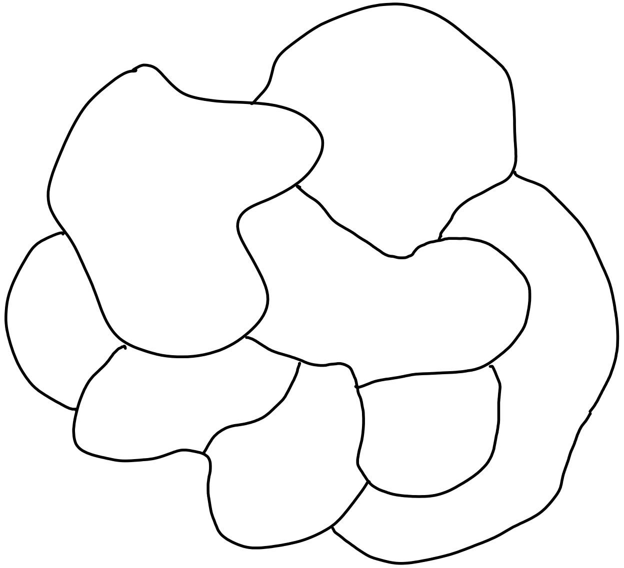

MAP COLORING

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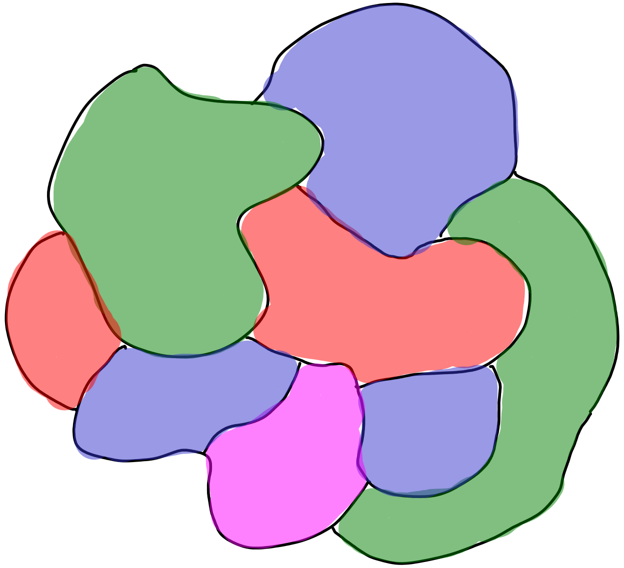

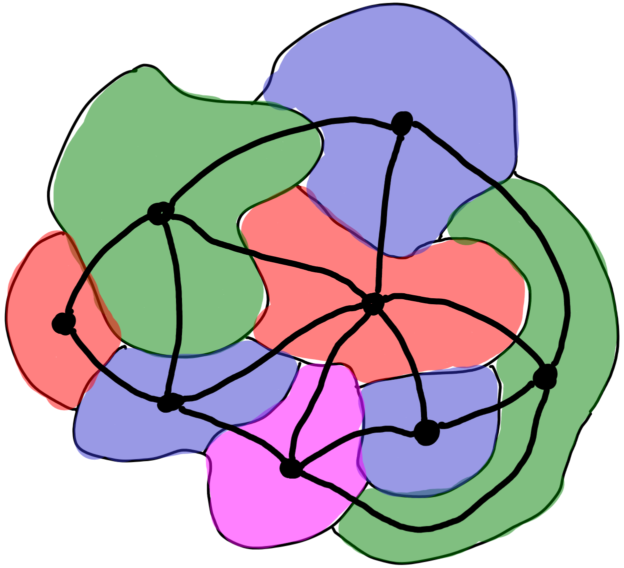

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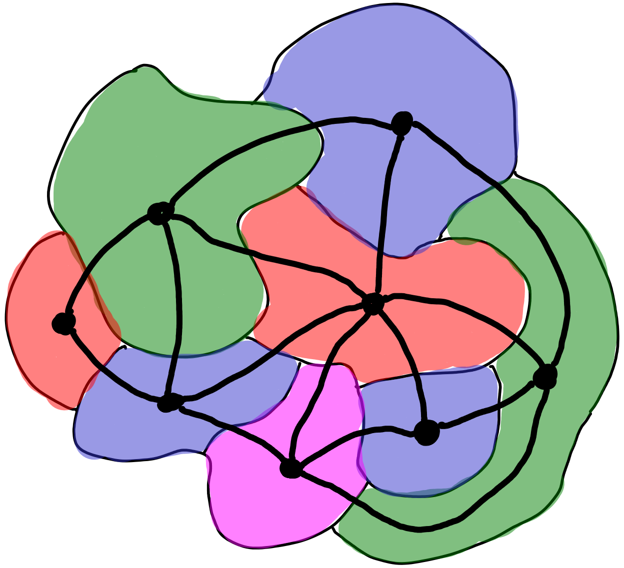
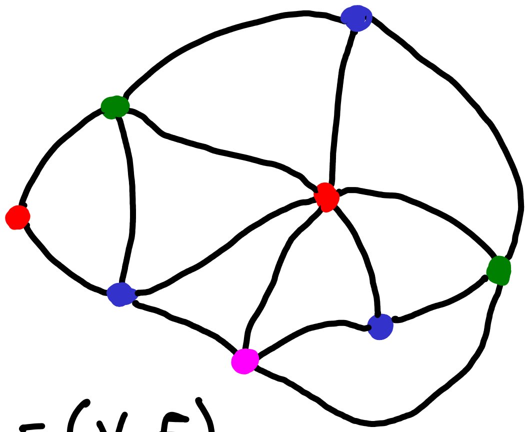


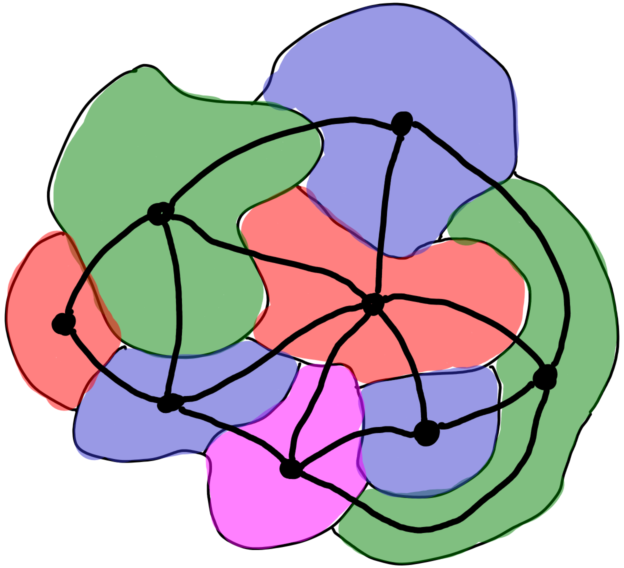
$\Rightarrow$



$G = (V, E)$

COLORING G → no adjacent vertices get same color

MAP COLORING



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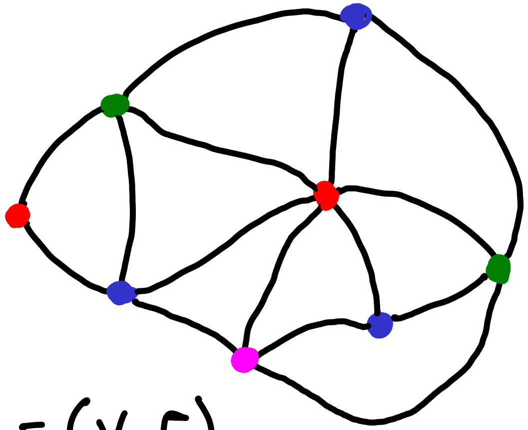

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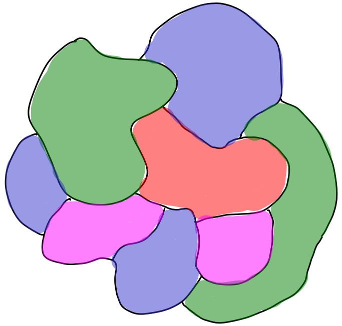

