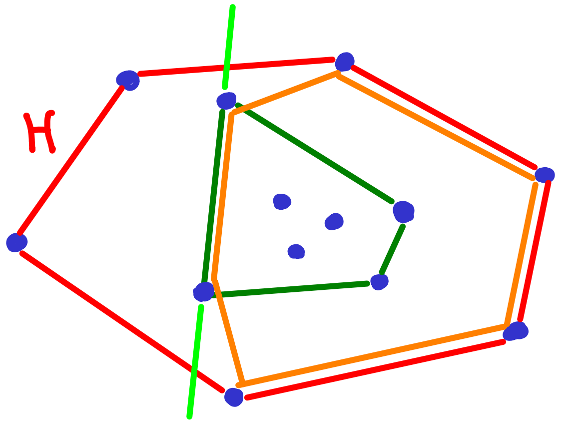
Choose a hexagon H containing min #pts.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.

  ↳ so every such segment isolates 1 or 2 points.



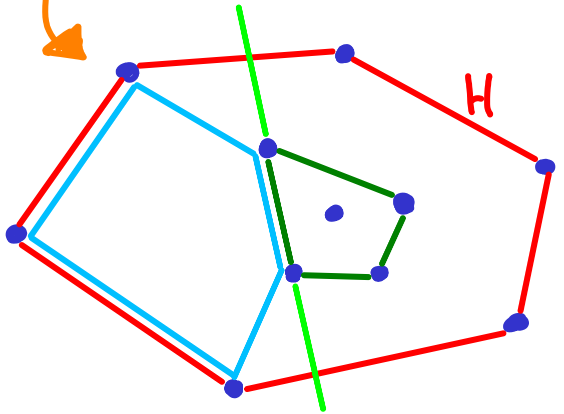
↳ use one segment
& form a hexagon
(why?)

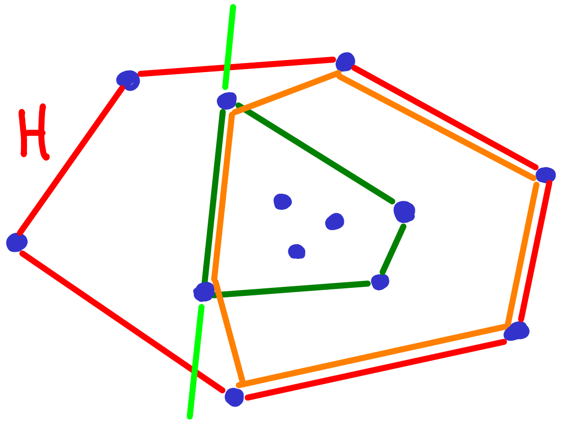


Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤ 1 points, DONE → so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.
  ↳ so every such segment isolates 1 or 2 points.
  ↳ use one segment & form a hexagon containing fewer points than H.
  (so?)



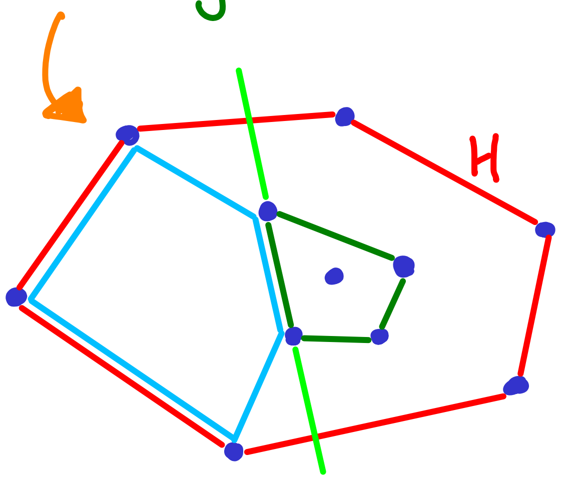


Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤1 points, DONE → so assume ≥2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.

↳ so every such segment isolates 1 or 2 points.



↳ use one segment & form a hexagon containing fewer points than H.

contradicts [H = "smallest"]