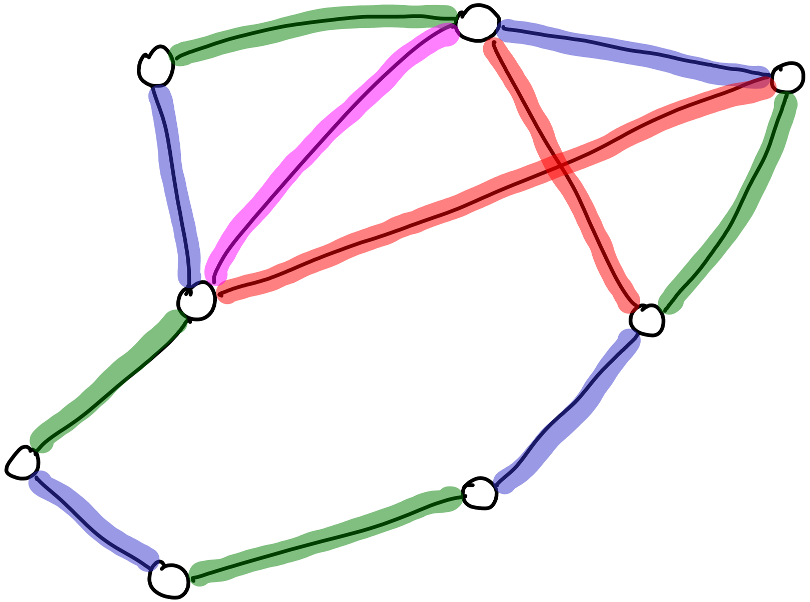
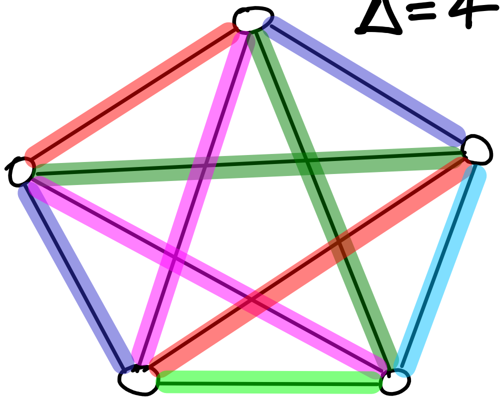
EDGE COLORING – no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



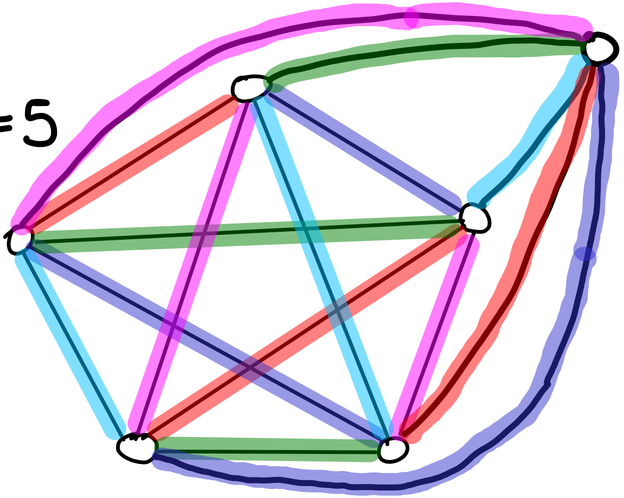
$\chi' = 4$
$\geq \Delta$

$\Delta = 4$



K_3	$\Delta = 2$	$\chi' = 3$
K_4	$\Delta = 3$	$\chi' = 4$
K_5	$\Delta = 4$	$\chi' = 5$
K_6	$\Delta = 5$	$\chi' = 5$!

$\chi' = 5$



K_7 can't continue extension

K_n : $f(n)$ and/or $f(\Delta)$?

Vizing's Thm: for all graphs, $\chi' \leq \Delta + 1$ (close to trivial lower bound)

There are proofs by induction on E, V and Δ

most common

Equivalent to algorithm that incrementally extends G.

Observation: If we are willing to use $\Delta+1$ colors, then

every vertex has ≥ 1 "missing" color. $d(v) \leq \Delta < \Delta+1$

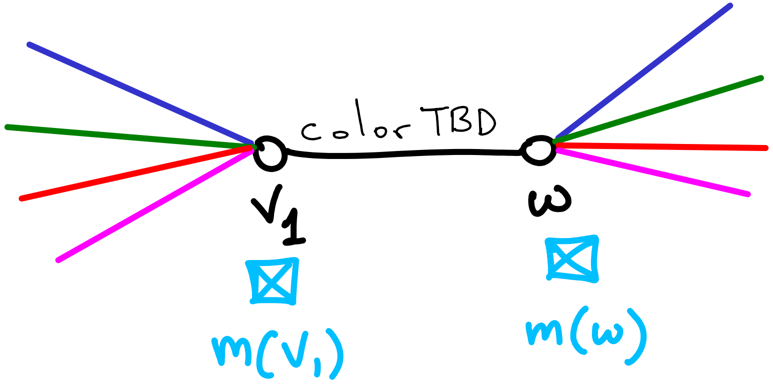
no incident edge has that color

Proof of Vizing's thm by induction on edges

(by constructive algorithm)

Assume we have colored $G - \{v_1, w\}$ $\qquad v_1 w =$ edge

We already know if $\exists$ $m(v_1) = m(w)$ then DONE.



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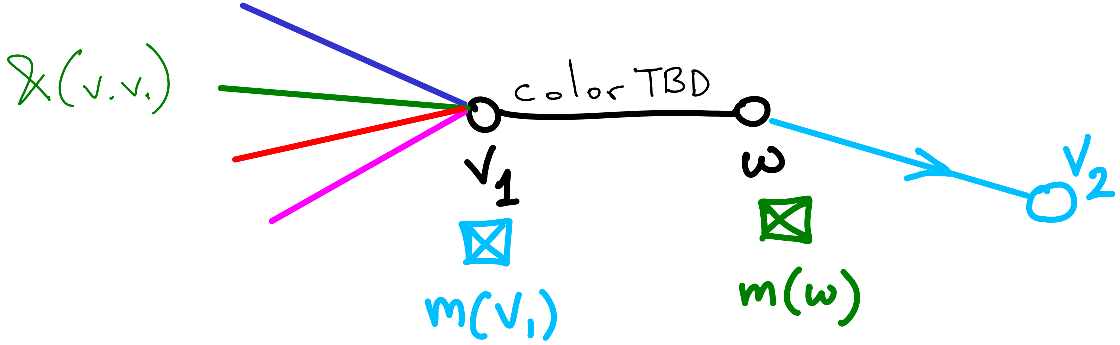
So pick any $m(v_1)$ ... w must have a neighbor v_2 with color $m(v_1)$



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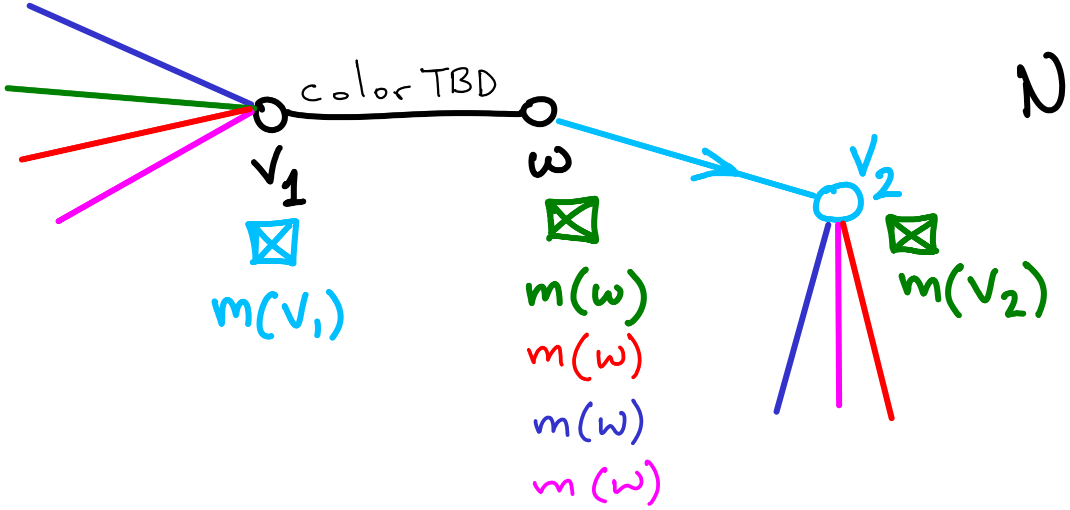

Now see if $\exists\ m(v_2) = m(w)$

↳ If yes, we're DONE...

$\Downarrow$

recolor $v_2 w \to v_2 w$

color $v_1 w \to v_1 w$

Proof of Vizing's thm by induction on edges

(by constructive algorithm)

Assume we have colored $G - \{v_1, w\}$ $v_1 w$ = edge

We already know if $\exists\ m(v_1) = m(w)$ then DONE.

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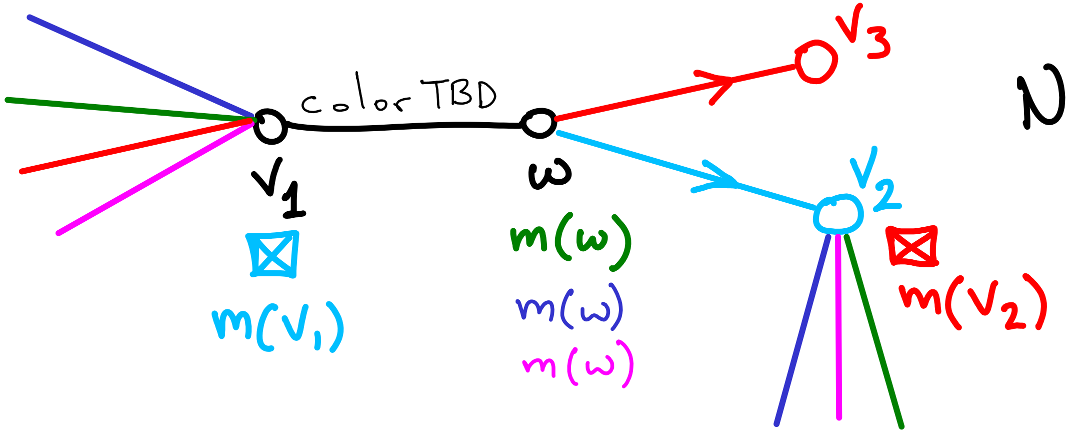


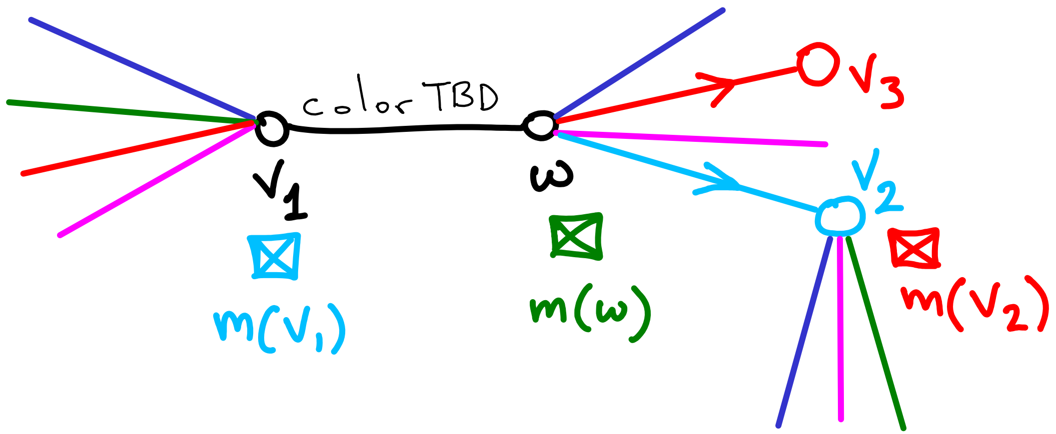
Now see if $\exists\ m(v_2) = m(w)$

↳ Either we're DONE, or

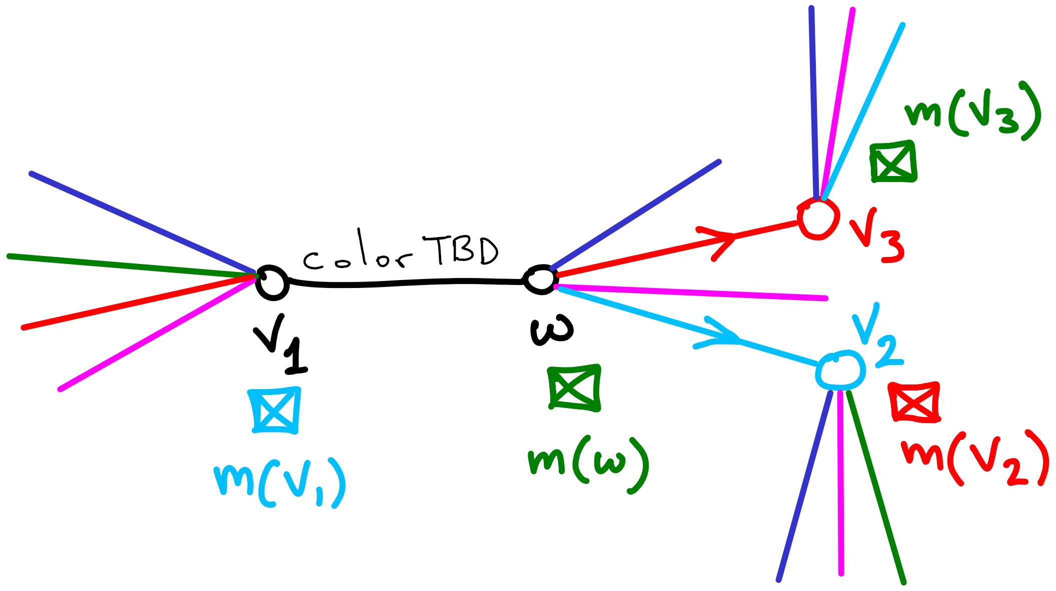
$\exists\ m(v_2) \neq m(w) \neq m(v_1)$

s.t. w has

a neighbor v_3 ...

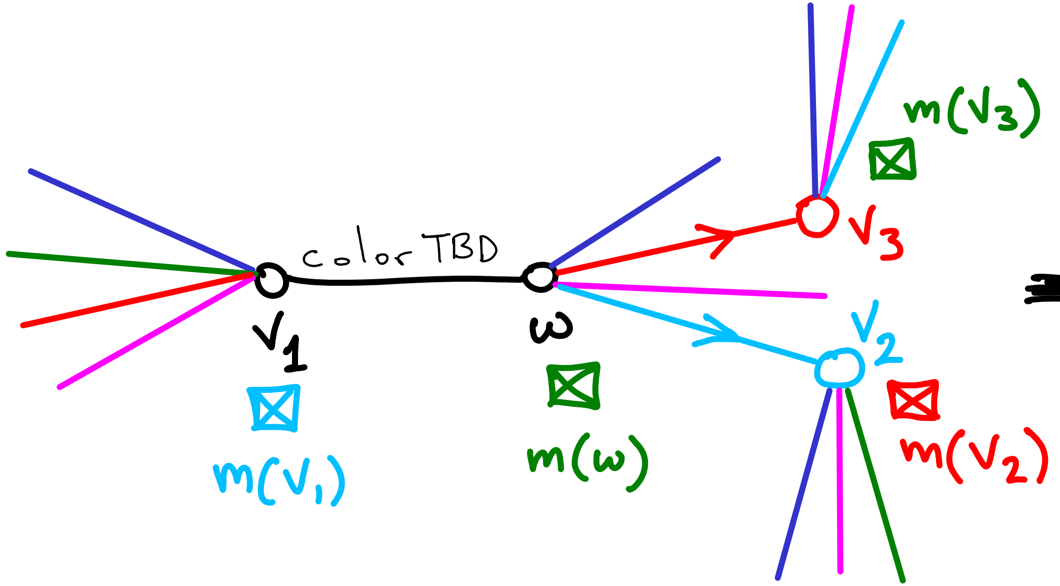


Check if $\exists\ m(v_3) = m(w)$



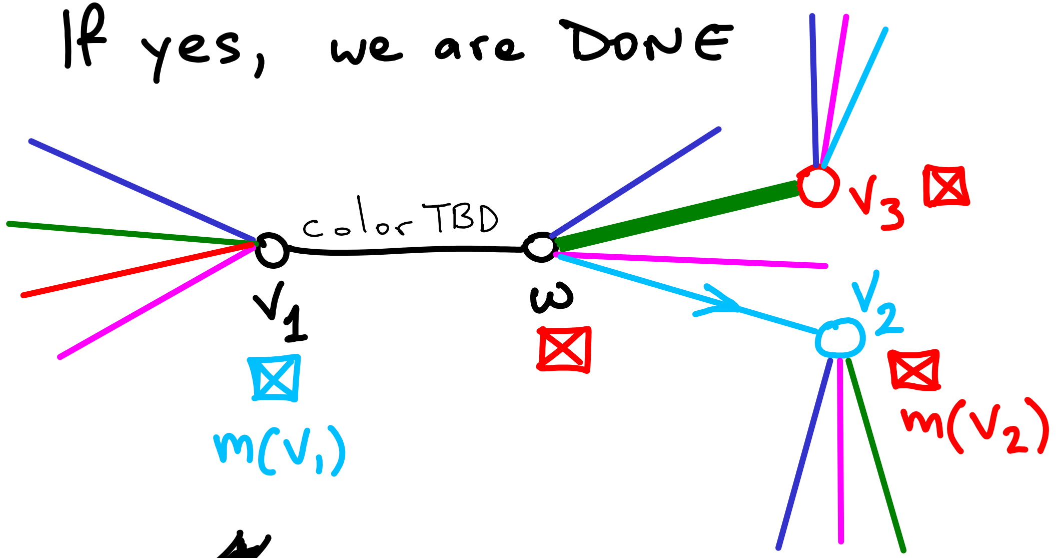
Check if $\exists$ $m(v_3) = m(w)$

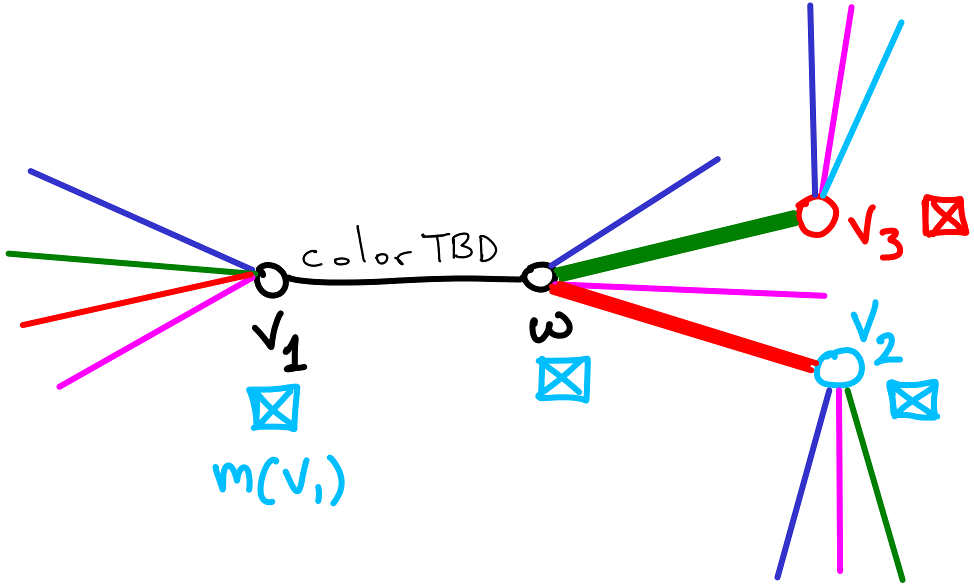
If yes, we are DONE ...