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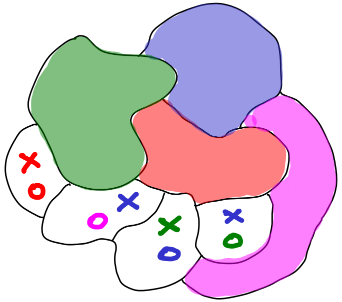

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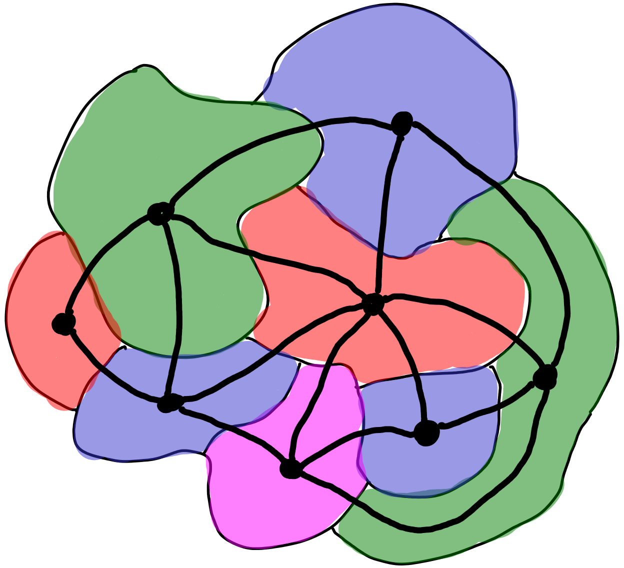

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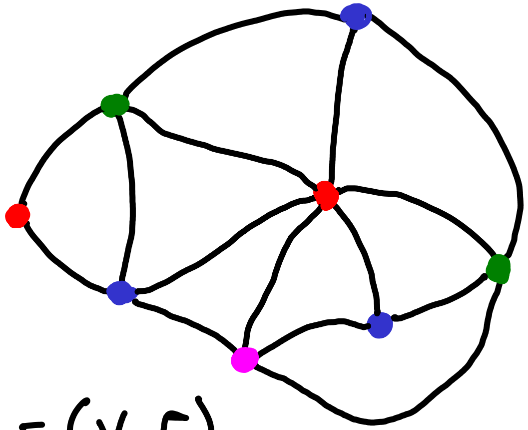

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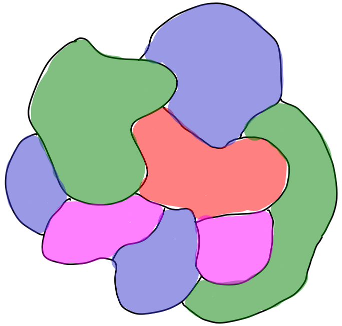

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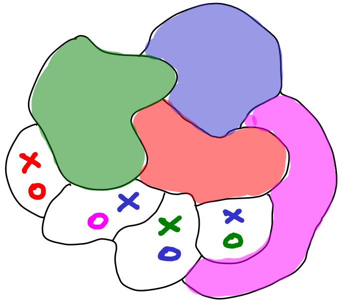

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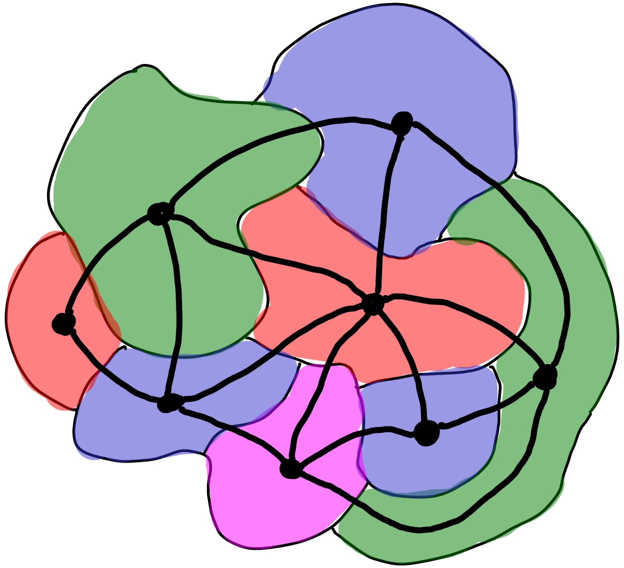

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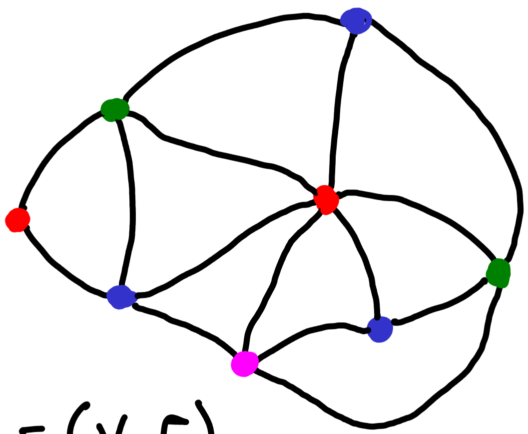

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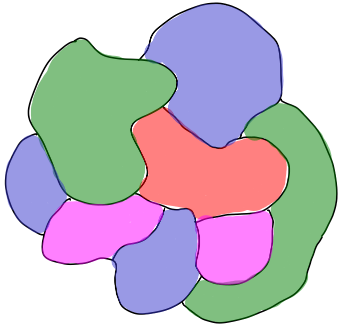

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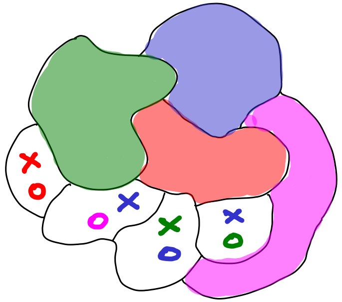

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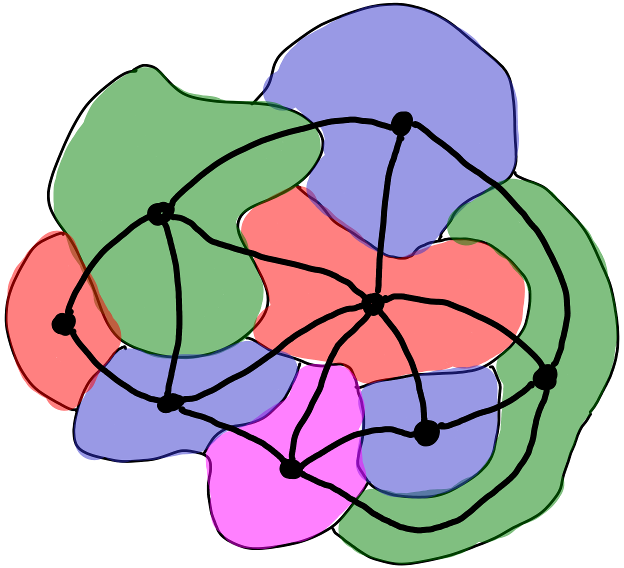
chromatic number

χρῶμα = color

MAP COLORING



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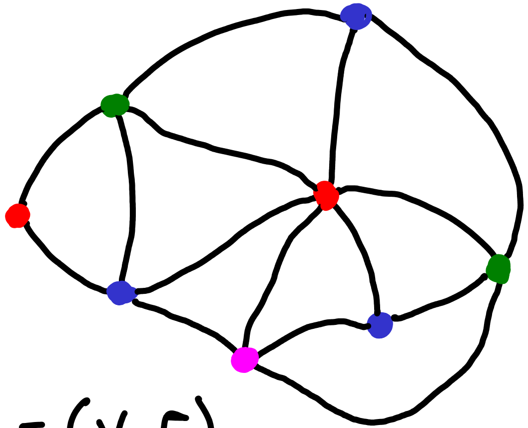

