CONTRAPOSITIVE

if you came to the first lecture, you've seen me before (fact)
(for Spring'14)

$\updownarrow$ equivalent $\updownarrow$

if you've never seen me before, you didn't come to the first lecture

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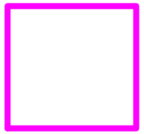
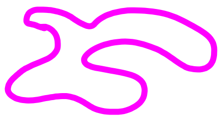
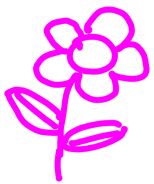
if you are a square, you have corners  (fact)

$\updownarrow$ equivalent $\updownarrow$

if you don't have corners, you are not a square  

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if A then B = if not B, then not A

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \quad = \quad \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's ok; doesn't contradict the above. It's not IFF

a	b	$a \to b$ valid?	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$ valid?
T	T	✓	F	F	✓
T	F	✗	T	F	✗
F	T	✓	F	T	✓
F	F	✓	T	T	✓

don't contradict

don't contradict

Context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct

$$7x+9 = 2a \quad // a: \text{integer} \to 7x+9: \text{even}$$
$$x = 2a-6x-9$$
$$x = 2a-6x-10+1$$
$$x = 2(a-3x-5)+1$$
$$x = 2b+1 \ (\text{odd}) \ (b = a-3x-5) \quad \square$$

contrapositive

Suppose x is not odd: $x = 2c$

$$7x+9 = 7 \cdot 2c + 9$$
$$= 14c + 8 + 1$$
$$= 2 \cdot (7c+4) + 1$$
$$= 2 \cdot d + 1 \quad (d = 7c+4)$$
$$7x+9 = \text{odd} \quad \square$$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

$$x^2 - 6x + 5 = 2a$$

direct

$$x^2 - 6x + (5 - 2a) = 0$$

$$x = \frac{6 \pm \sqrt{36 + 8a - 20}}{2}$$

$$x = 3 \pm \sqrt{4 + 2a}$$

$\vdots$?

$$x = 2b + 1$$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

direct

$x^2 - 6x + 5 = 2a$

$\vdots$?

$x = 2b + 1$

contrapositive

Suppose x is not odd: $x = 2c$

$$\begin{aligned} x^2 - 6x + 5 &= (2c)^2 - 6 \cdot 2c + 5 \\ &= 4c^2 - 12c + 5 \\ &= 4c^2 - 12c + 4 + 1 \\ &= 2 \cdot (2c^2 - 6c + 2) + 1 \\ &= 2 \cdot d + 1 \qquad (d = 2c^2 - 6c + 2) \\ &= \text{not even} \quad \square \end{aligned}$$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if x is irrational then $\sqrt{x}$ is irrational

direct

???

contrapositive

Suppose $\sqrt{x}$ is not irrational

$$\sqrt{x} = \frac{a}{b}$$

$$x = \frac{a^2}{b^2} \quad : \text{ not irrational} \quad \square$$

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving if A then B for now

$$(a+b)\cdot(a-b) \longleftrightarrow a^2 - ab + ba - b^2 \longleftrightarrow a^2 - b^2$$

You can prove something directly (in one direction)

or work in both directions

instead of starting w/ $\neg B$ & leading to $\neg A$

(which contradicts $A \to \neg B$)

$$\boxed{\text{assume both } A \text{ and } \neg B \text{ are true \& arrive at some contradicting statement}}$$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

$$\downarrow \qquad\qquad \downarrow$$

(a: int.) $x = 2a$ $\qquad$ $x = 2b+1$ (b: int.)

$$2a = 2b+1$$

$$a = b + \tfrac{1}{2}$$

impossible/absurd/contradiction

Notice we met halfway at an incorrect statement.

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $az + b = 0$.

(if $\underbrace{ax+b=0}_{A}$ then for $y \neq x$, $\underbrace{ay+b \neq 0}_{B}$)

Assume $A \wedge \neg B$: $\quad ax+b=0 \quad \& \quad ay+b=0$

$$ax+b = ay+b$$

$$ax = ay$$

$$x = y \text{ — contradicts}$$

$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong
4) conclude that (2) is false thus the initial claim is true

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ <u>no common divisor</u>

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even
(a: even)

$a = 2c$ $\{c: \text{int}\}$ $\Rightarrow 2b^2 = 4c^2 \Rightarrow b$: even

b=2d

$\sqrt{2} = \frac{a}{b} = \frac{2c}{2d}$ contradiction

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

- Assume that #primes is finite: $p_1, p_2, \dots, p_n$
- Let $t = 1 + \prod_{i=1}^{n} p_i$ (i.e., $1 + p_1 \times p_2 \times \dots \times p_n$)
- Notice $t > p_i$ for all i. So if t is prime, contradiction. (to the first assumption)
- If t is not prime then $\exists$ prime factor $q \neq t$ of t $\left.\right\}$ q is prime, divides t, and $1 < q < n$

  (you've seen this, t is composite)
  - if $q \neq p_i$ (for all i), contradiction. (to the first assumption)
  - if $q = p_j$, we know q divides $\prod_{i=1}^{n} p_i$ (because q is a factor of t)

    but then it can't also divide $1 + \prod_{i=1}^{n} p_i$ (contr.)