Check if $\exists\ m(v_3) = m(w)$

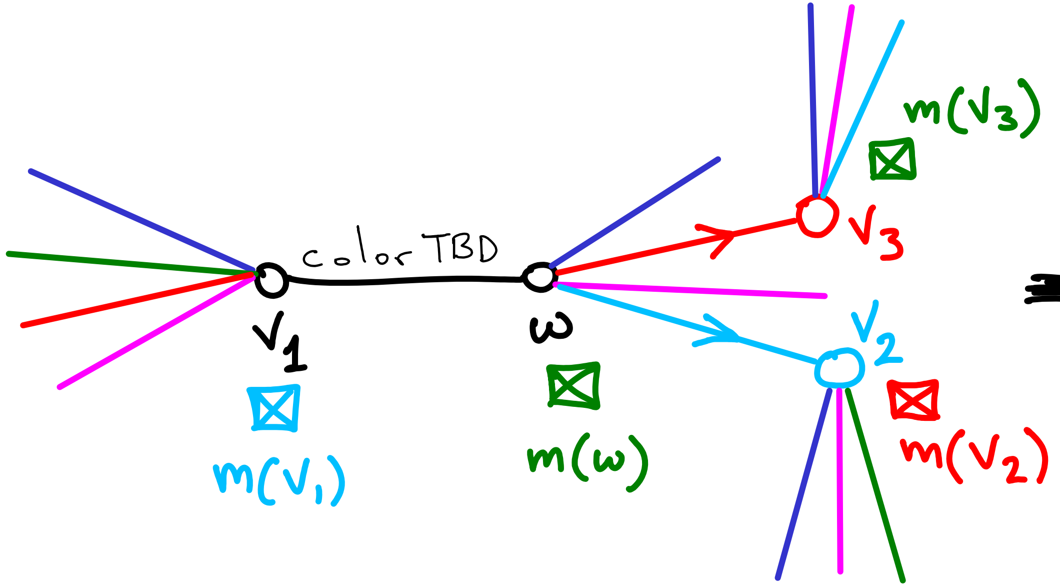
If yes, we are DONE





1 recolor $v_3w \to v_3w$

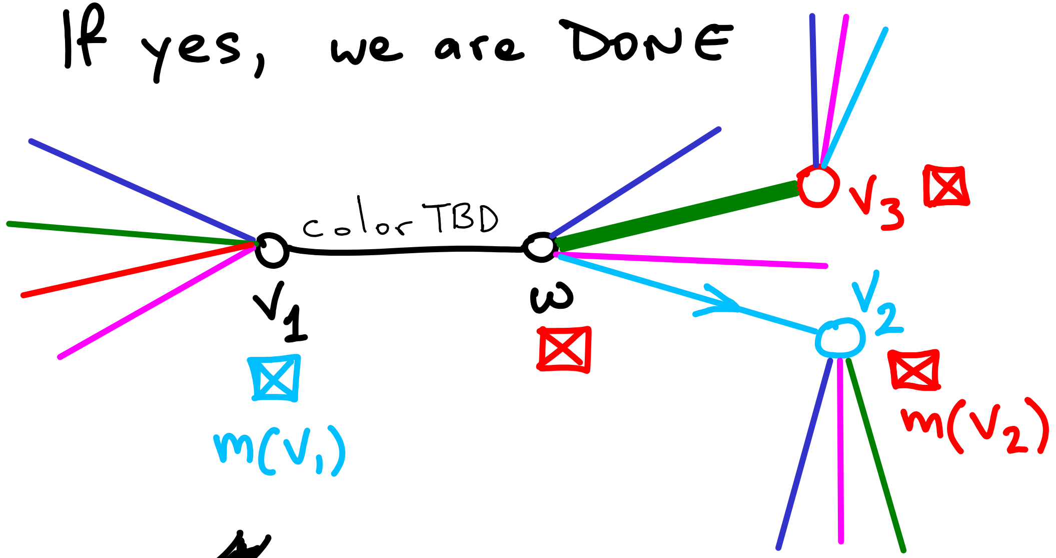
2 recolor $v_2w \to v_2w$



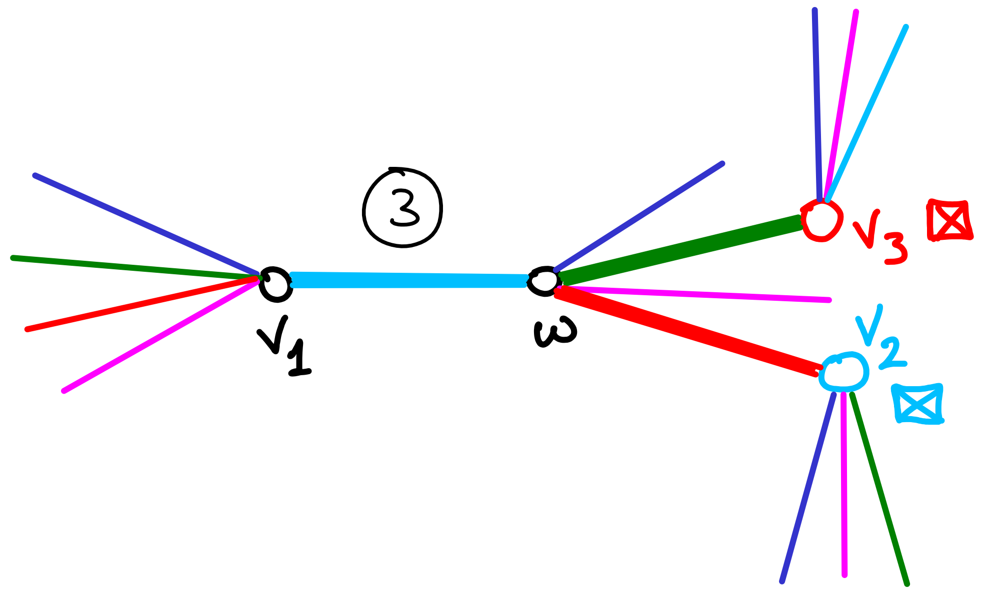
①

Check if $\exists$ $m(v_3) = m(w)$

If yes, we are DONE



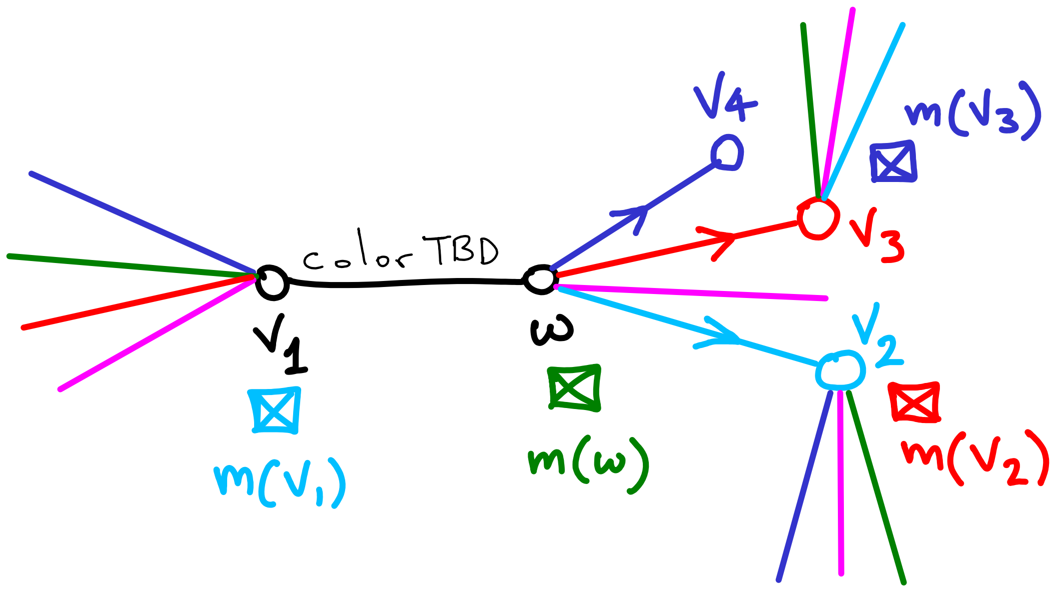
②



① recolor $v_3w \rightarrow v_3w$

② recolor $v_2w \rightarrow v_2w$

③ color $v_1w \rightarrow v_1w$



Recap:

Check if $\exists$ $m(v_3) = m(w)$

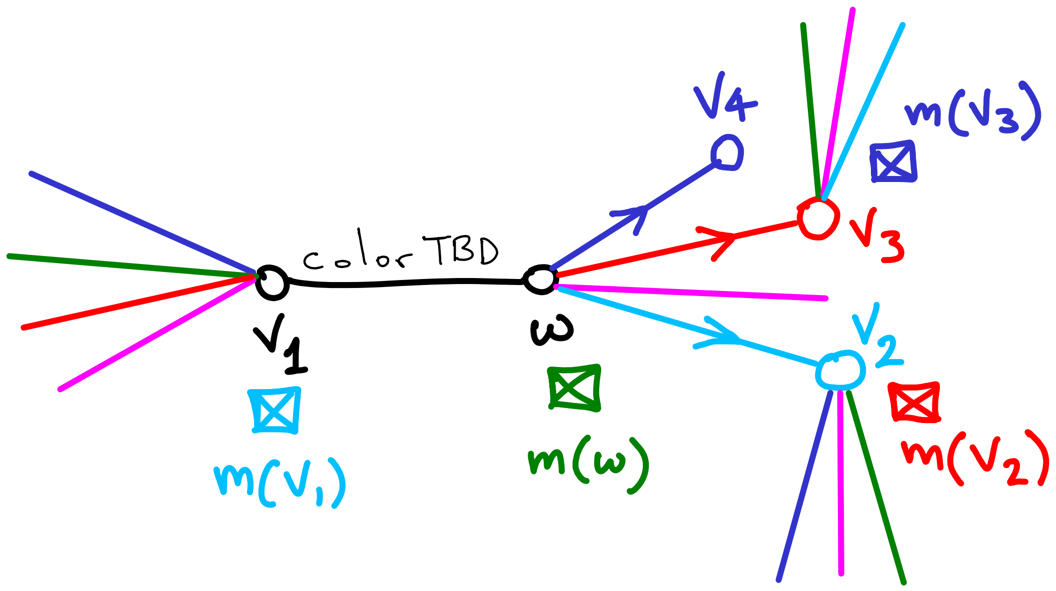
Either we are DONE or

$\exists$ v_4 s.t edge wv_4 has

color $m(v_3)$

& we keep going

(but not forever)



Check if $\exists$ $m(v_3) = m(w)$

Either we are DONE or

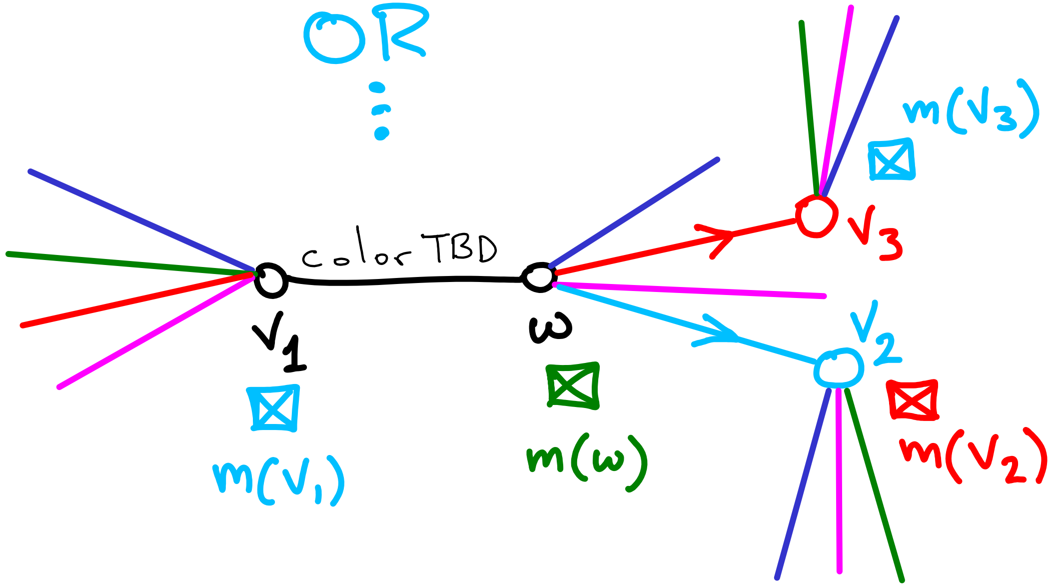
$\exists$ v_4 s.t edge wv_4 has color $m(v_3)$