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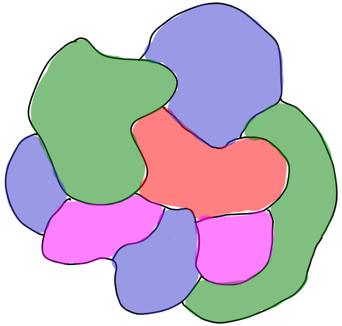

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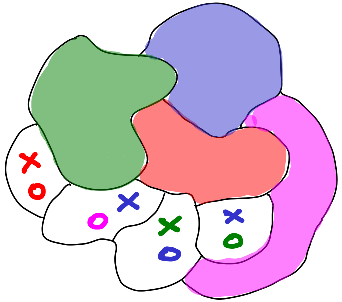

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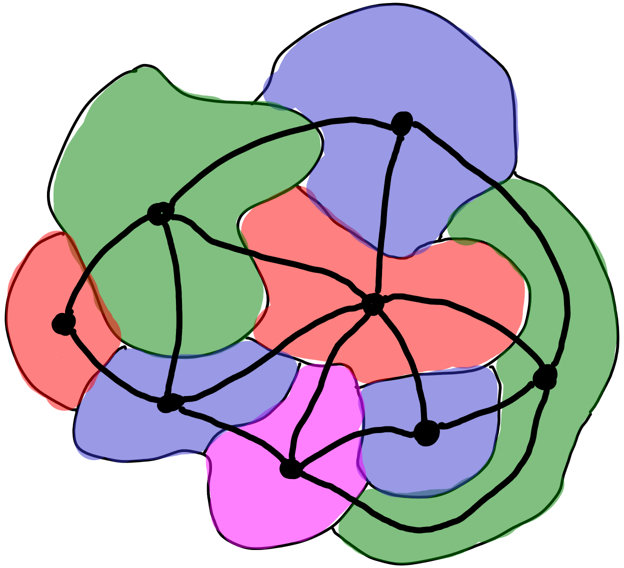

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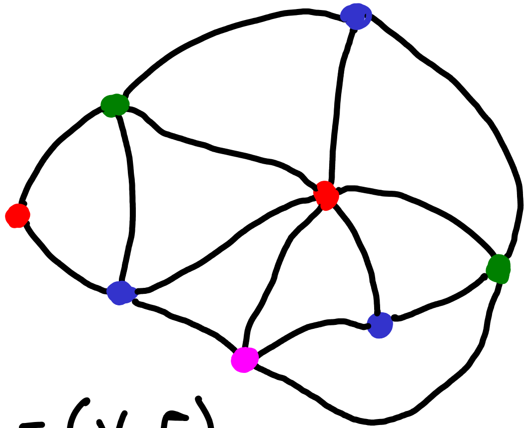

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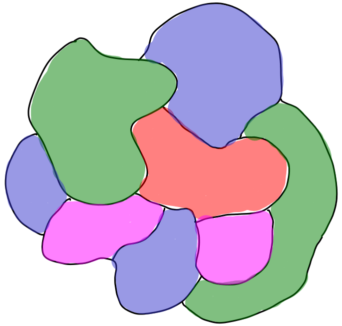

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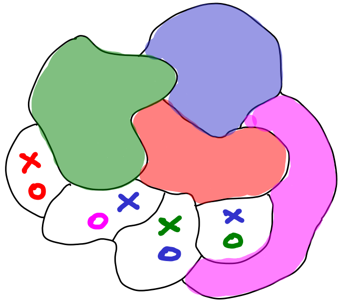

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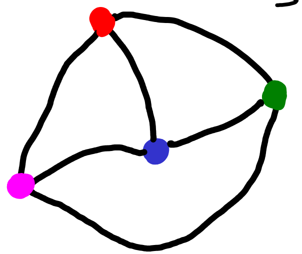

subgraph K_4

so $\chi \geq 4$

...but not 3-colorable

EXAM SCHEDULING

students : s_1 s_2 s_3 s_4 s_5 classes c_1 c_2 c_3 c_4 c_5

EXAM SCHEDULING

students : s_1 s_2 s_3 s_4 s_5 classes c_1 c_2 c_3 c_4 c_5

s_1	s_2	s_3	s_4	s_5
c_1 c_2 c_3	c_1 c_4 c_5	c_1 c_5	c_2 c_3	c_1 c_2 c_4

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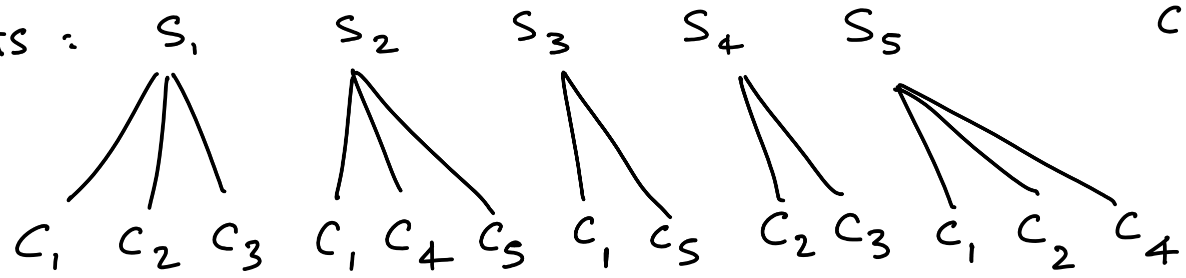

classes c_1 c_2 c_3 c_4 c_5

Can't schedule exam simultaneously for classes taken by s_i

Want to minimize exam slots.

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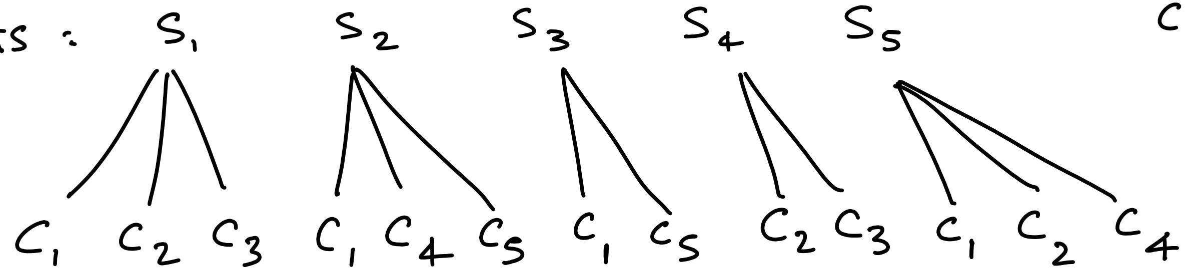

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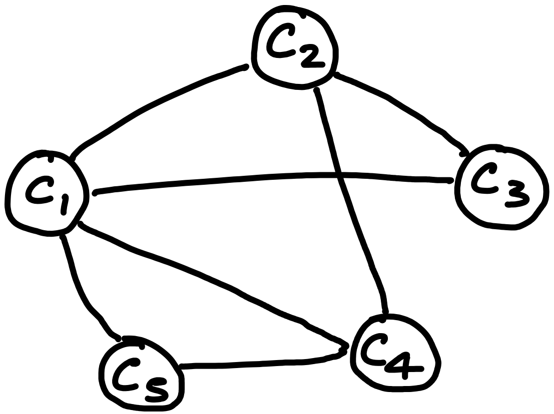
Make G: V = classes E = conflicts



