SMALLEST COUNTEREXAMPLE

Prove that the first n odd natural numbers sum to n^2.

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. Now what?

We saw the claim is true for small i.

If the claim is false, there must be some j for which this happens.

Focus on the smallest such j & on j-1

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$?

if false, then $\exists x$ for which it is false & $\boxed{\underbrace{x-1}_{} \text{ for which it is true}}$

↳ in fact for all $i < x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i = x-1} = (x-1)^2$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\boxed{\exists x \text{ for which it is false}}$ & $\underbrace{x-1}$ for which it is true

in fact for all $i<x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

$$1 + 3 + 5 + \cdots + (2x-3) + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}$ for which it is true

↳ in fact for all $i<x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

← either this should have been $\neq$

$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2$$

← or this should have been =

$$(x-1)^2 + 2x-1 \neq x^2$$

$$x^2 \cancel{-2x} \cancel{+1} \cancel{+2x} \cancel{-1} \neq x^2$$

contradiction

... which contradicts the smallest counterexample assumption, i.e.,

THERE IS NO (SMALLEST) COUNTEREXAMPLE

↳ CLAIM IS TRUE

SMALLEST COUNTEREXAMPLE recap

see p. 131 "well-ordering principle"

- be able to "count" & "order" instances of the claim

  (case / example)

- prove the claim for smallest instance

- assume the claim is false: then there is a smallest example, E_i, for which it is false

  (smallest counterexample)

- this implies the claim is true for the next smallest example, E_{i-1}.

- use E_i & E_{i-1} to get a contradiction (to the existence of any counterexample)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice
$$\left\{\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array}\right.$$

- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. $\Big\}$ $\boxed{2^x \leq x^2}$ $(x > 5)$

  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

- focus on $x-1$: $\boxed{2^{x-1} > (x-1)^2}$ → combine to get contradiction

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

because x is the smallest counterexample

and not the smallest case

!

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

if $x^2 - 4x + 2 \geqslant 0$ we will get a contradiction

$$\Updownarrow$$

$$(x-2)\cdot(x-2) \geqslant 2$$

true for $x \geqslant 4$

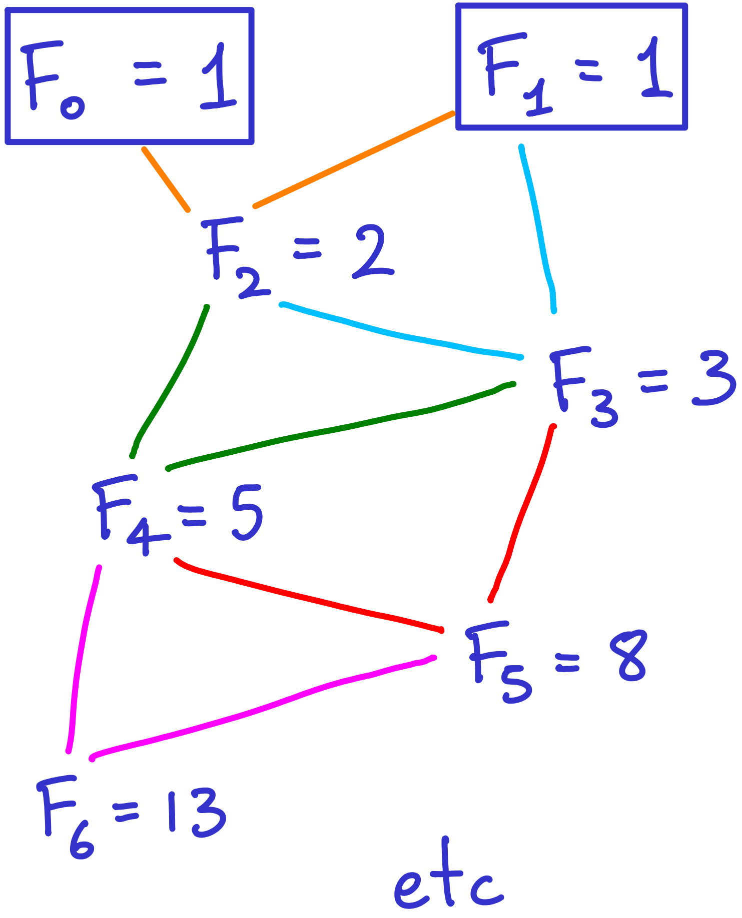
DONE

conclusion

For $n \in \mathbb{Z}$, $n \geqslant 5$, $2^n > n^2$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$



Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

suppose smallest counterexample is $n = x$

$\hookrightarrow F_x > 1.7^x$

we want a contradiction, so most likely this will involve F_{x-1}

but it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geq 2$

$\hookrightarrow$ is $F_0 \leq 1.7^0$? yes. is $F_1 \leq 1.7^1$? yes. OK!