⋮

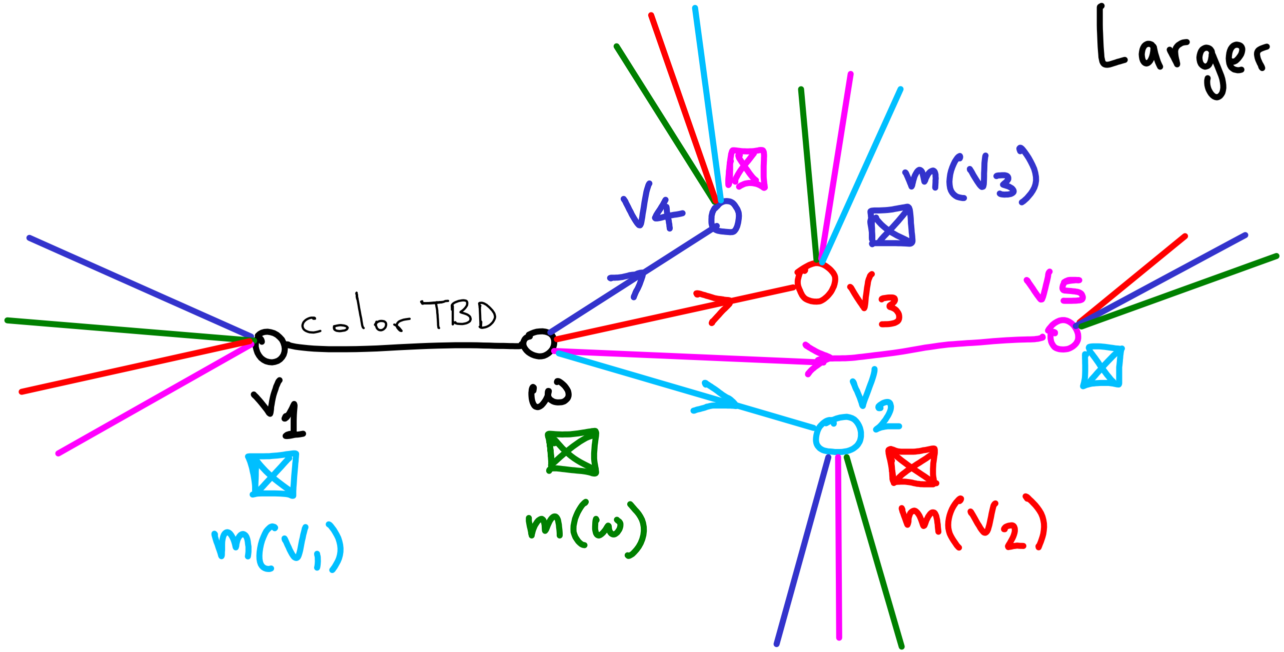
OR

⋮

We're back to a missing color that we saw already

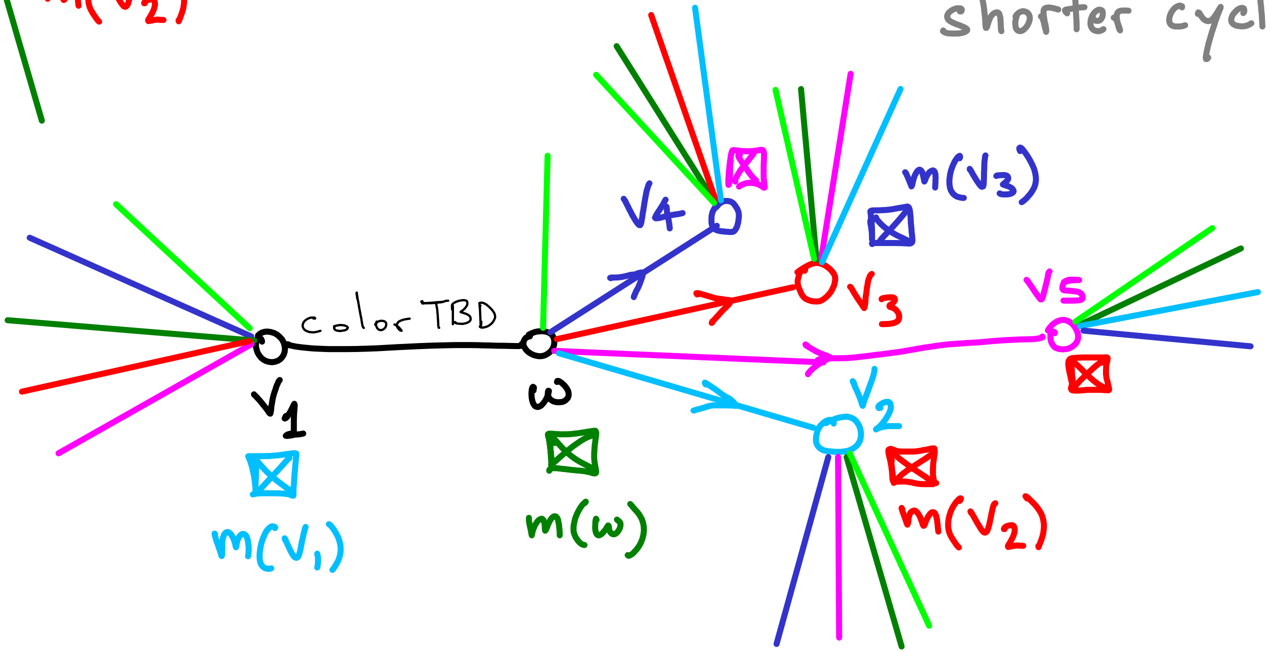


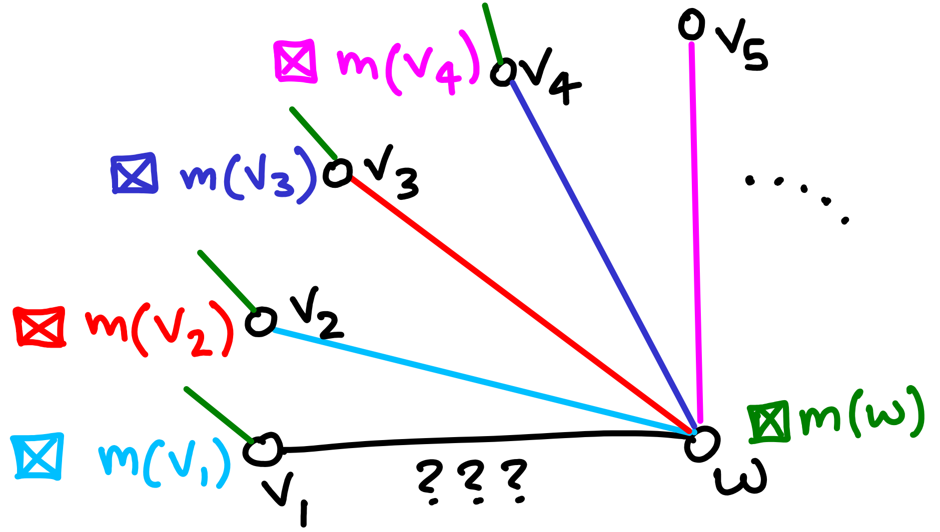
Larger examples of inevitable cycle



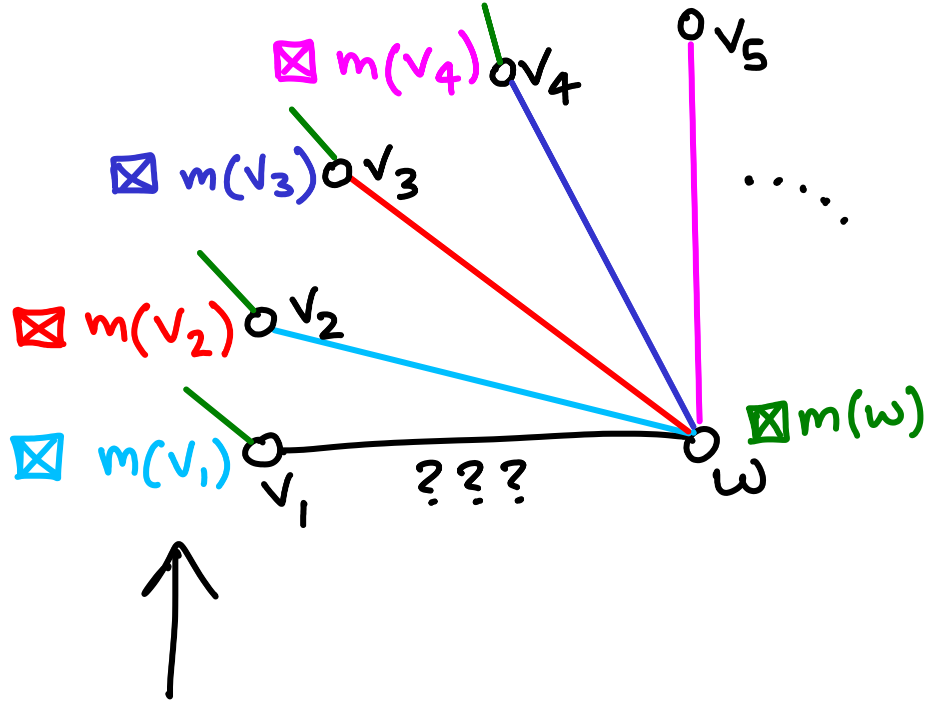
back to square one

shorter cycle





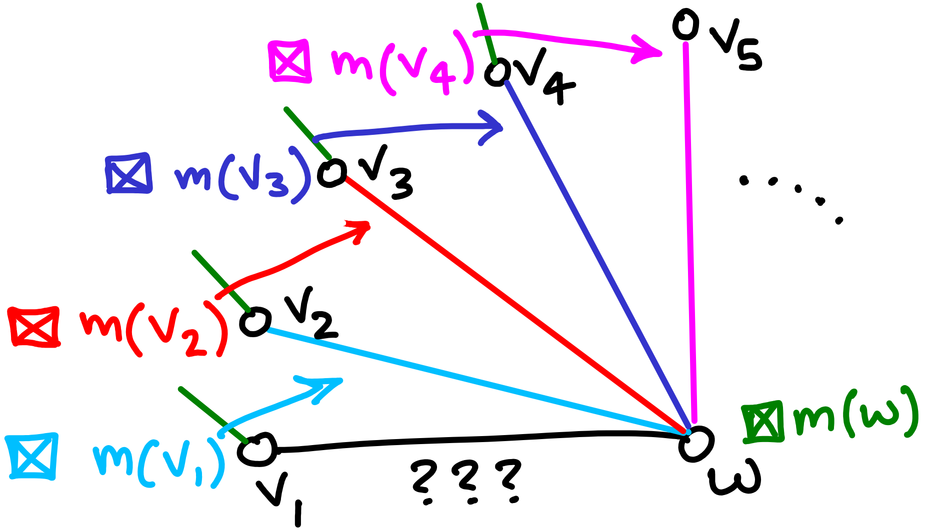
Fan(w) has h neighbors of w: $v_1 \ldots v_h$
& $h-1$ colors used.



Fan(w) has h neighbors of w: $v_1 \ldots v_h$
& $h-1$ colors used.

for $i=1$ to $h-1$

- v_i is missing color $m(v_i) = m_i$



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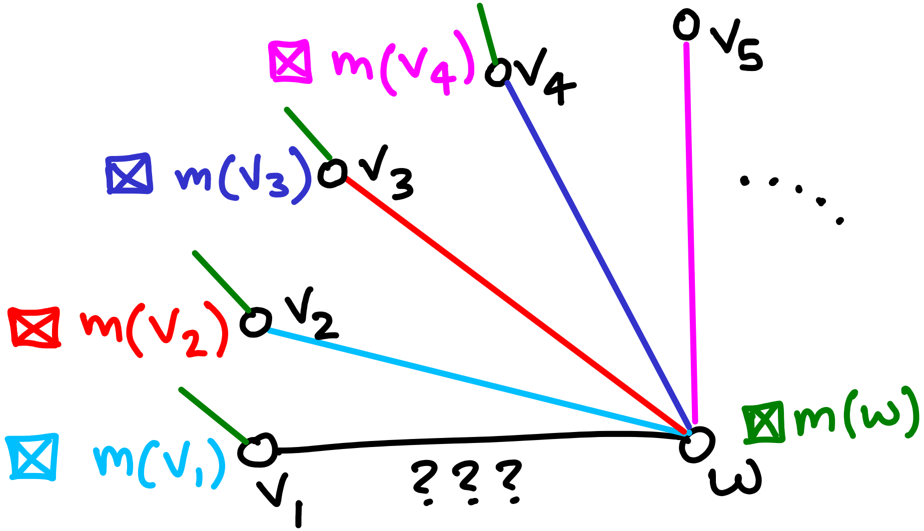
for i=1 to h-1

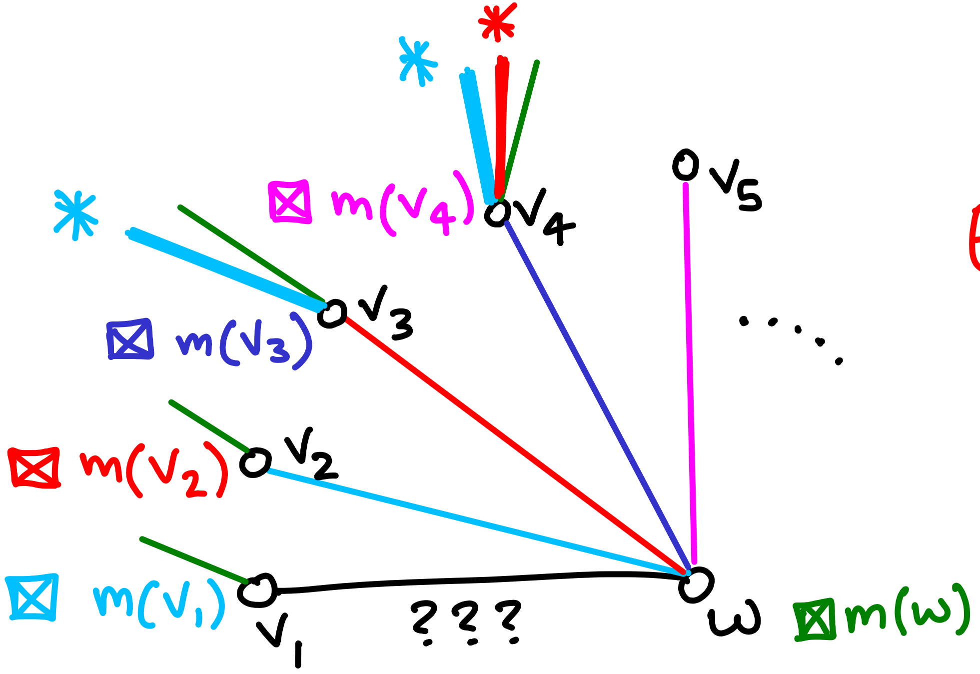
- v_i is missing color $m(v_i) = m_i$
- $\overline{v_{i+1} w}$ is colored m_i



Fan(w) has h neighbors of w: $v_1 \ldots v_h$
& h-1 colors used.

for $i=1$ to $h-1$

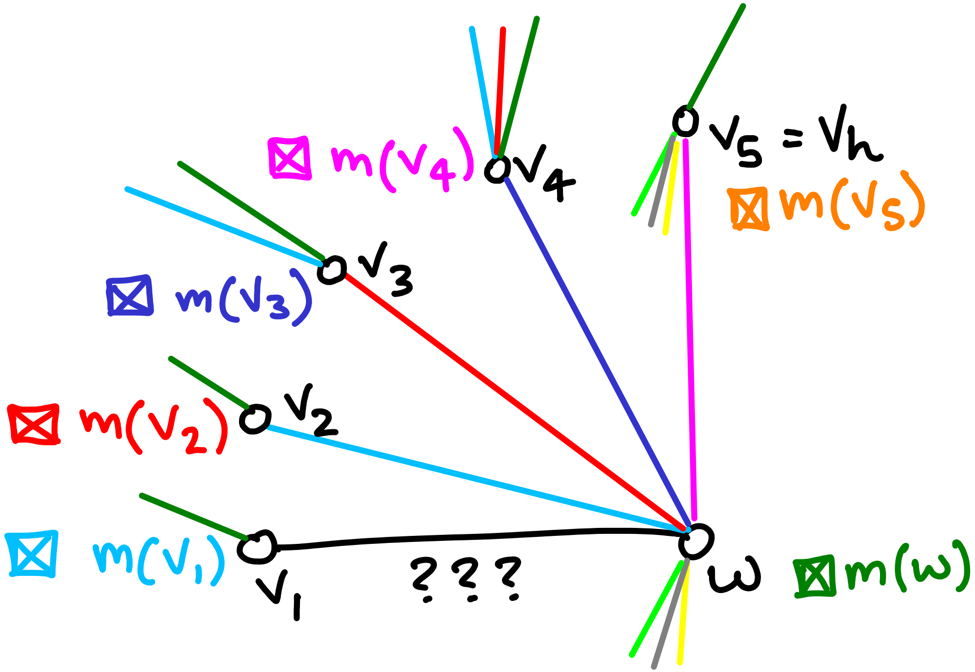
- v_i is missing color $m(v_i) = m_i$
- $\overline{v_{i+1} w}$ is colored m_i
- $m_i \neq m(w)$