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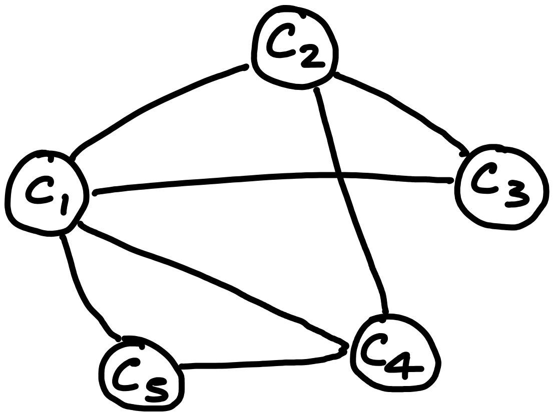
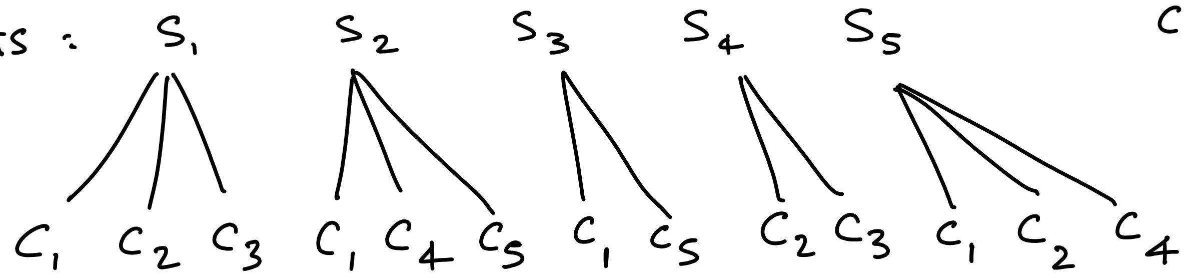


Colors = slots (minimize colors)

If no edge has same color at endpoints,
then no 2 classes are in same slot

EXAM SCHEDULING

students :



classes c_1 c_2 c_3 c_4 c_5

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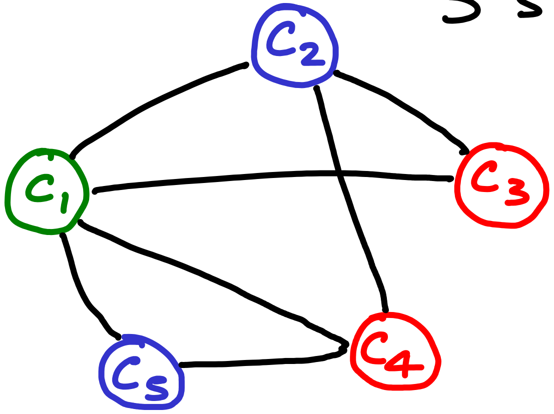
3 slots



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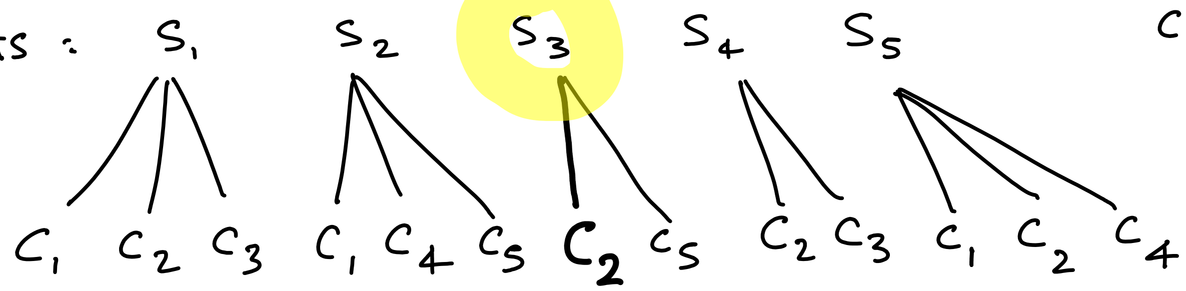

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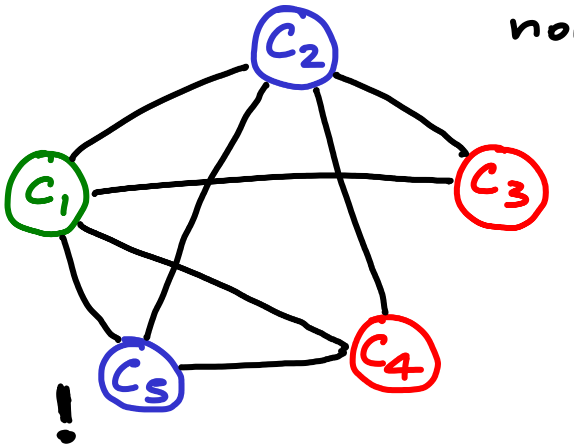
Need 4 now



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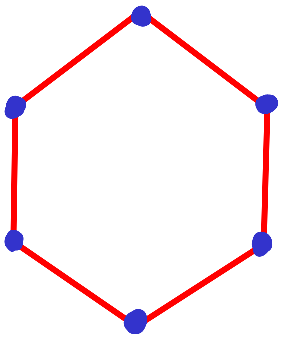
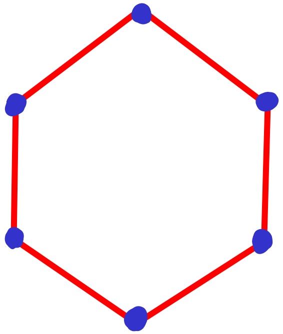


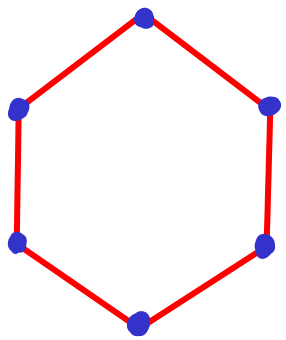
What is χ for cycles?



What is χ for cycles?

$\chi = 2$ if V even

$= 3$ if V odd



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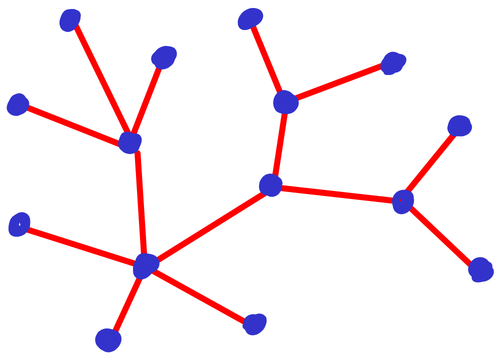
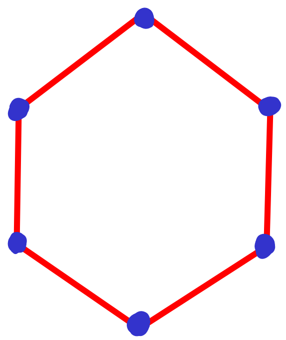


For trees?



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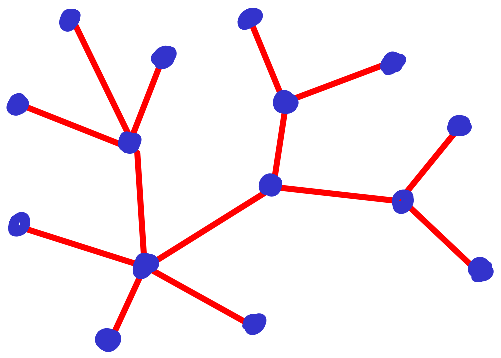
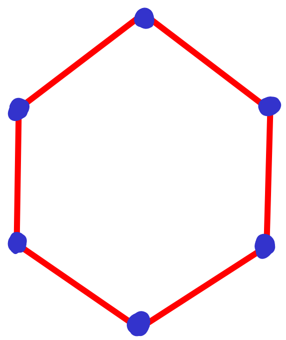


For trees?

Remove a leaf, v.

2-color the rest...

...