$F_0 = F_1 = 1$ // for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geqslant 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ $(x \geqslant 2)$

& we can safely assume $F_y \leq 1.7^y$ for $y < x$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

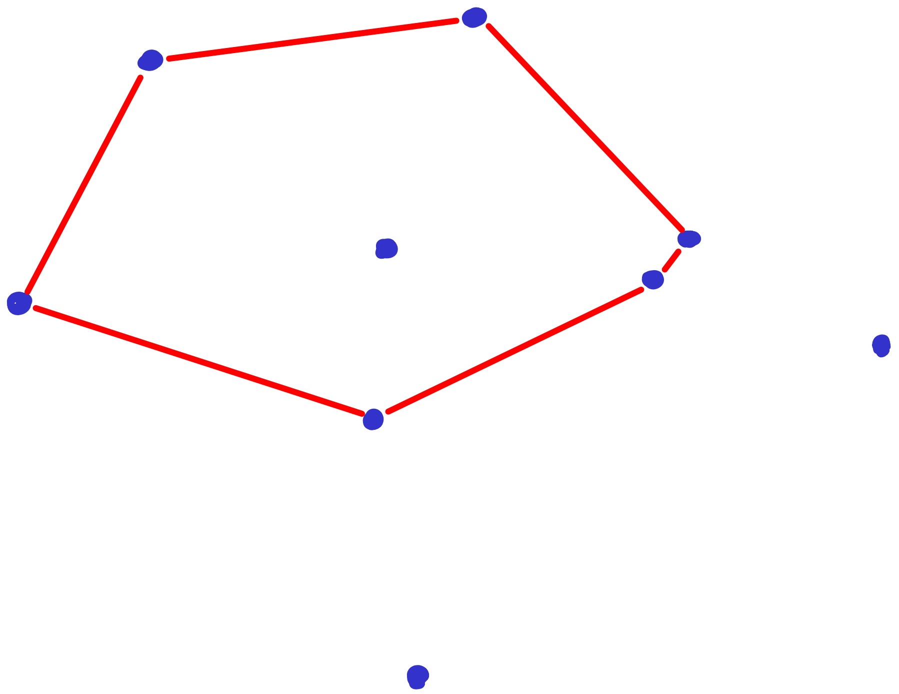
$$\begin{aligned} &= 1.7^{x-2} \cdot (1.7 + 1) \\ &= 1.7^{x-2} \cdot 2.7 \\ &< 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89] \\ &= 1.7^x \end{aligned}$$

so $F_x < 1.7^x$

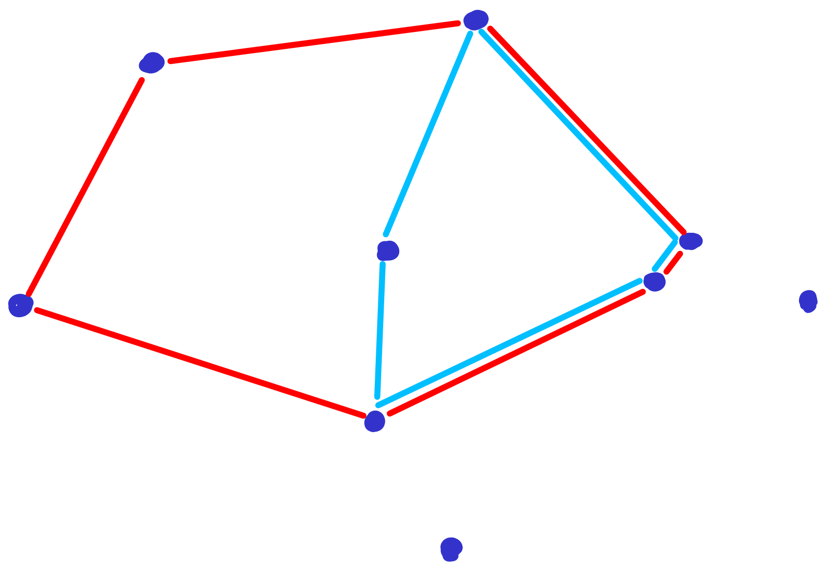
CONTRADICTION

Theorem: in $\mathbb{R}^2$, every set of $\geqslant 17$ points w/ no 3 on a line has 6 points in convex position.