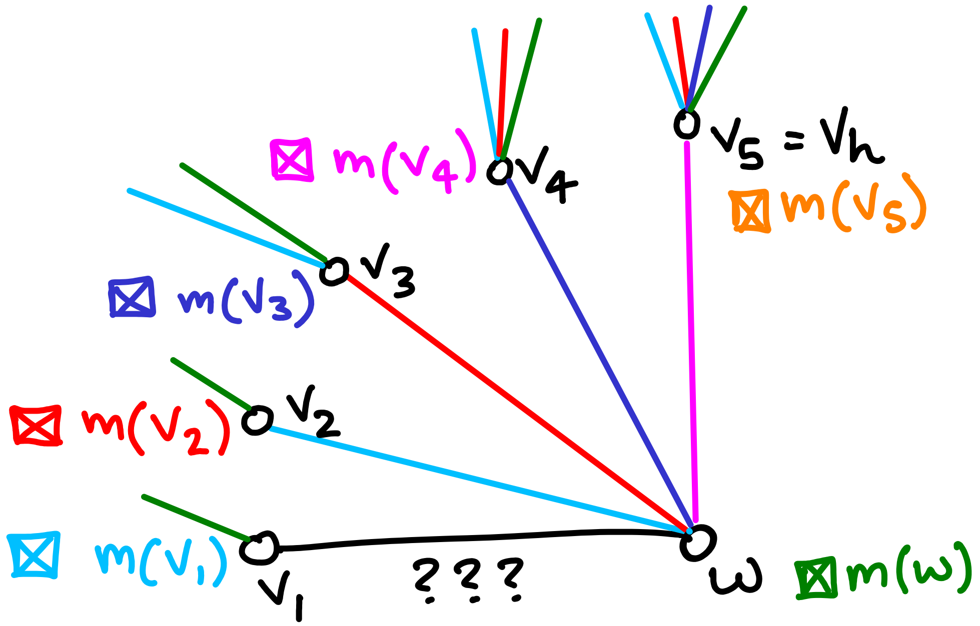
Extra requirement, for $i=2$ to $h-1$

- v_i does not miss any color already missed by v_j $(j<i)$

Our algo starts w/ a fan w/ $h \geq 3$



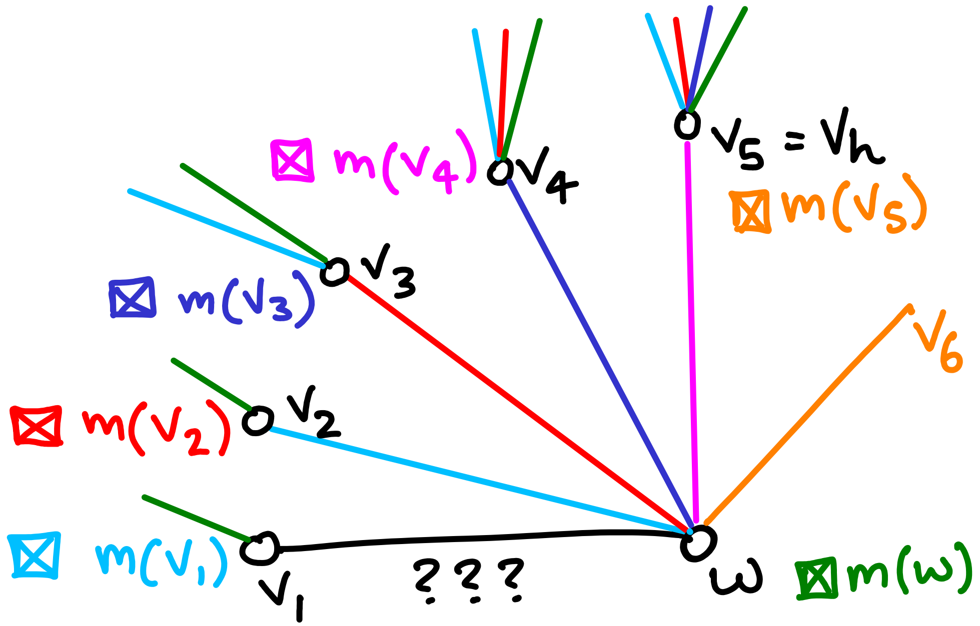
If v_h has no $m_h = m_w$



If v_h has no $m_h = m_w$ &

all previous colors on fan are incident,

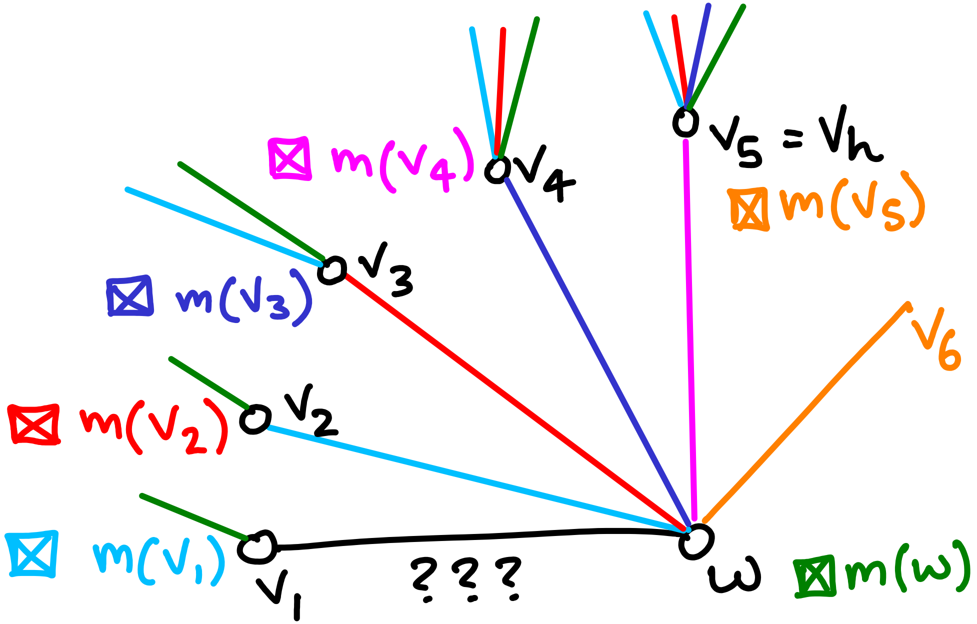
(i.e. no common missing colors)



If v_h has no $m_h = m_w$ &
all previous colors on fan are incident,
then there must be a v_6
but this can't continue forever.

[so one of two things happens]

(2 conditions set above)



If v_h has no $m_h = m_w$ &
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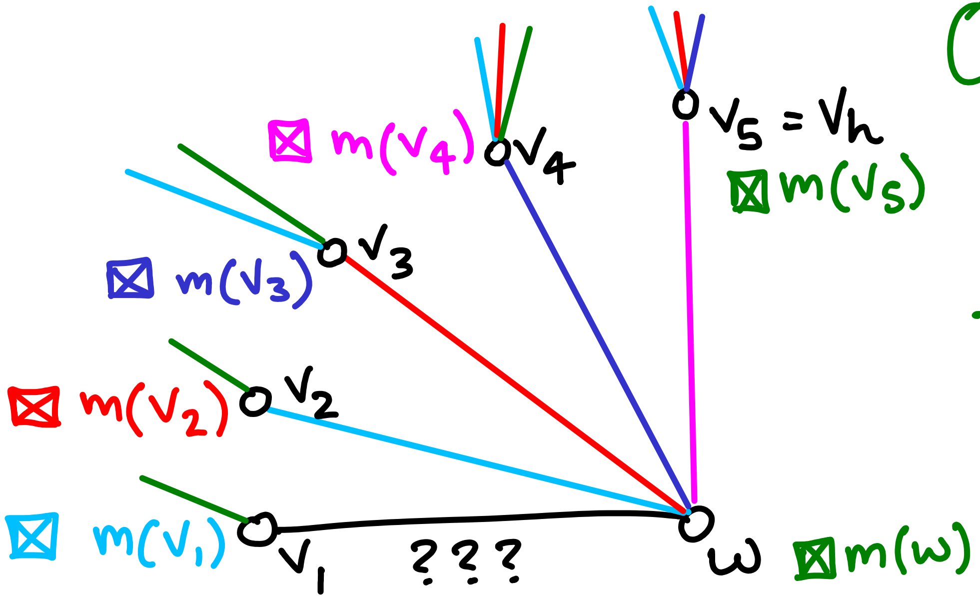
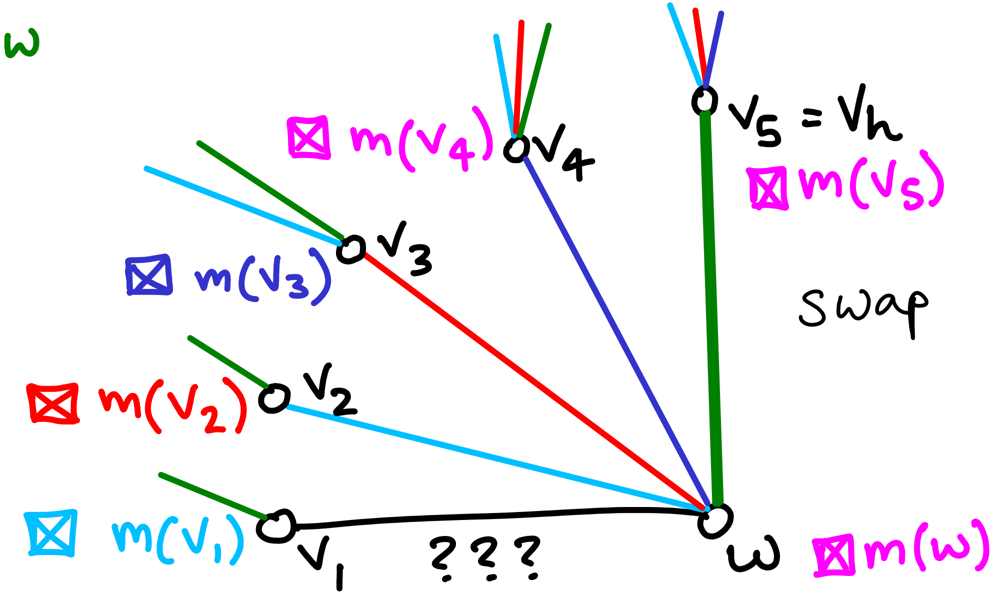
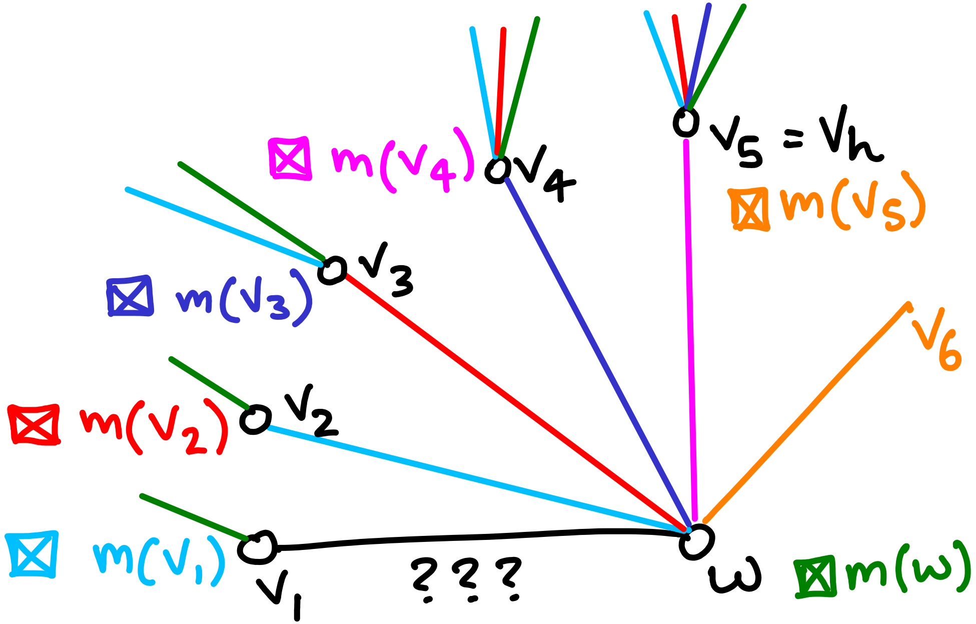
[so one of two things happens]

(i) $m_h = m_w$







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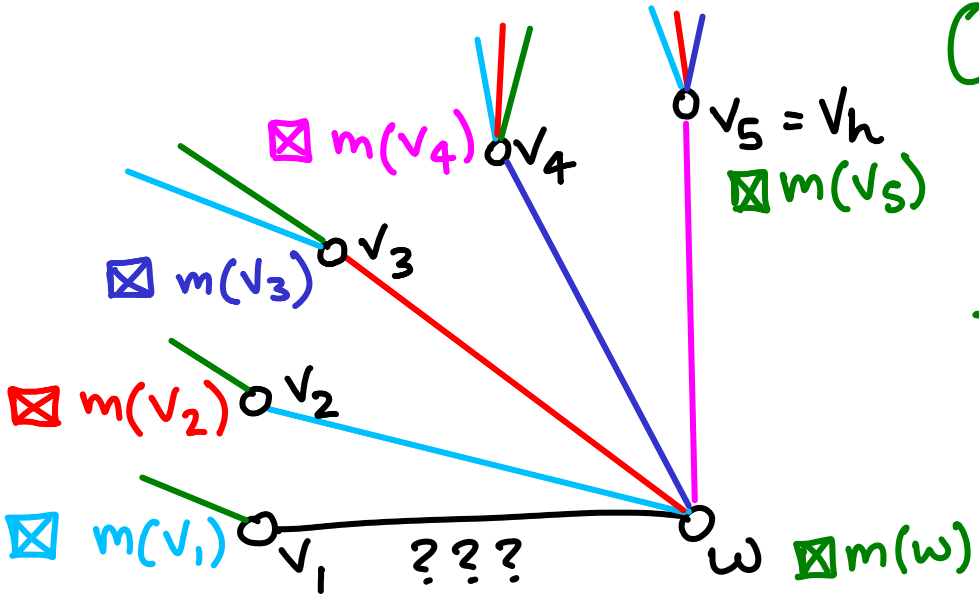

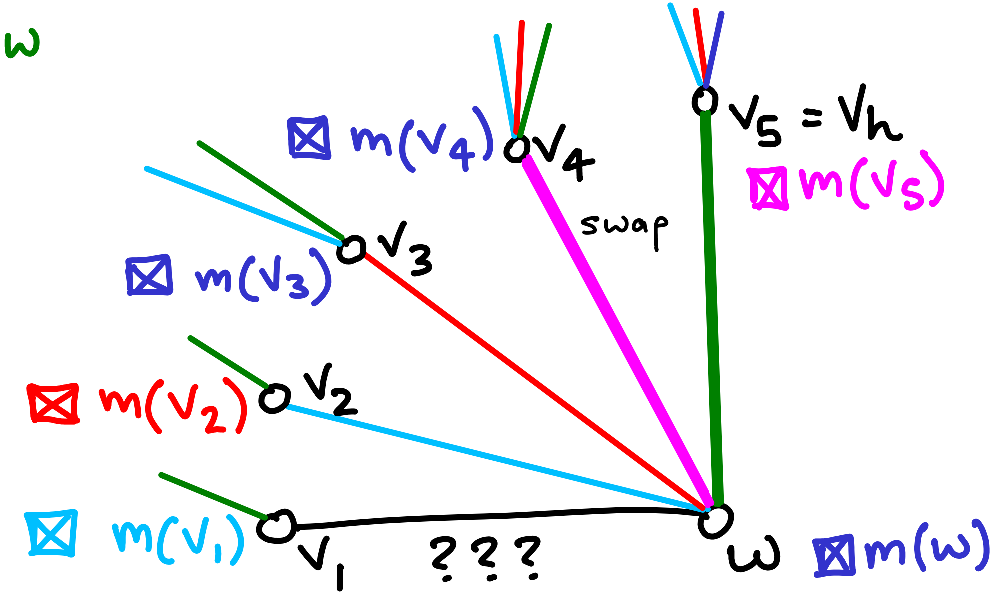

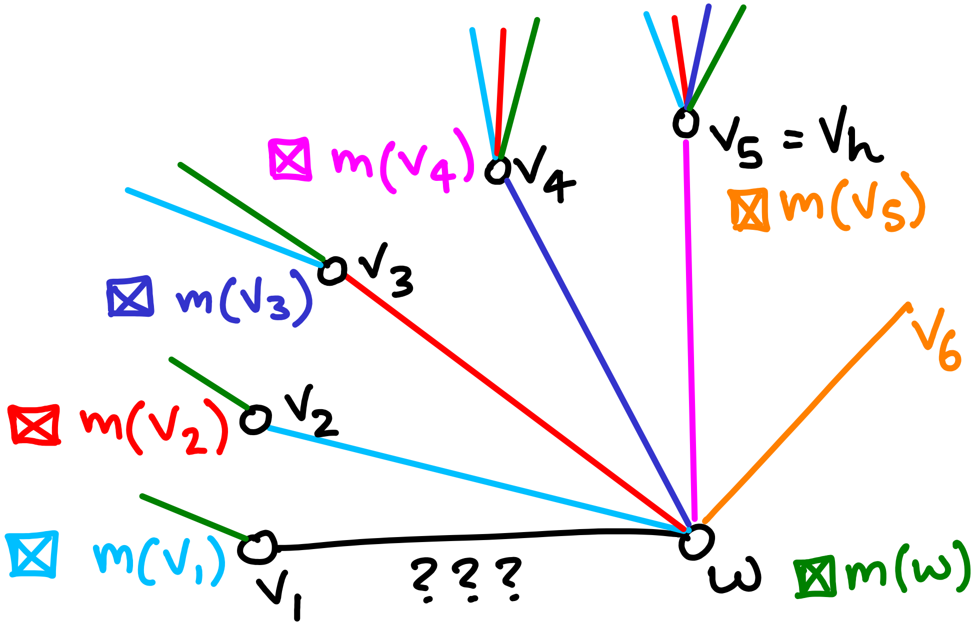

If v_h has no $m_h = m_w$ & all previous colors on fan are incident, then there must be a v_6 but this can't continue forever.