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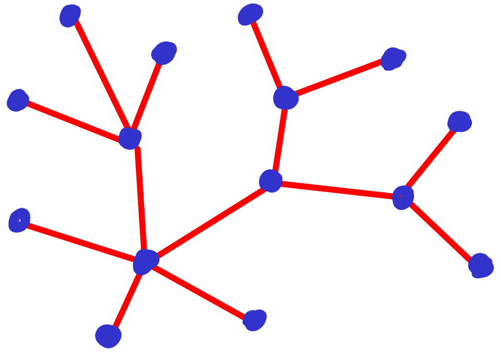

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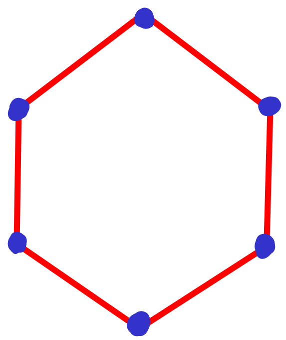
Color v opposite of $p(v)$

$\chi = 2$



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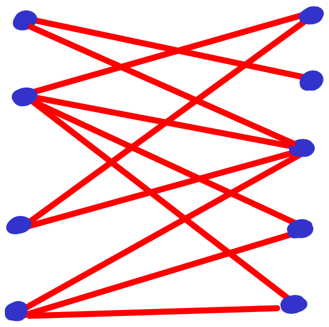
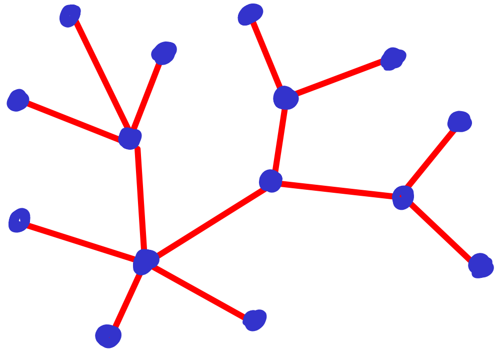


For bipartite graphs?



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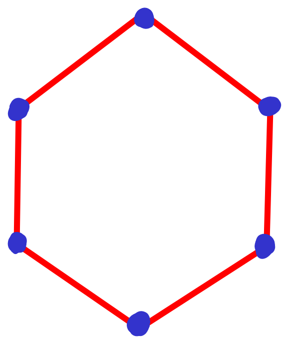

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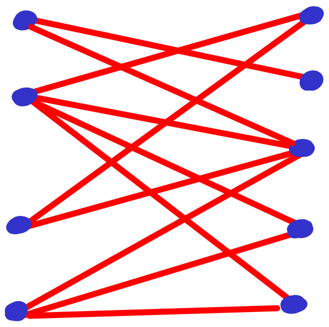

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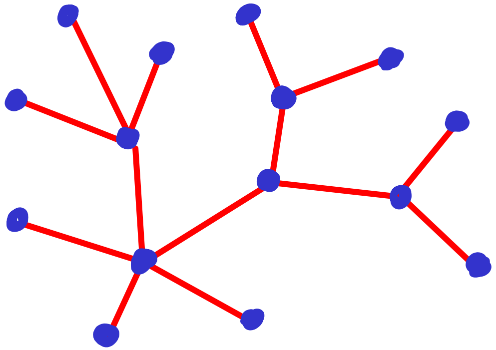

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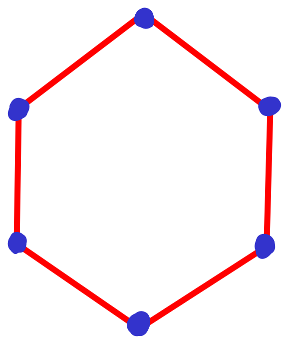

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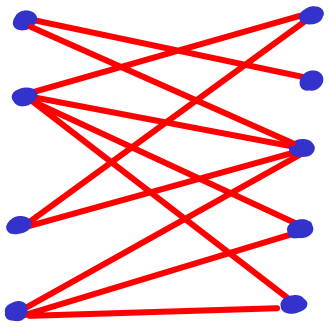

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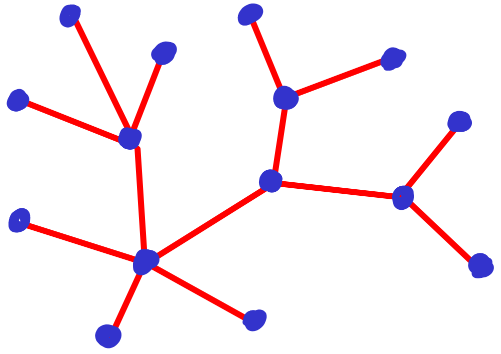
In fact if $\chi(G) = 2$ then G is bipartite by definition



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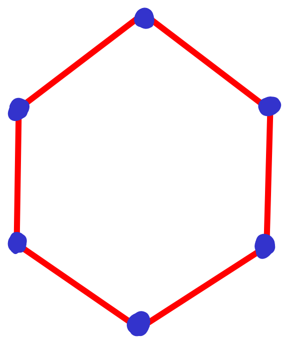

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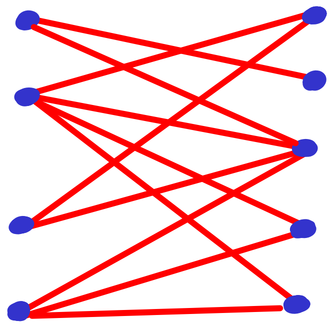
Claim:

G is bipartite

$\iff$

G contains no odd cycle



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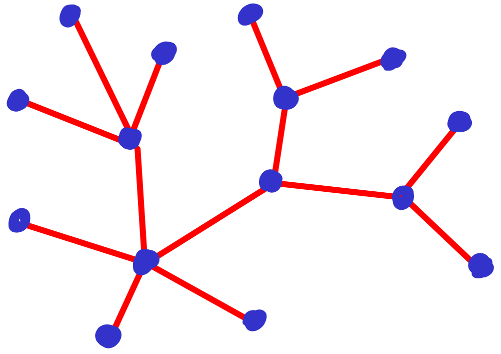

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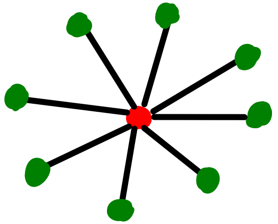
Trivial bounds : $\chi \leq n$ $[K_n ; \Delta = n]$ & $\chi \geq 2$  $\Delta = n-1$

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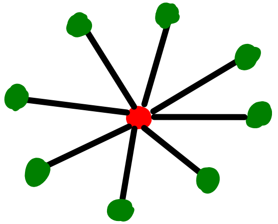
Trivial bounds : $\chi \leq n$ $[K_n ; \Delta = n]$ & $\chi \geq 2$  $\Delta = n-1$

Claim $\chi \leq \Delta + 1$

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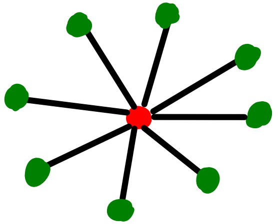

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Claim $\chi \leq \Delta + 1$

Incrementally "add" vertices.

Look at colors of their neighbors.

Always have ≥ 1 color available.

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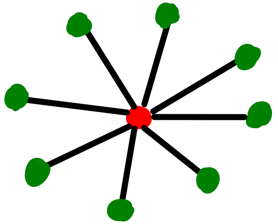
Trivial bounds : $\chi \leq n$ $[K_n ; \Delta = n]$ & $\chi \geq 2$  $\Delta = n-1$

Claim $\chi \leq \Delta + 1$

Incrementally "add" vertices.
Look at colors of their neighbors.
Always have ≥ 1 color available.

$\longleftrightarrow$

Remove any vertex v.
Color $G-v$ by induction.
Re-insert v

Back to $\chi = 2$. $\iff$ bipartite graphs

We can test for this efficiently: greedy BFS

Back to $\chi = 2$. $\Leftrightarrow$ bipartite graphs

We can test for this efficiently: greedy BFS

↳ Color any vertex blue

Then its neighbors must be red.

Back to $\chi=2$. $\iff$ bipartite graphs

We can test for this efficiently: greedy BFS

↳ Color any vertex blue

Then its neighbors must be red.

All their neighbors must be blue, etc

Back to $\chi=2$. $\iff$ bipartite graphs

We can test for this efficiently: greedy BFS

↳ Color any vertex blue

Then its neighbors must be red.