"has a hexagon"
(not necessarily regular)



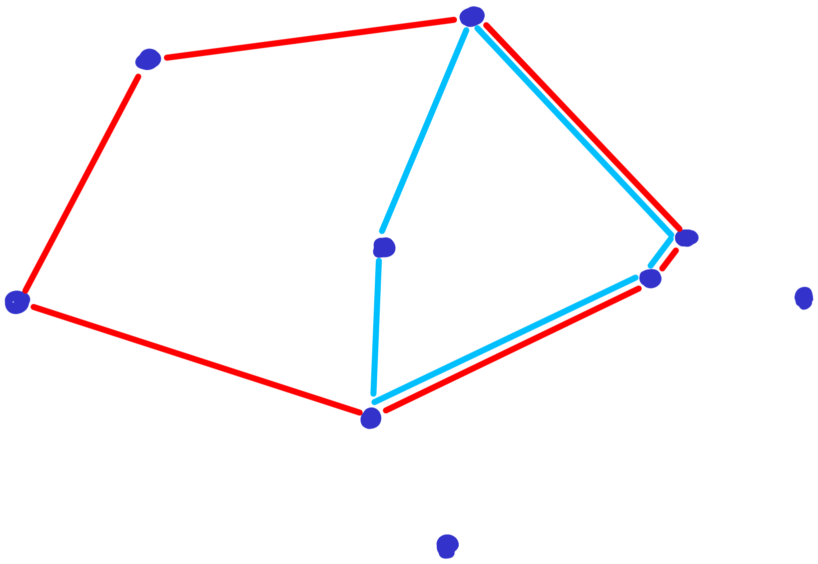
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line, if P has 6 points forming a hexagon then P has 5 points forming an empty pentagon.



Stronger claim: the empty pentagon is inside a hexagon

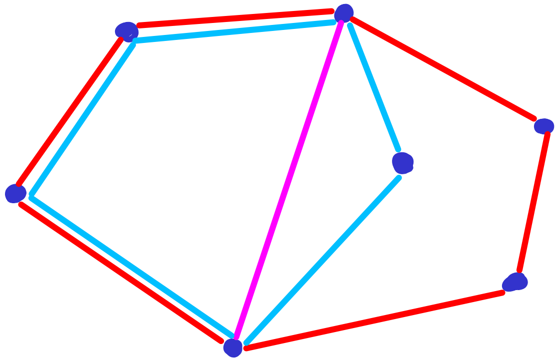
In fact, inside the hexagon containing the fewest number of points

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.



PROOF SKETCH:

Choose a hexagon H containing min # pts.
- if H is empty, DONE.
- if H contains exactly 1 point,
"split" H and then we are DONE.



cont'd ...

(so if there is a counterexample to our claim, then H must contain at least 2 points)

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.

Note: we will next assume that there is some counterexample, for the sake of contradiction.
A counterexample will consist of a point set where H does not contain an empty pentagon.

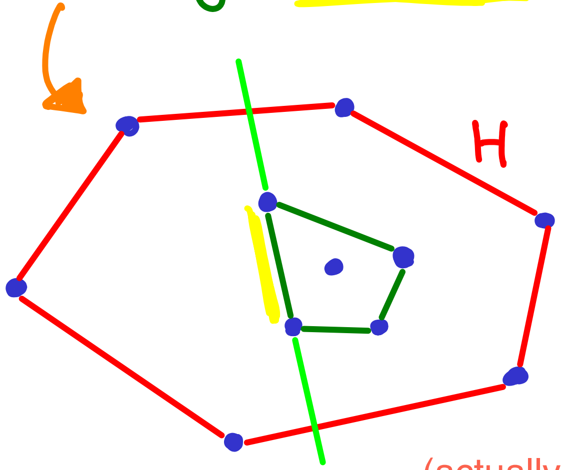
We order all possible occurrences of H, according to the number of points, n, in H.
We've seen that if n=0 or 1, there is no counterexample. So we have a base case.
Next we assume that there is some smallest counterexample, where H contains x>1 points.
So we will draw a hexagon H, with the constraint that it contains at least 2 points, and that
it is the hexagon with fewest number of points inside.

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.

- if any "extreme segment" of interior points "isolates" 3 points of H ...