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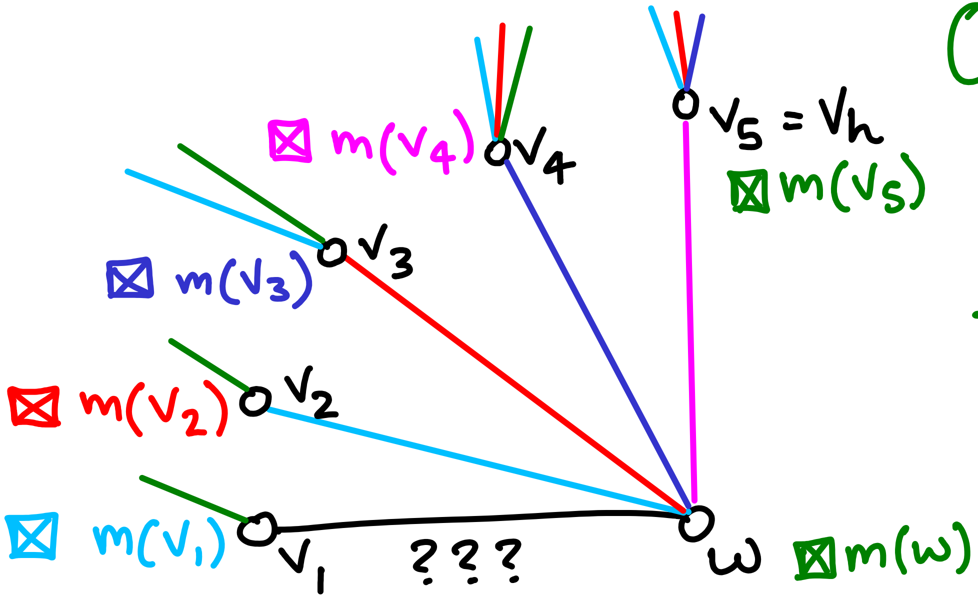
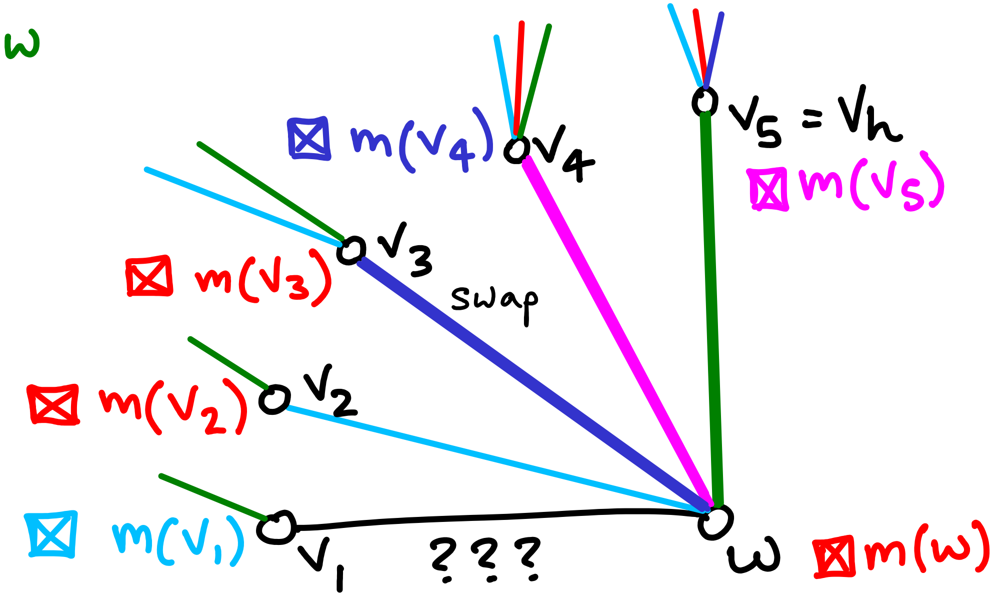
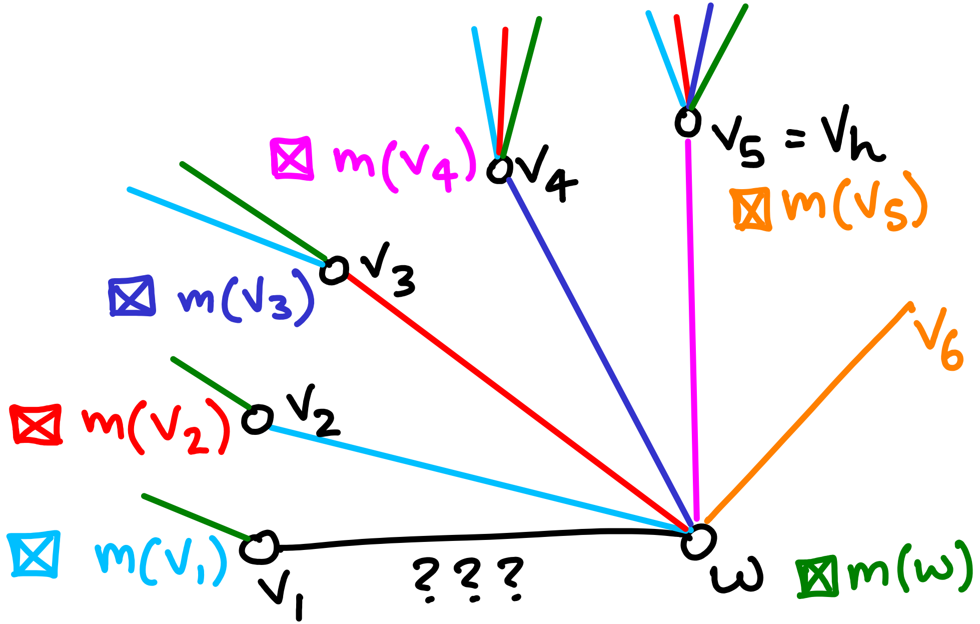




If v_h has no $m_h = m_w$ & all previous colors on fan are incident, then there must be a v_6 but this can't continue forever.

[so one of two things happens]



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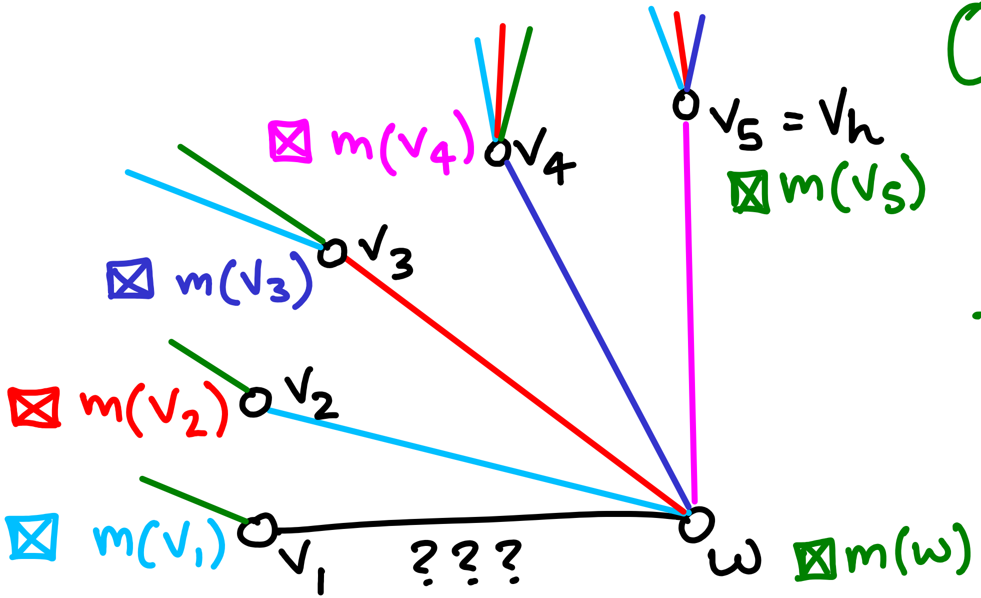
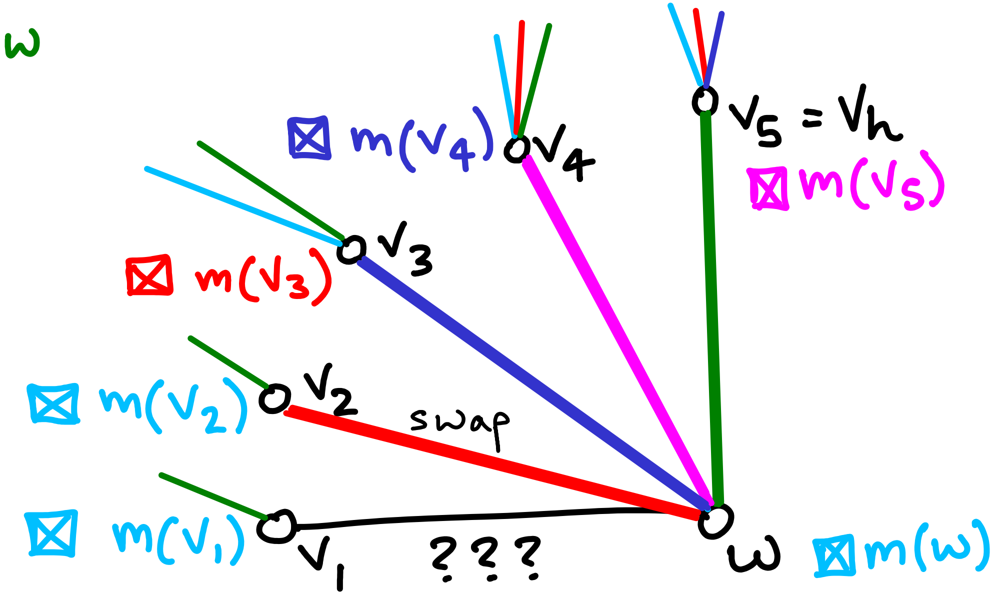
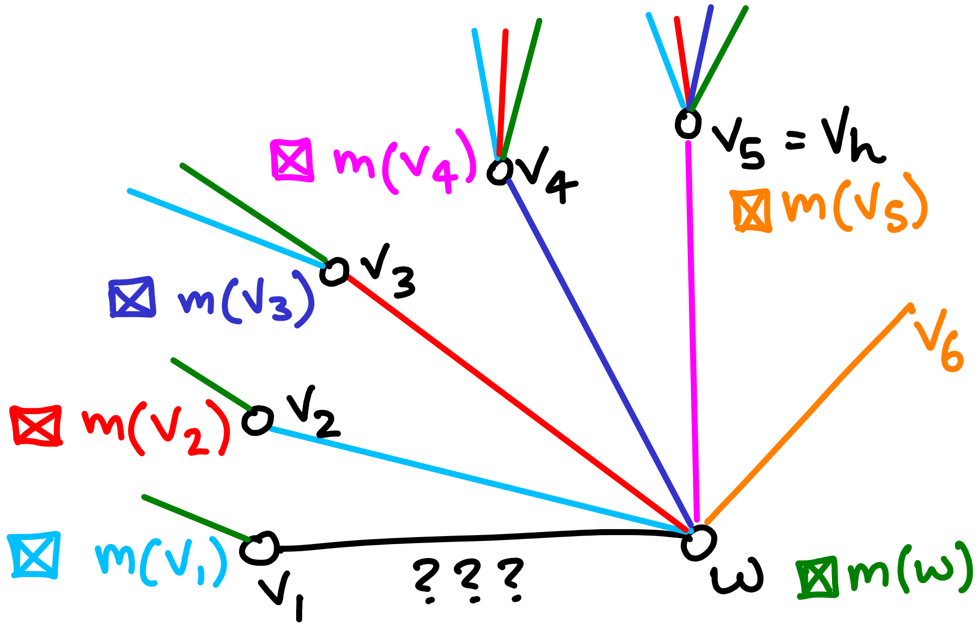




If v_h has no $m_h = m_w$ & all previous colors on fan are incident, then there must be a v_6 but this can't continue forever. [so one of two things happens]



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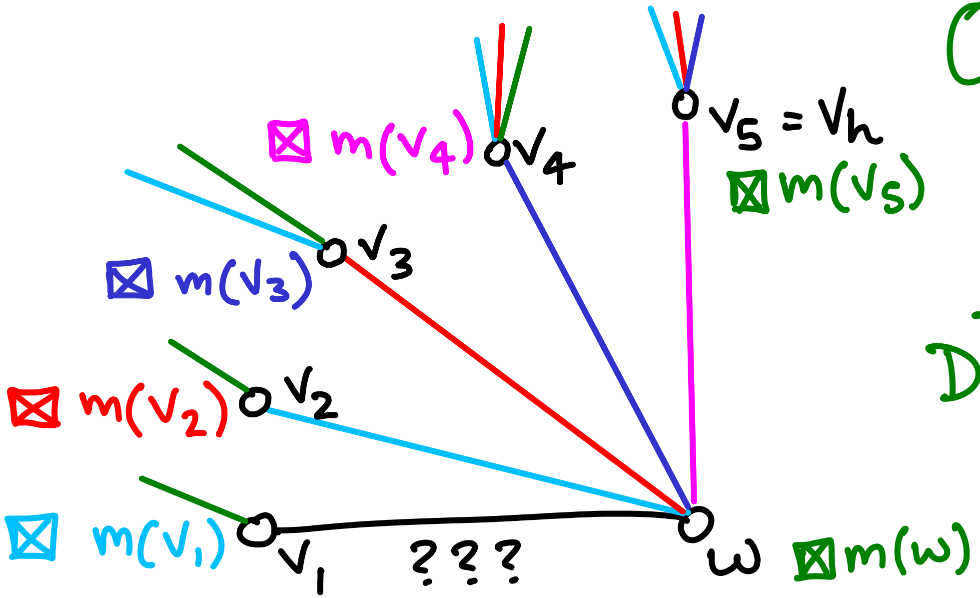
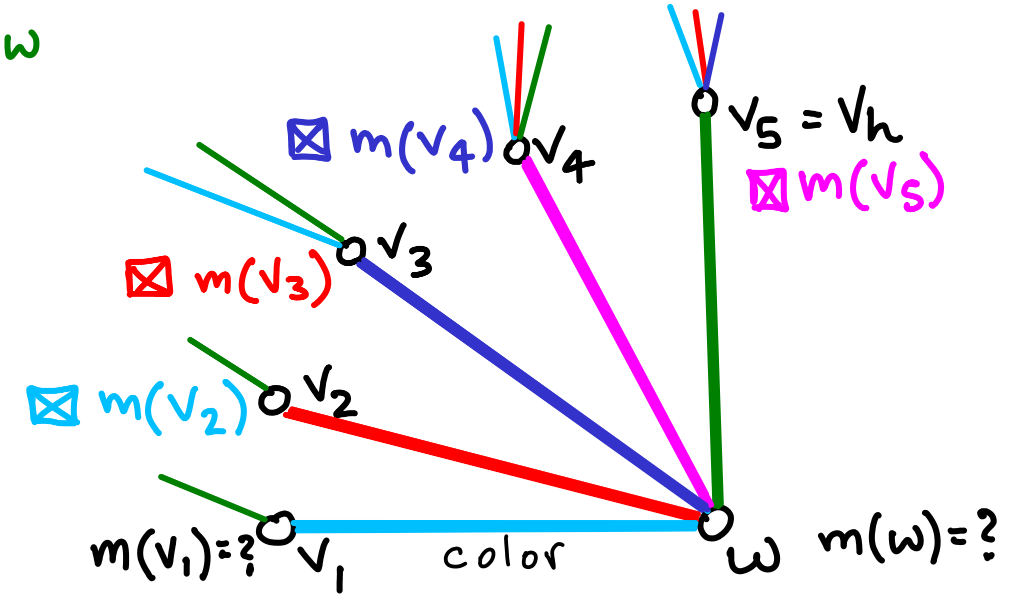
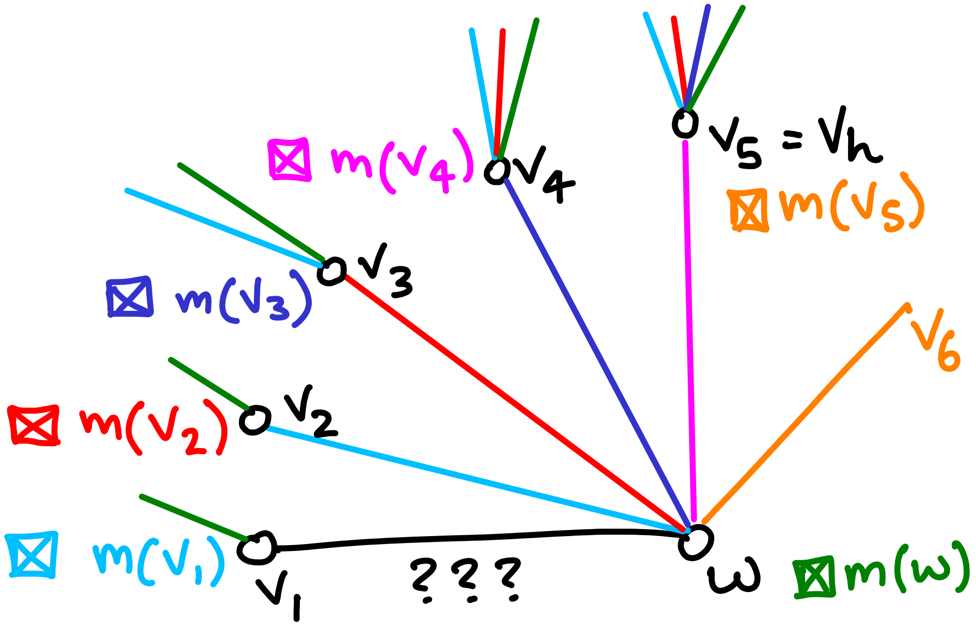


DONE





If v_h has no $m_h = m_w$ &
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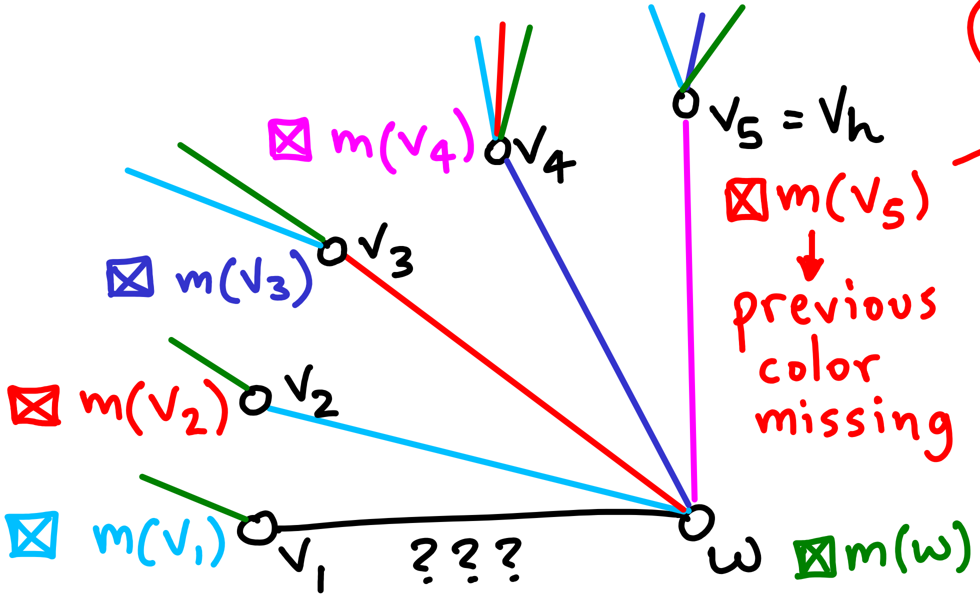
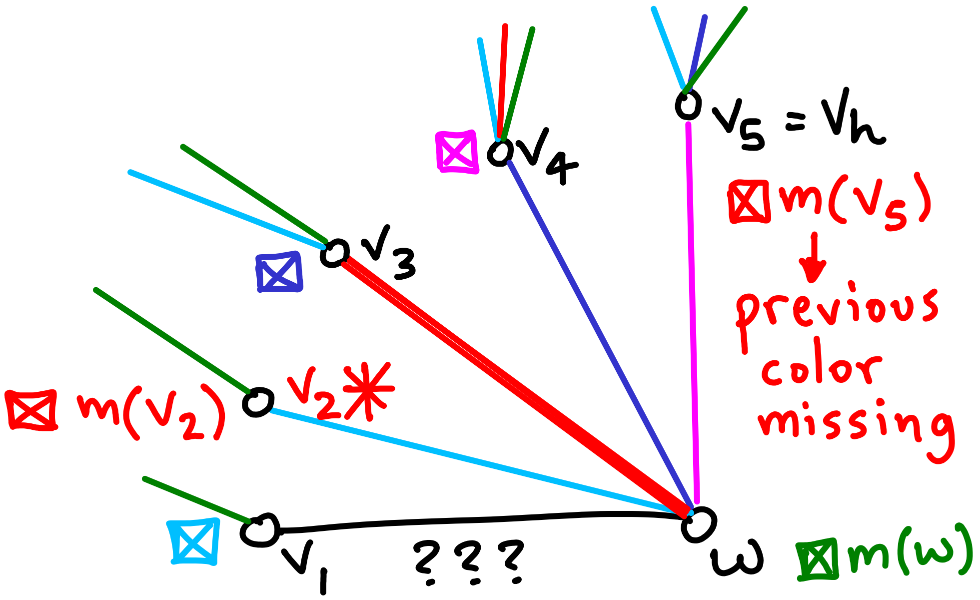
2



previous color missing $\Rightarrow m_h = m_{previous}$

If v_h ($= v_5$) is missing a previously used color, on $v_i w$ ($= v_3 w$)
then $\exists$ v_{i-1} (v_2) also missing that color, i.e. $m_h = m_{i-1}$