All their neighbors must be blue, etc

If this search finds an already colored vertex,

either it matches what we would have colored it

OR we conclude that $\chi > 2$

Back to $\chi=2$. $\Leftrightarrow$ bipartite graphs

We can test for this efficiently: greedy BFS

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Testing if $\chi \leq 3$ is NP-complete! (or if $\chi \leq$ any constant)

COLORING PLANAR GRAPHS (like map duals)

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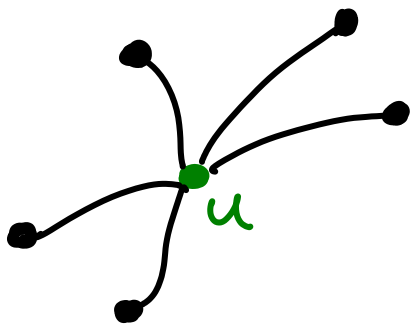

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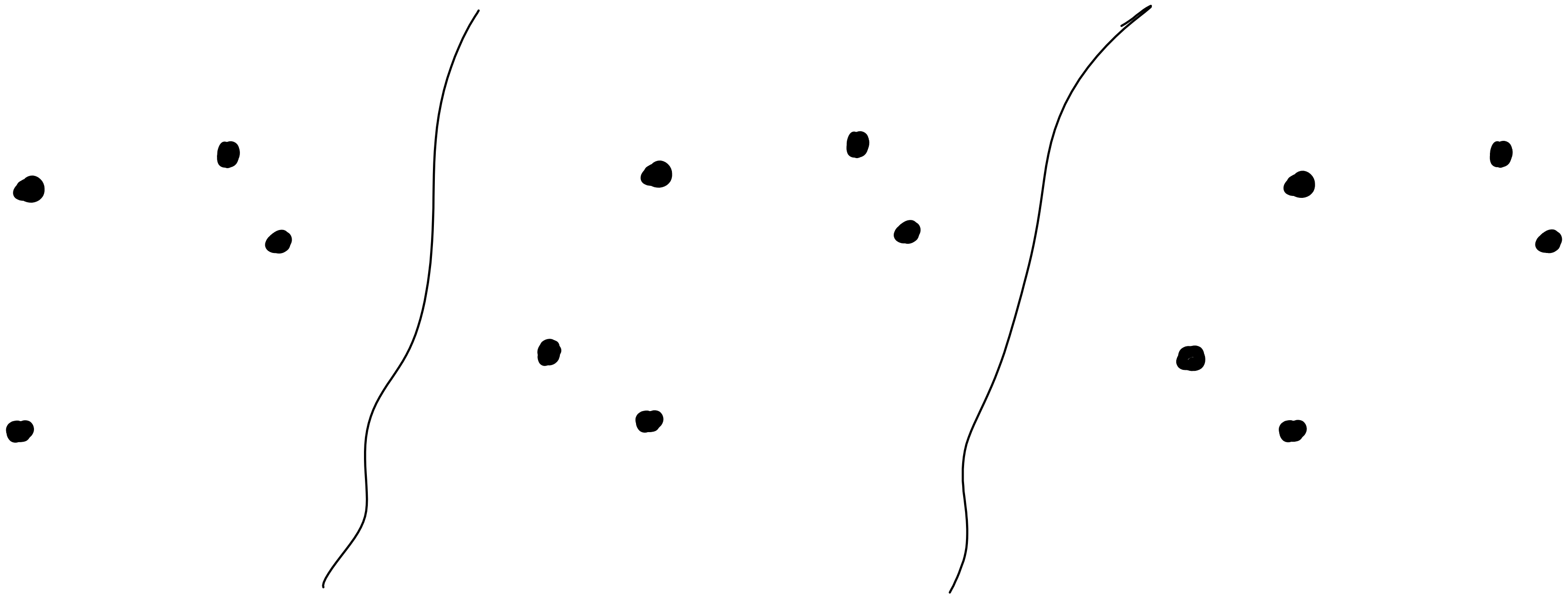

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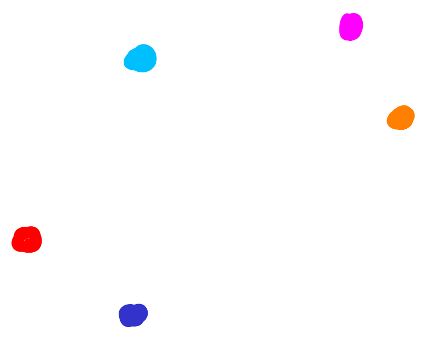

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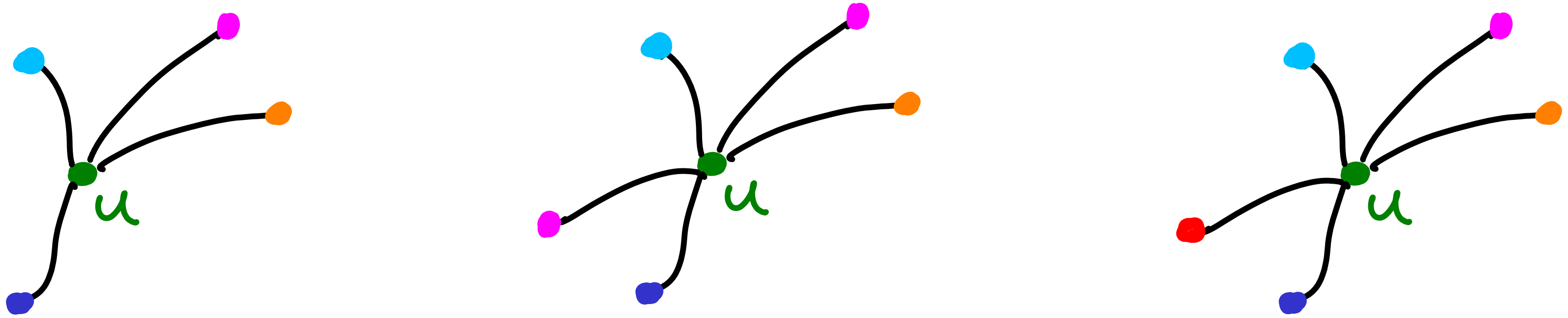
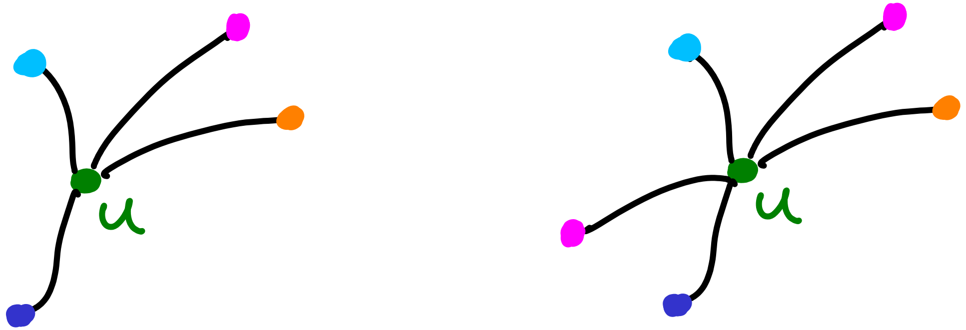


re-insert u: give it a color not used by neighbors $\square$

Claim: $\chi \leq 5$ ... trivial if ???