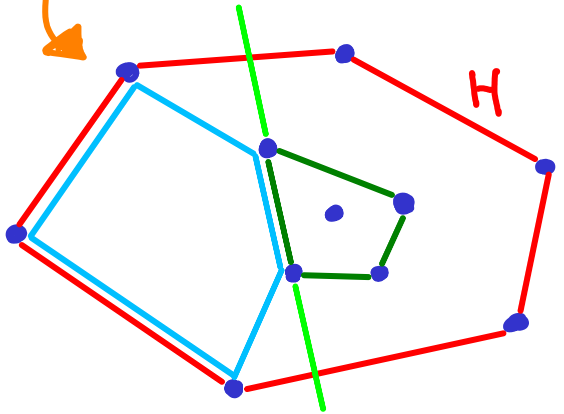
As explained in class:
There are at least 2 points strictly inside the hexagon H.
Form a tight "fence" around those points. (fyi: this is actually called the "convex hull").
For every "extreme segment" on that fence (shown in yellow here), extend into a line (green). That line has all of the interior points on one side, by its definition.
On the other side of the line are some number of points from H.
I say that those points are "isolated". In the picture here, 3 points are isolated.

(actually, if H contains exactly 2 points, then there are no other interior points to be "on one side" of the green line. In this case we must isolate at least 3 points on one side of the green line)

Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min #pts.

- shown: if H contains ≤ 1 points, DONE → so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.



… because we isolated 3 points using an extreme segment,
we find an empty pentagon.

This contradicts any attempt to claim that H is a (smallest) counterexample.
So if H has any hope of being a counterexample, then no extreme segment can isolate >2 points of H. So we will proceed by assuming this.

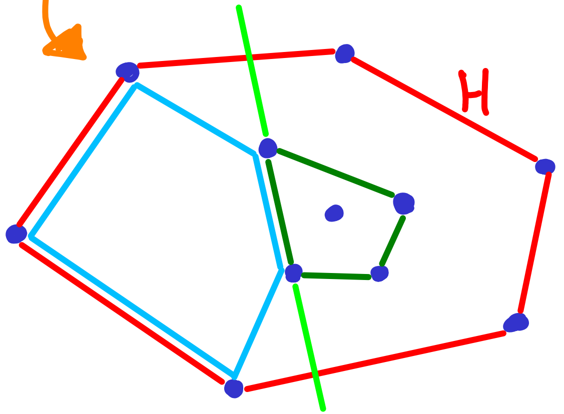
Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line, if P has 6 points forming a hexagon then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

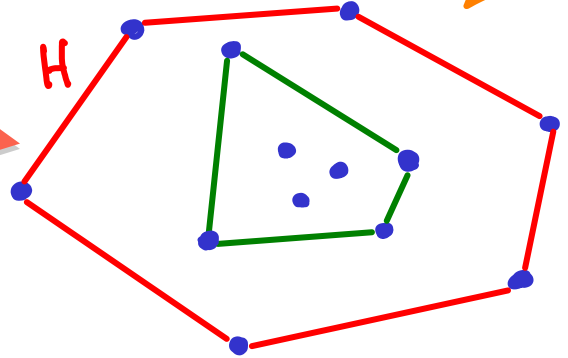
- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.

- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.

$\hookrightarrow$ so every such segment isolates 1 or 2 points.



(note that I have drawn a picture where every extreme segment isolates 2 points from H.
If only 1 point is isolated, we'll get the same conclusion. Convince yourself that every extreme segment of the interior points isolates at least one point of H.)

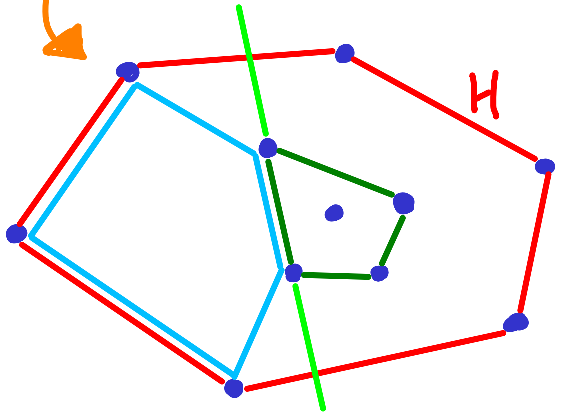


Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.

Choose a hexagon H containing min # pts.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.
- if any "extreme segment" of interior points "isolates" 3 points of H, DONE.

↳ so every such segment isolates 1 or 2 points.



↳ use one segment & form a hexagon containing fewer points than H.

contradicts [H = "smallest"]