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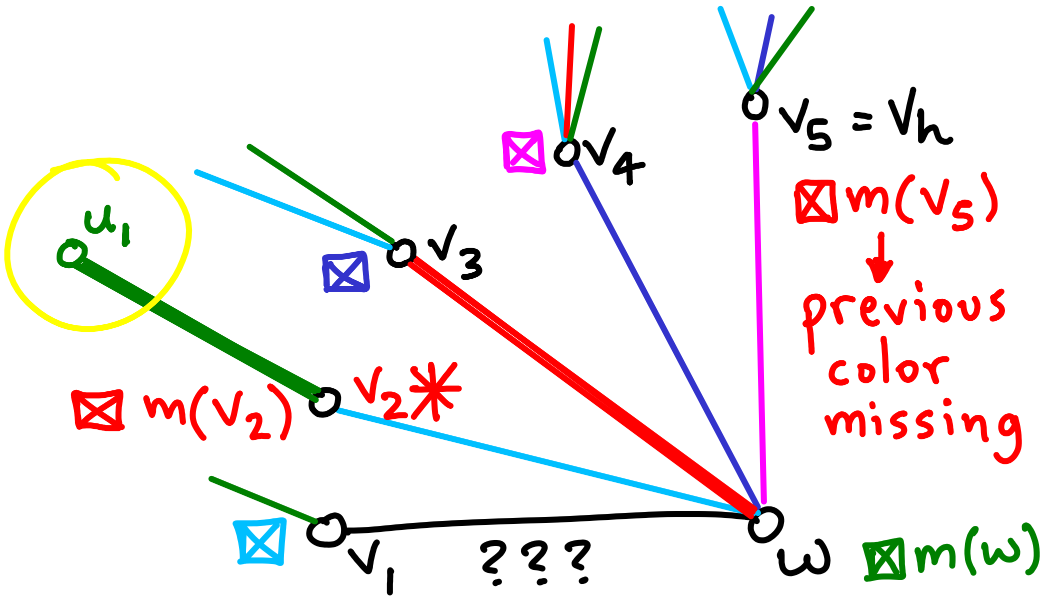
then $\exists$ v_{i-1} (v_2) also missing that color, i.e. $m_h = m_{i-1}$

We also know that v_{i-1} has an edge of color $m(w)$ [as do all v_j]
Let this edge lead to u_1.



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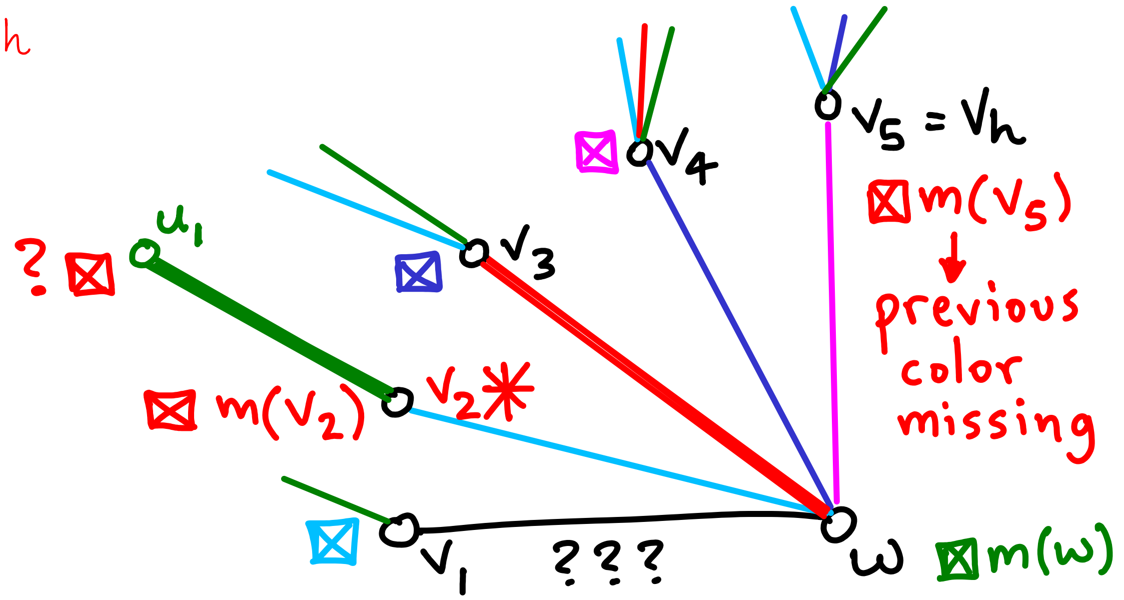
We also know that v_{i-1} has an edge of color $m(w)$ [as do all v_j]
Let this edge lead to u_1.

What happens if u_1 misses m_{i-1} ?

i.e m_h



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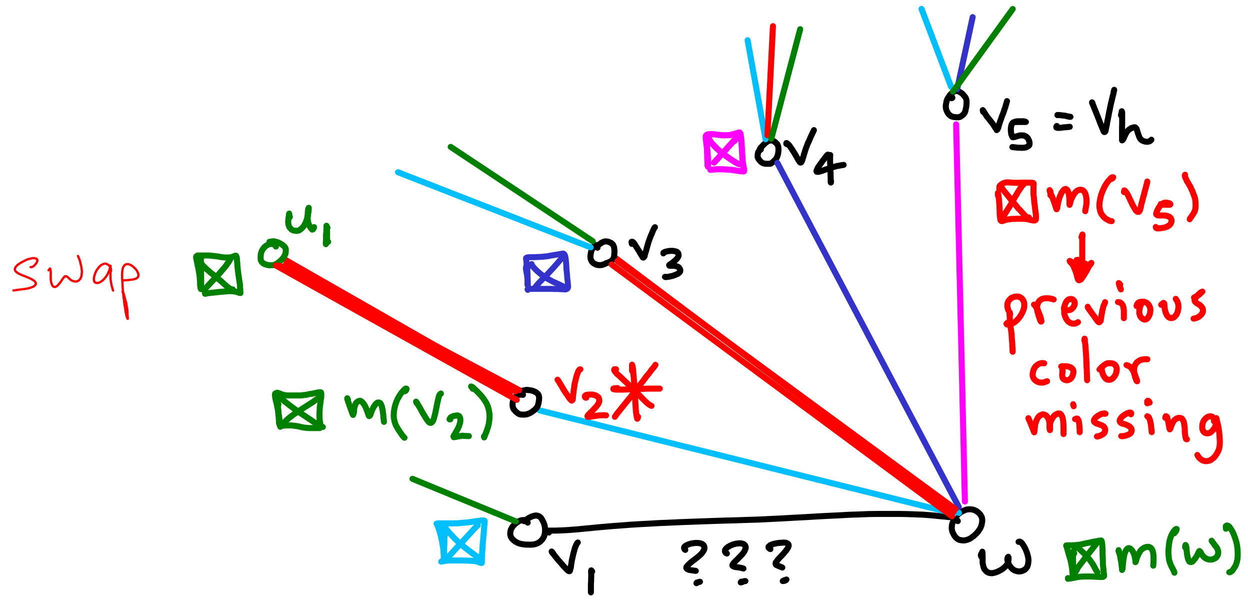

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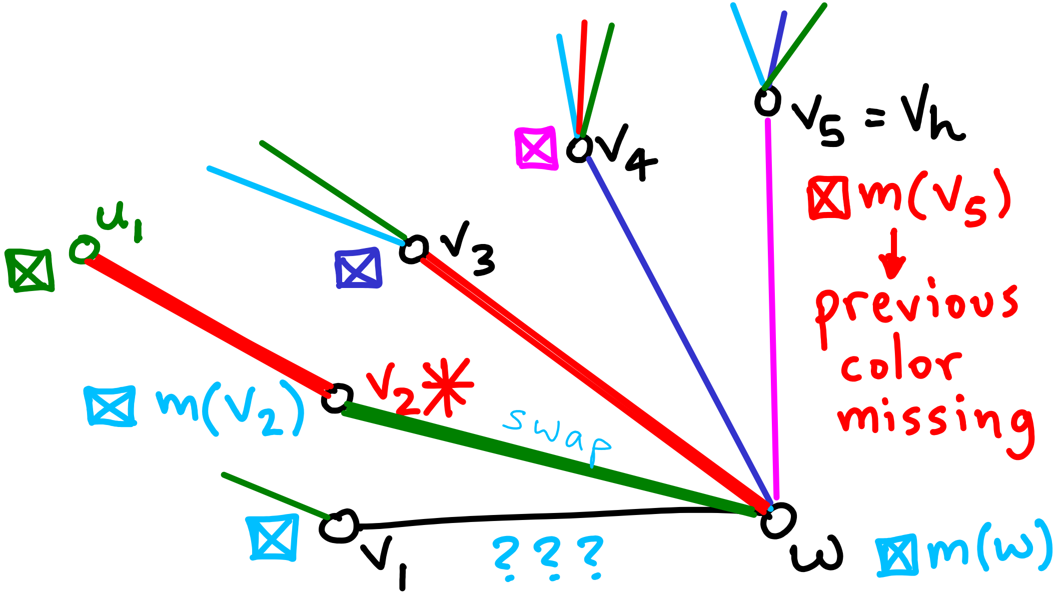
We also know that v_{i-1} has an edge of color $m(w)$ [as do all v_j]

Let this edge lead to u_1.

What happens if u_1 misses m_{i-1} ?

↓

We get a chain reaction & color $v_1 w = m(v_1)$



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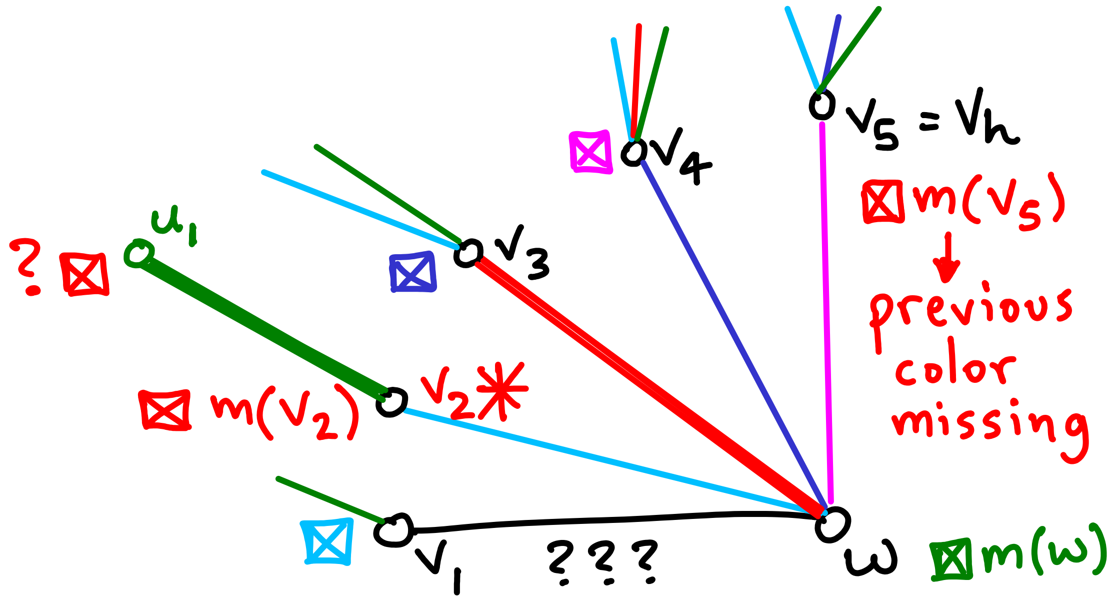
then $\exists$ v_{i-1} (v_2) also missing that color, i.e. $m_h = m_{i-1}$

We also know that v_{i-1} has an edge of color $m(w)$ [as do all v_j]

Let this edge lead to u_1.

What happens if u_1 misses m_{i-1} ?

↳ We would be DONE



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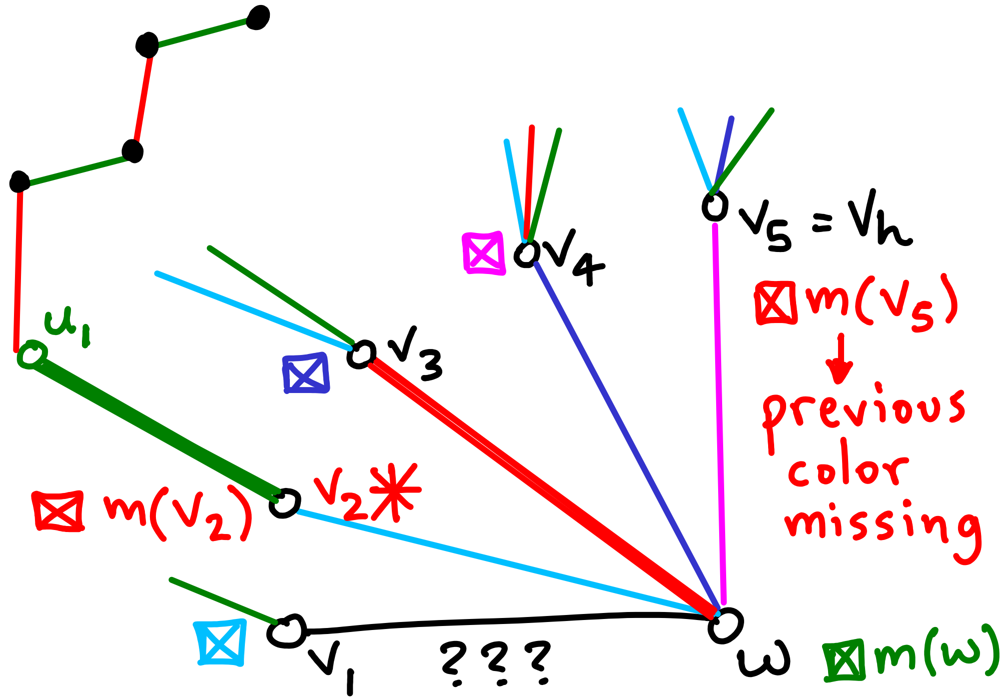
Let this edge lead to u_1.

What happens if u_1 misses m_{i-1} ?

↳ We would be DONE

So we get a path v_2 ... }

↳ not a cycle: can't return to v_2 or anywhere on path.



If v_h ($= v_5$) is missing a previously used color, on $v_i w$ ($= v_3 w$)

then $\exists$ v_{i-1} (v_2) also missing that color, i.e. $m_h = m_{i-1}$

We also know that v_{i-1} has an edge of color $m(w)$ [as do all v_j]

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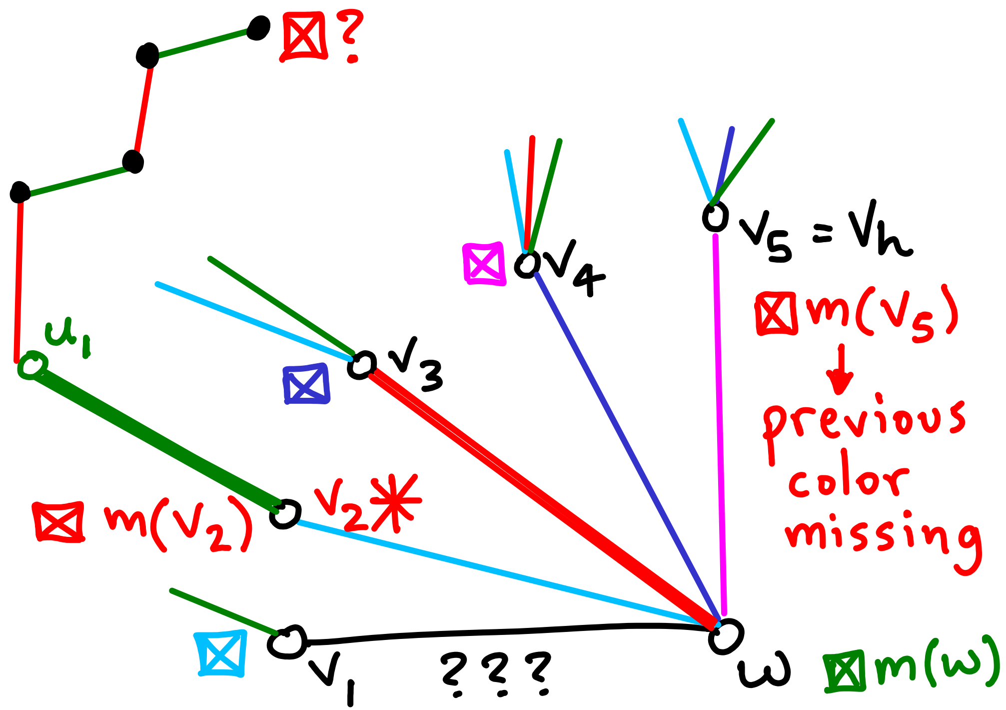
If the path ends w/o touching w...