Claim: $\chi \leq 5$ ... trivial if $V \leq 5$



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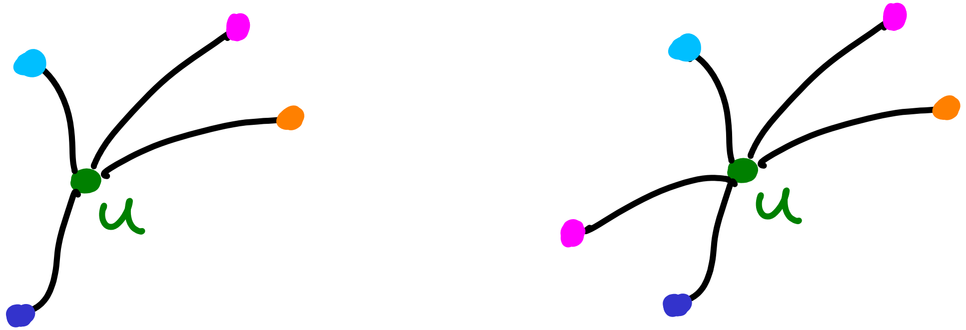

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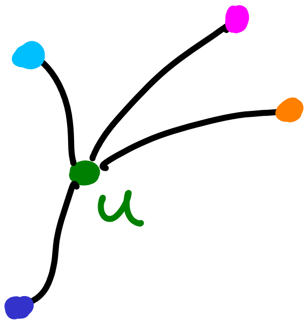

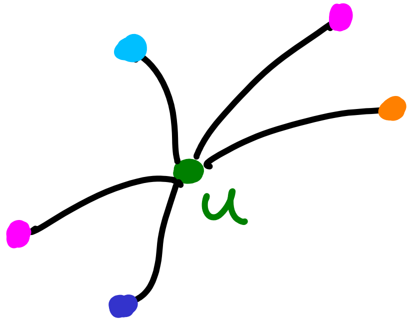

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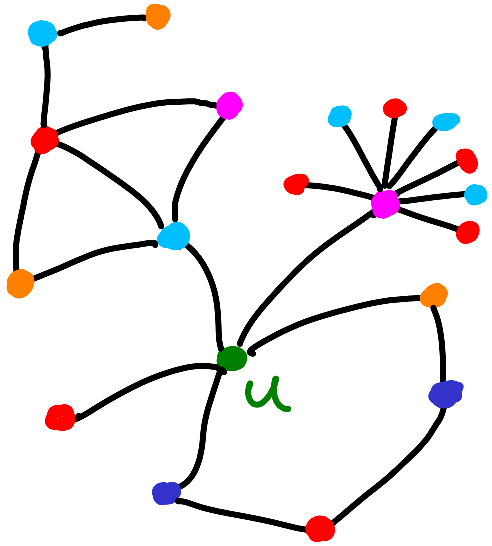
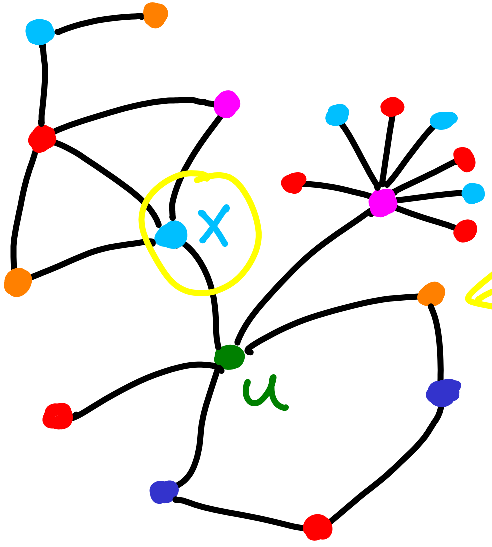


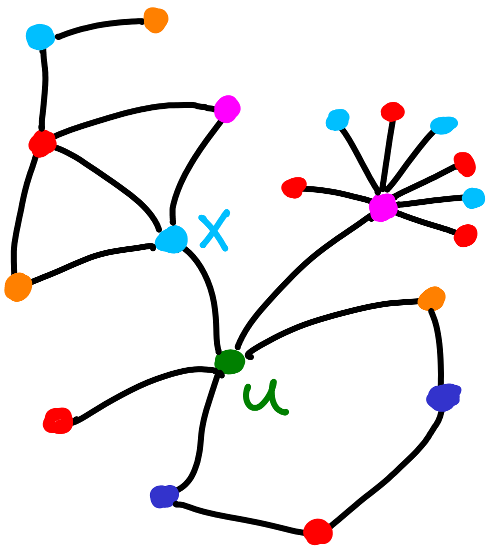
Consider any embedding of G

We need a neighbor of u to change color

Try to change × from ● to ●

[specifically skipping 2 over in adj(u)]





Try to change x from ● to ●

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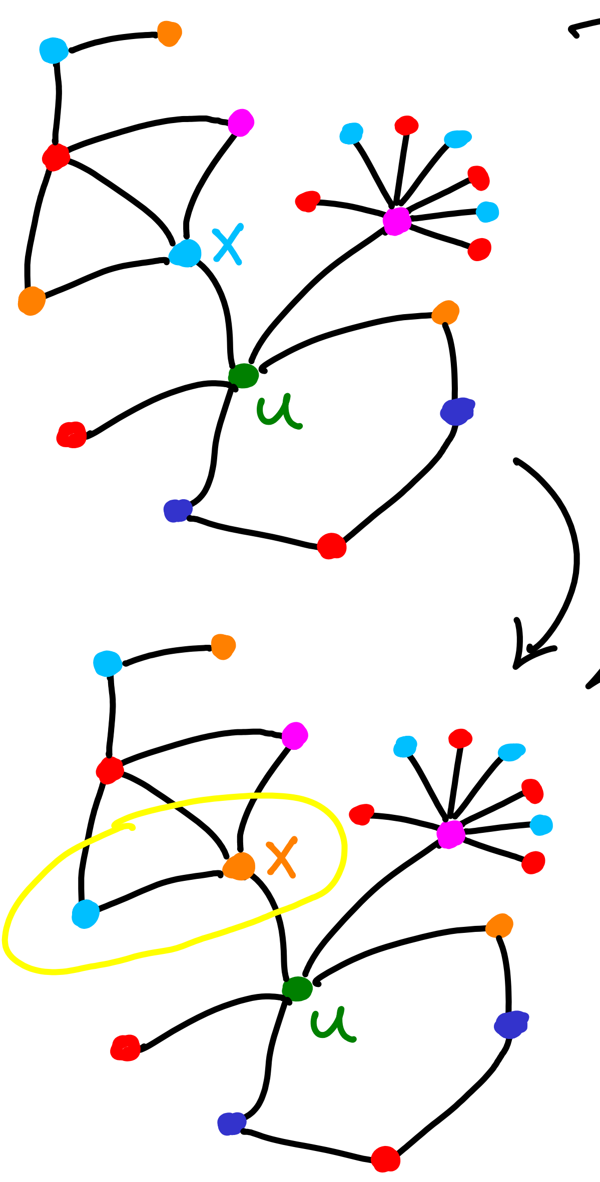
↳ This works if x has no ● neighbors



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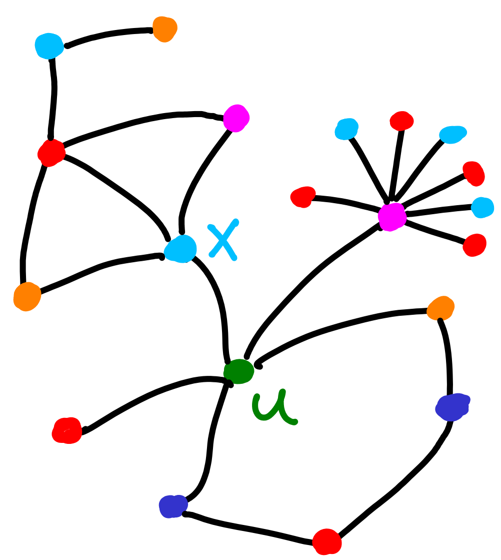

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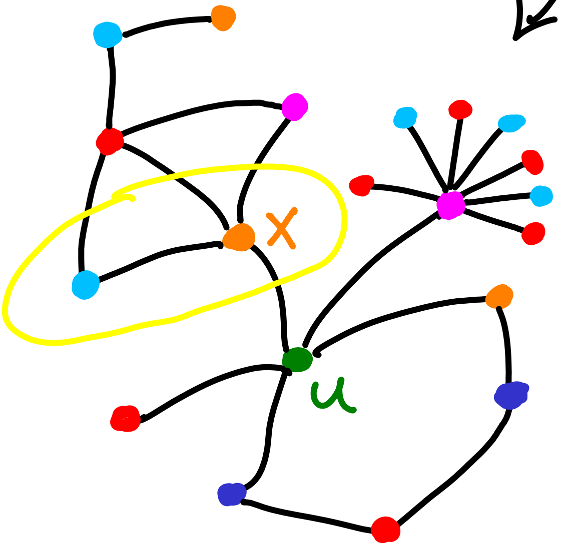