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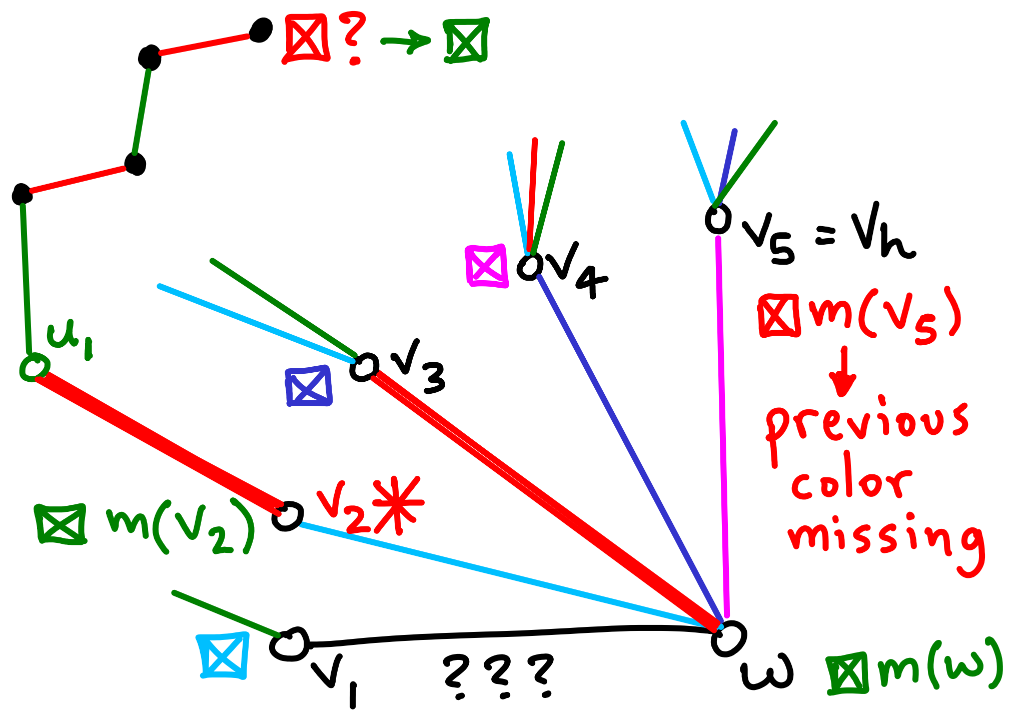

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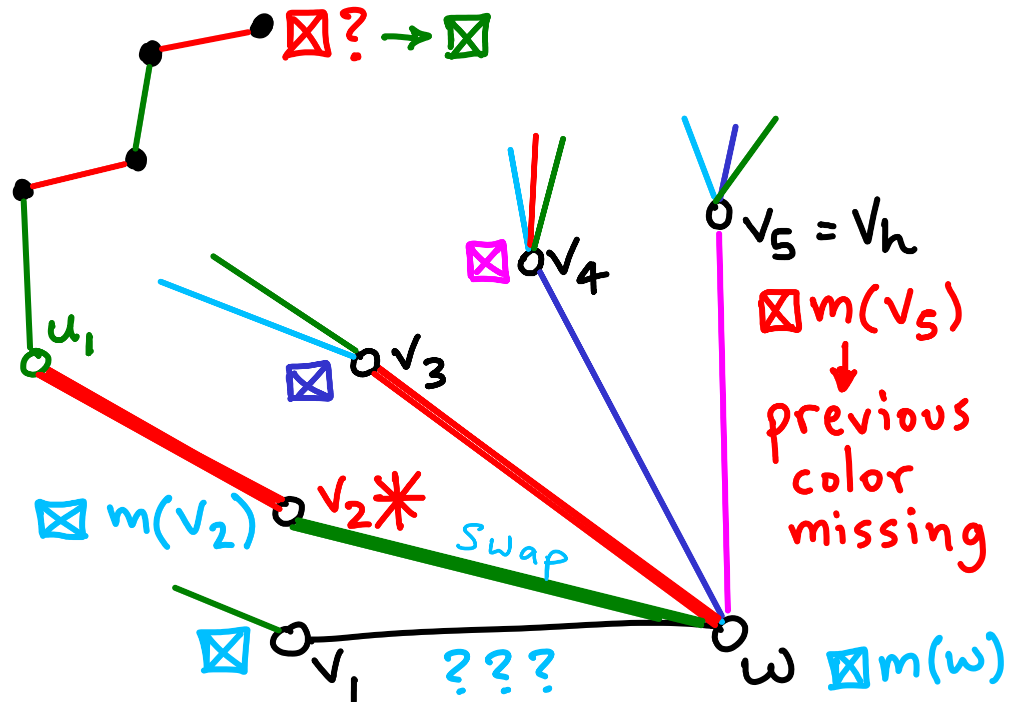
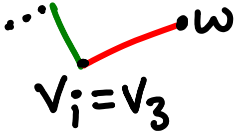
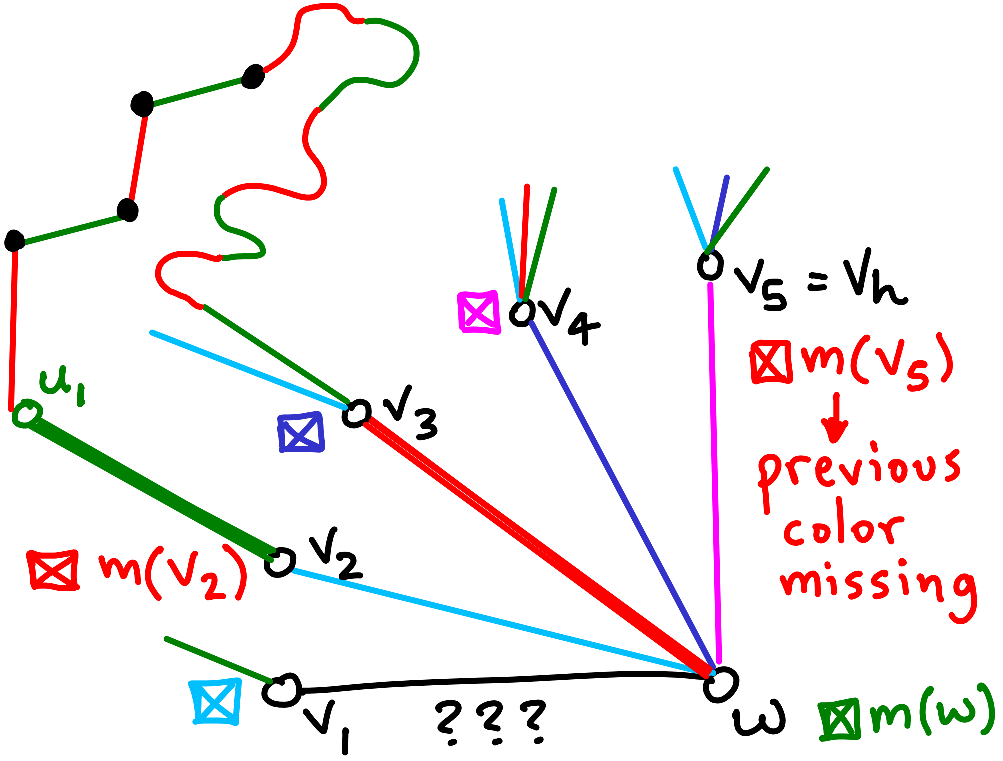


Final case: path ends at w. (it can't go through w because $\boxtimes\, m(w)$)

So path ends w/ 



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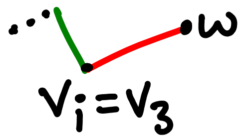
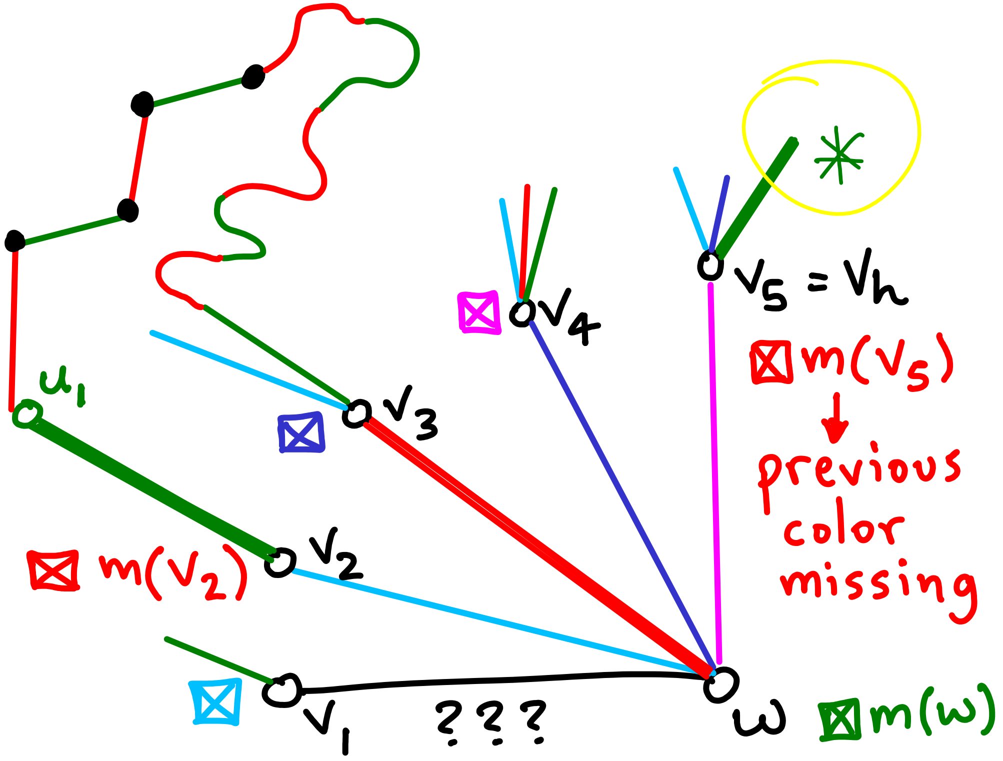
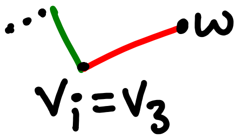
So path ends w/ 

Now look at edge from v_h with color $m(w)$

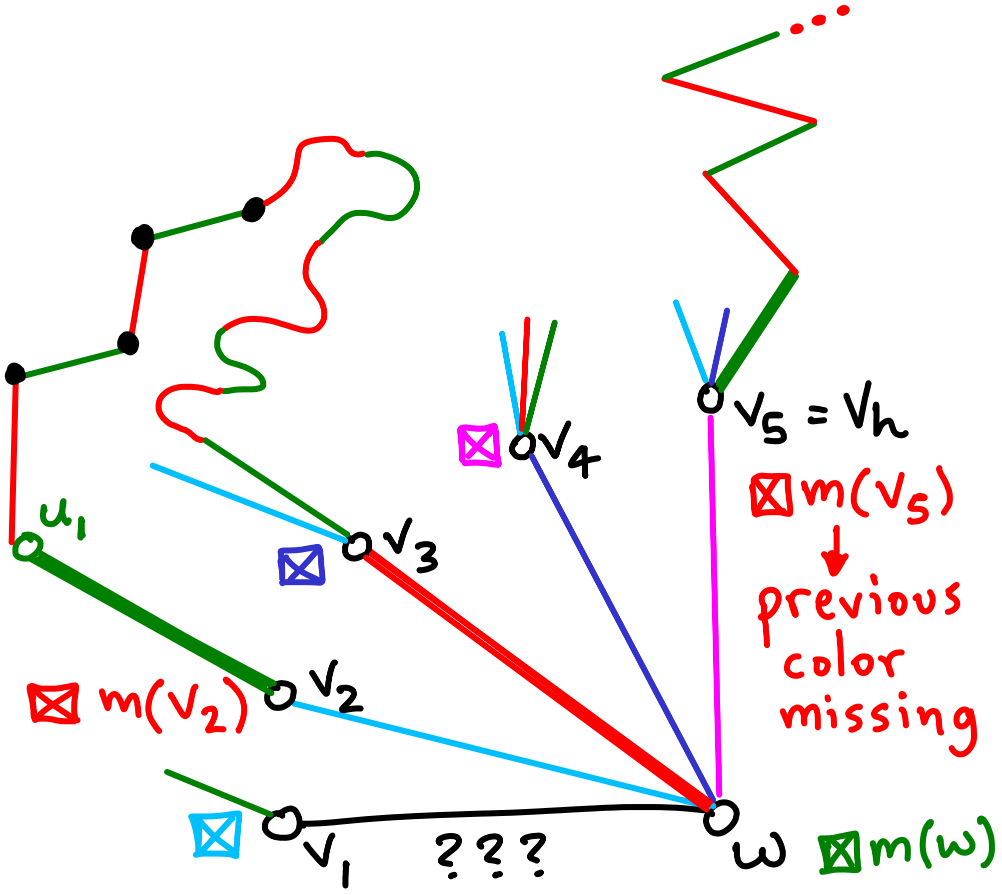


Final case: path ends at w. (it can't go through w because ⊠ m(w))

So path ends w/ 

Now look at edge from v_h with color m(w)

Extend to a path with same colors as previous path.



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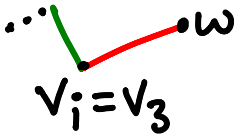

Now look at edge from v_h with color $m(w)$

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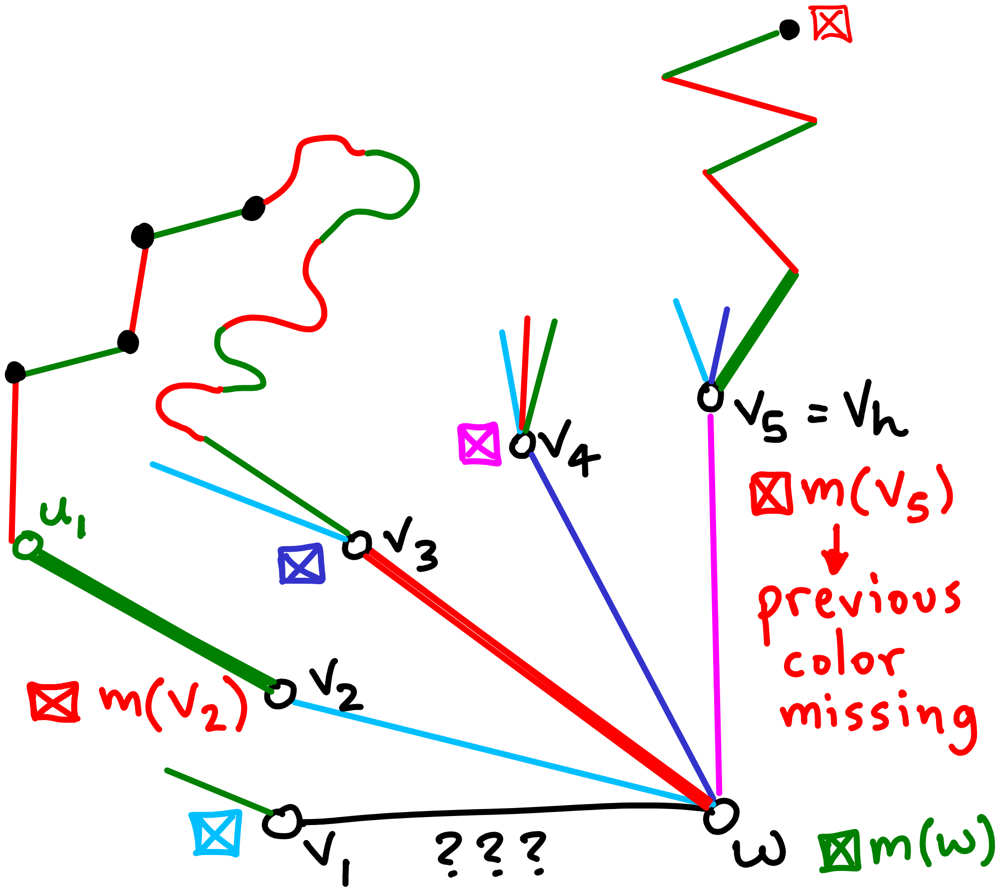
↳ must be simple (no cycles)

& <u>can't use w at all</u>

(actually it can't touch previous path)



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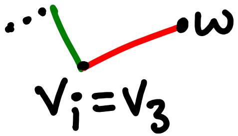
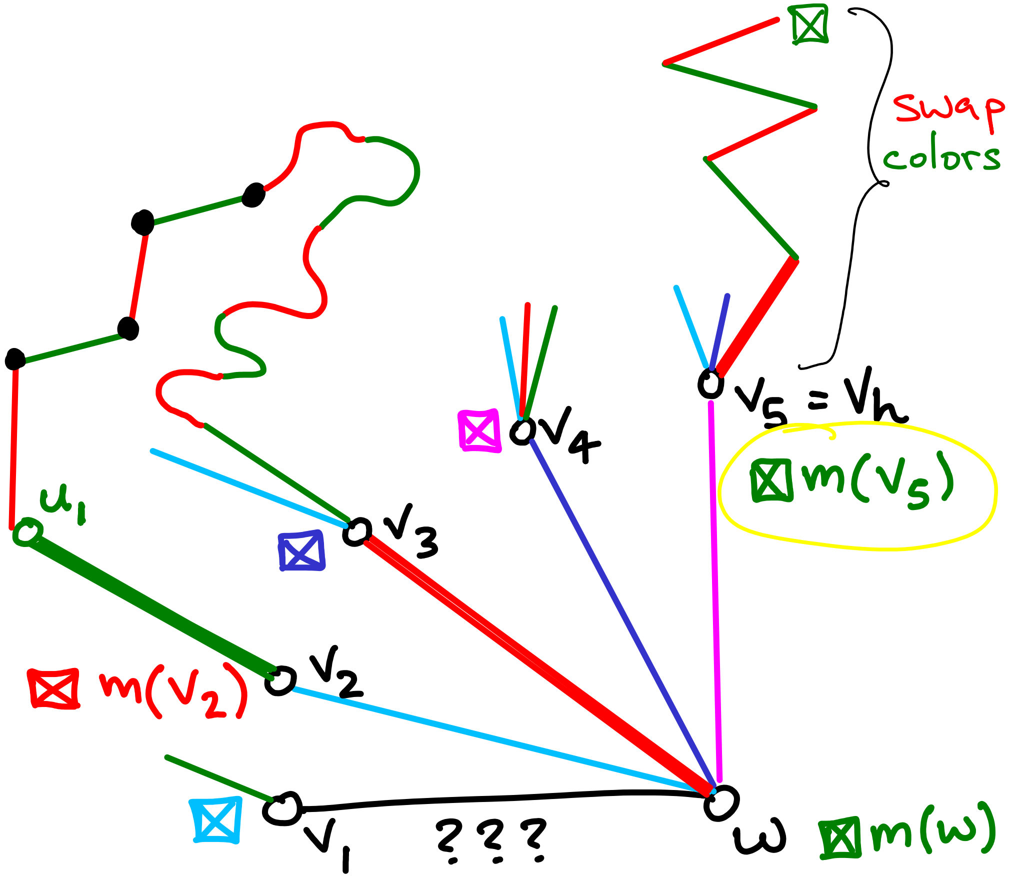
So path ends w/ 

Now look at edge from v_h with color m(w)

Extend to a path with same colors as previous path.

↳ must be simple (no cycles) & can't use w at all

Then swap colors on the v_h path



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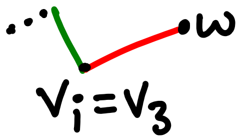
So path ends w/ 

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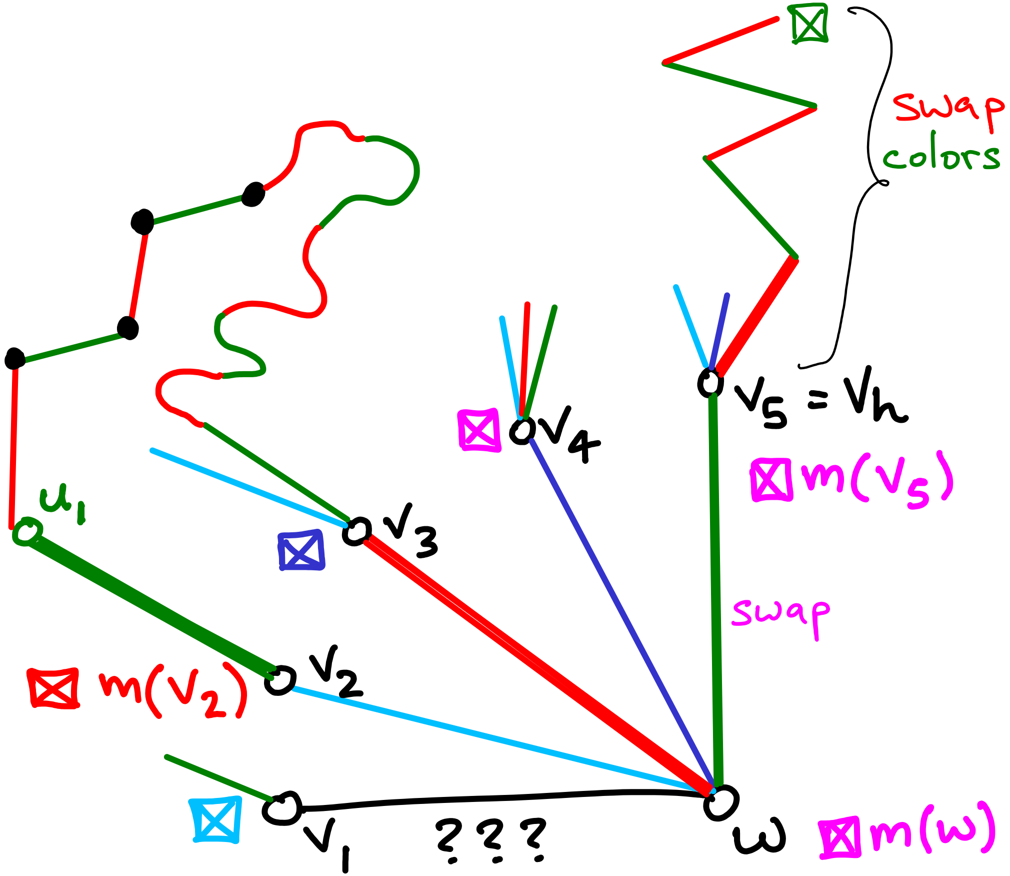
Then swap colors on the v_h path

This starts a chain reaction on the fan



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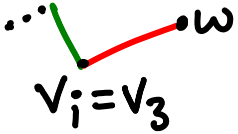
So path ends w/ 

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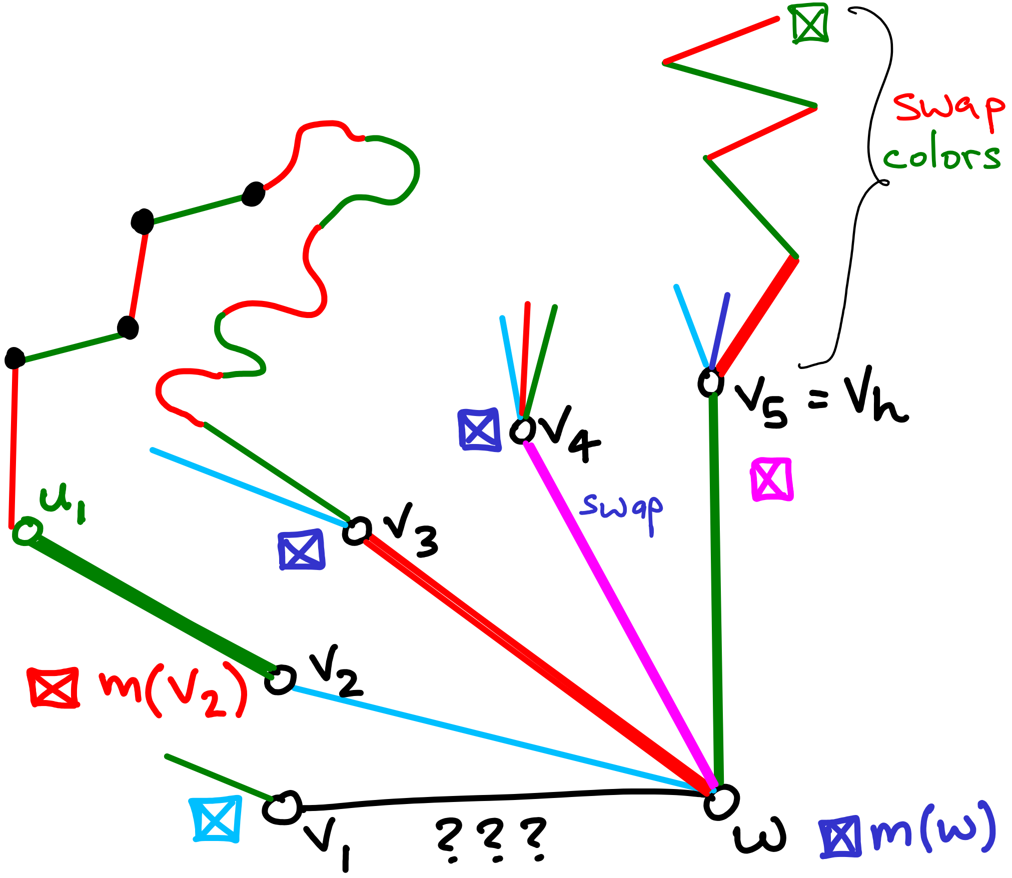

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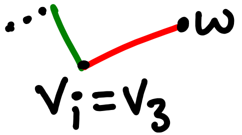

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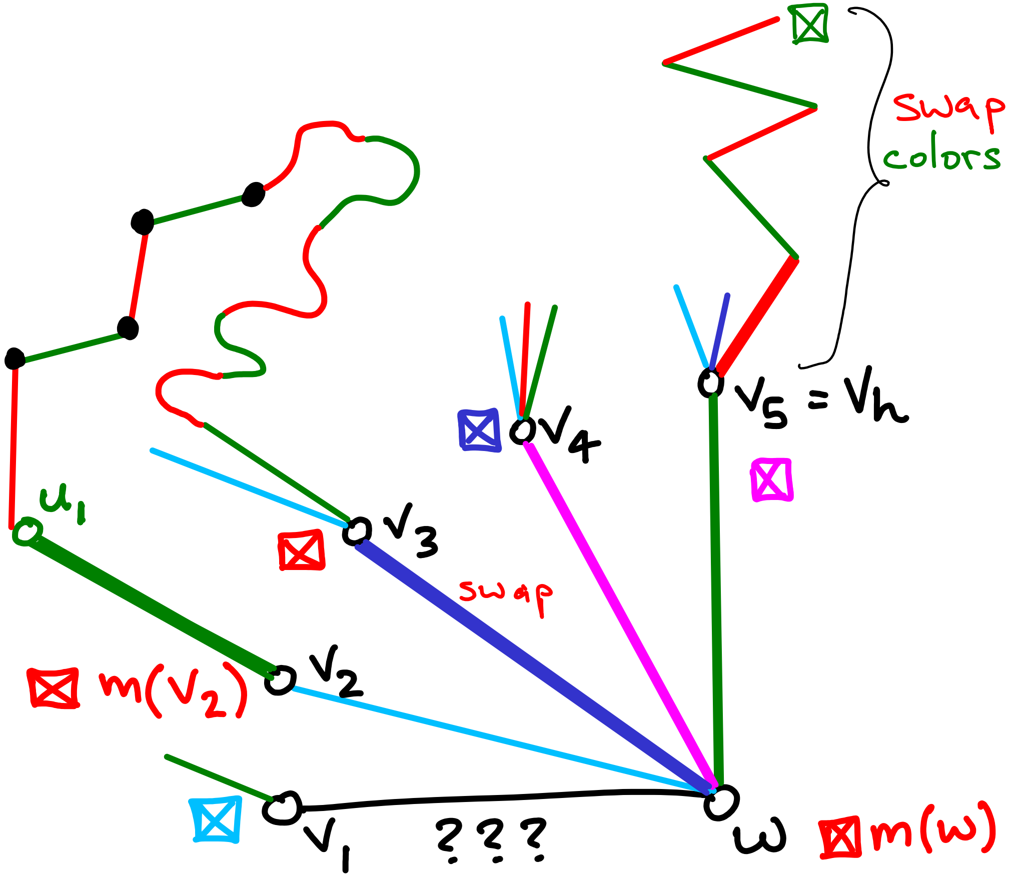

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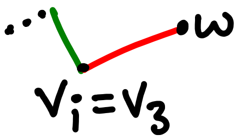
So path ends w/ 

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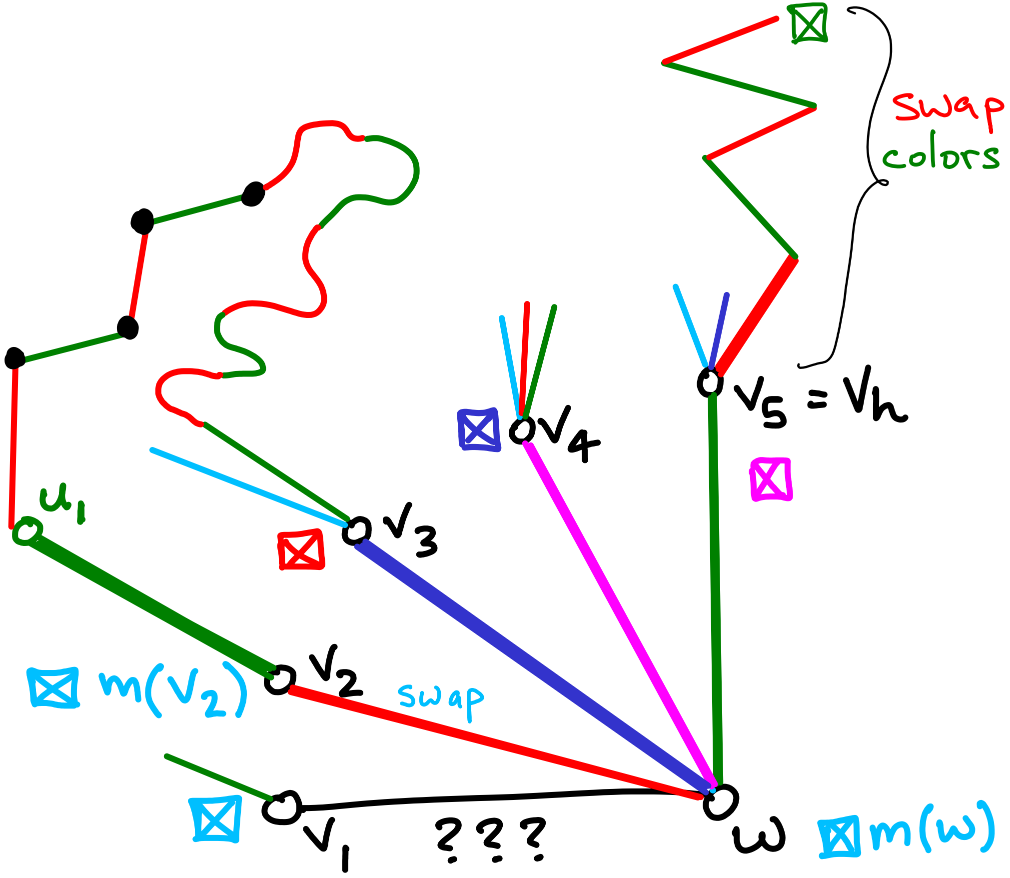

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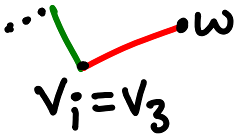
So path ends w/ 

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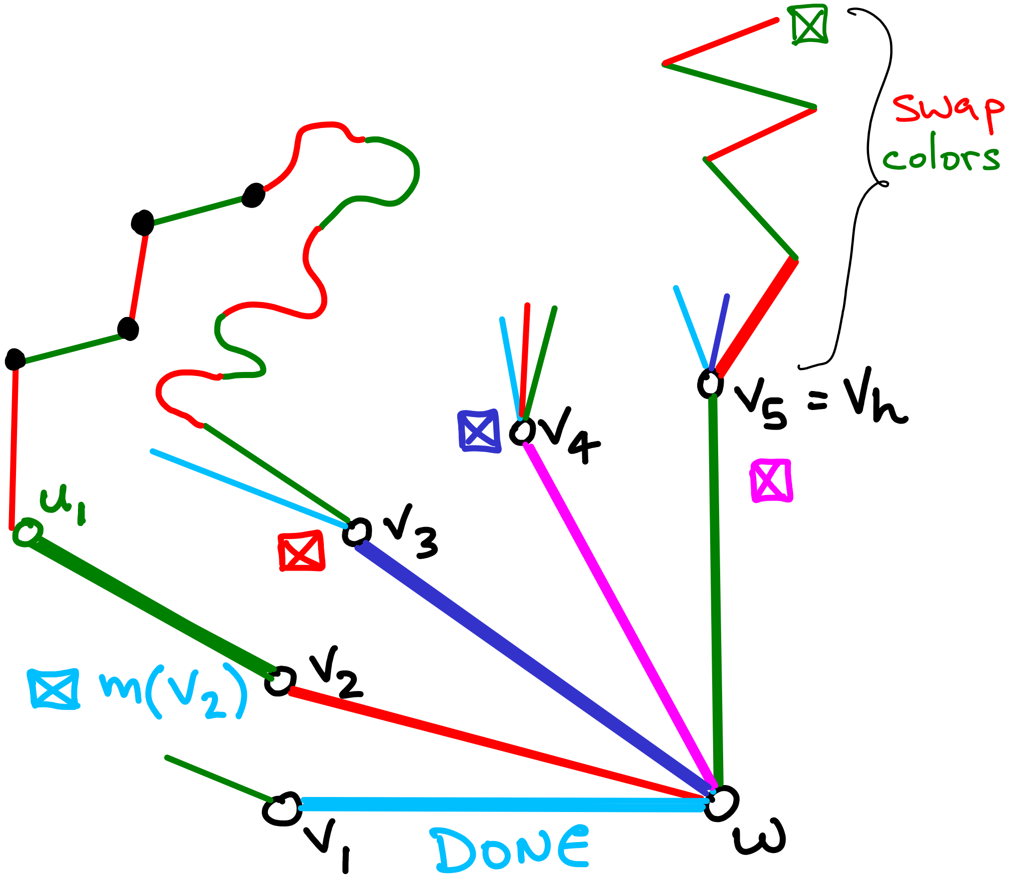

Conclusion: every graph w/ max degree Δ
can be edge-colored w/ $\leq \Delta+1$ colors : $\chi' \leq \Delta+1$

Remarkably $\Delta \leq \chi' \leq \Delta+1$

Amazingly (?) it is NP-hard to decide if any given graph has $\chi' = \Delta$ or $\chi' = \Delta+1$!

(there are polynomial algorithms for specific classes of graphs)