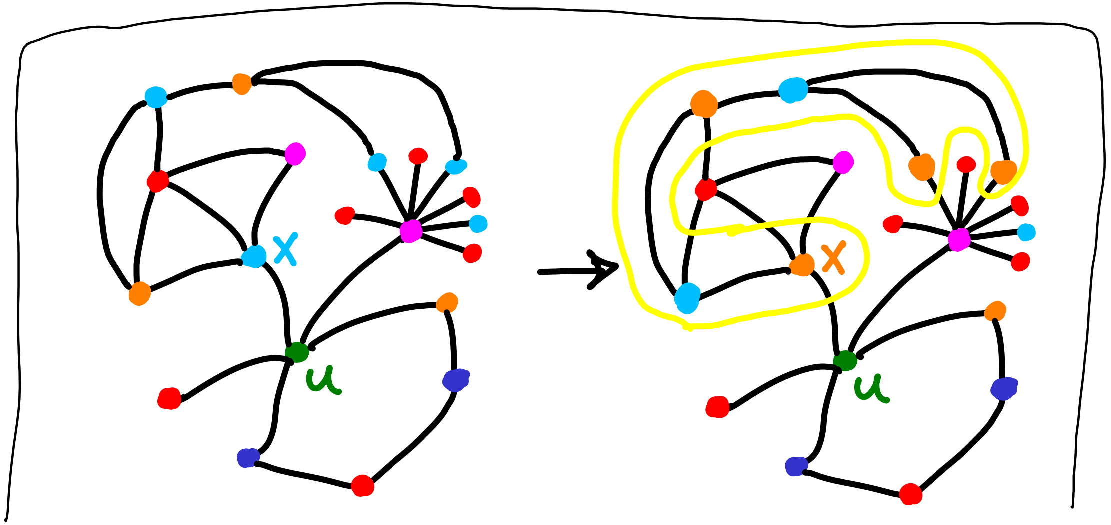

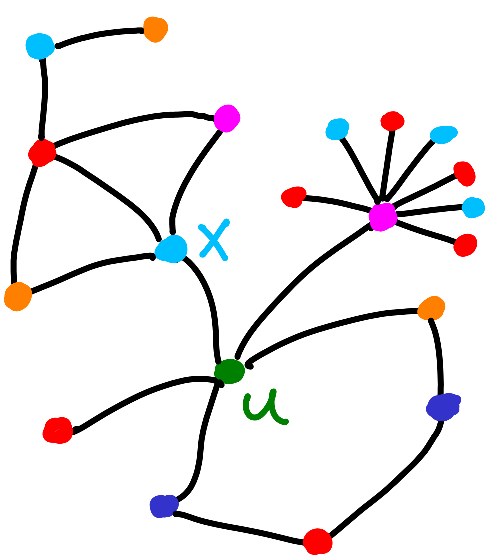

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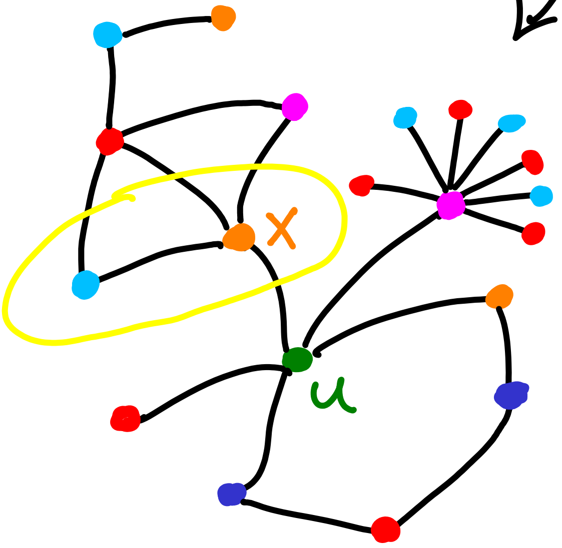
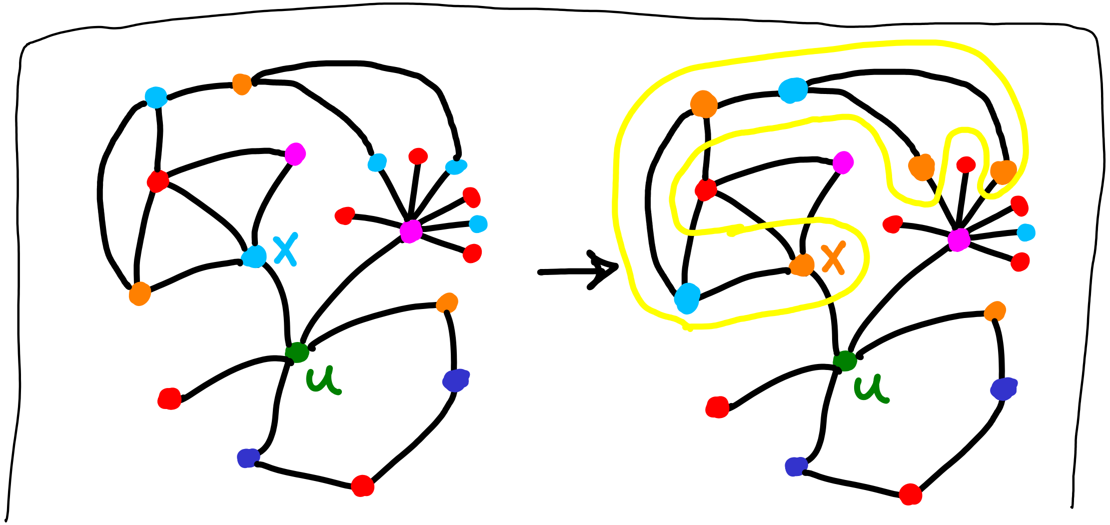
ONE PROBLEM

?





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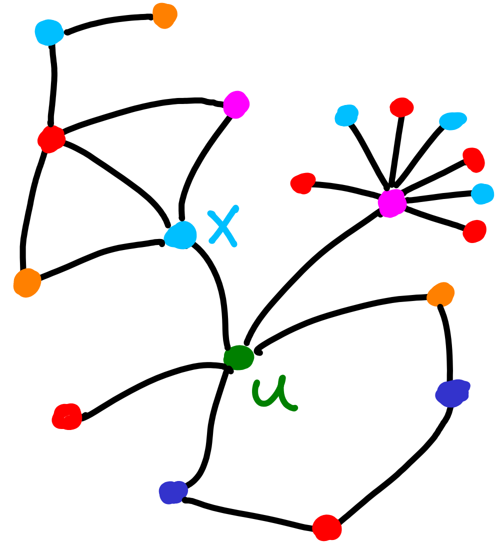

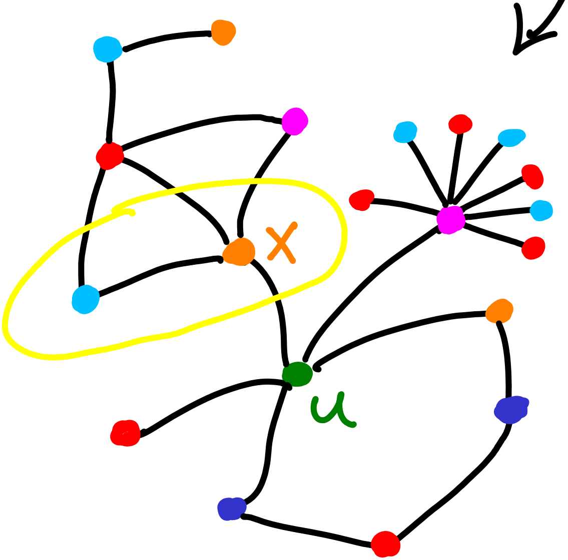

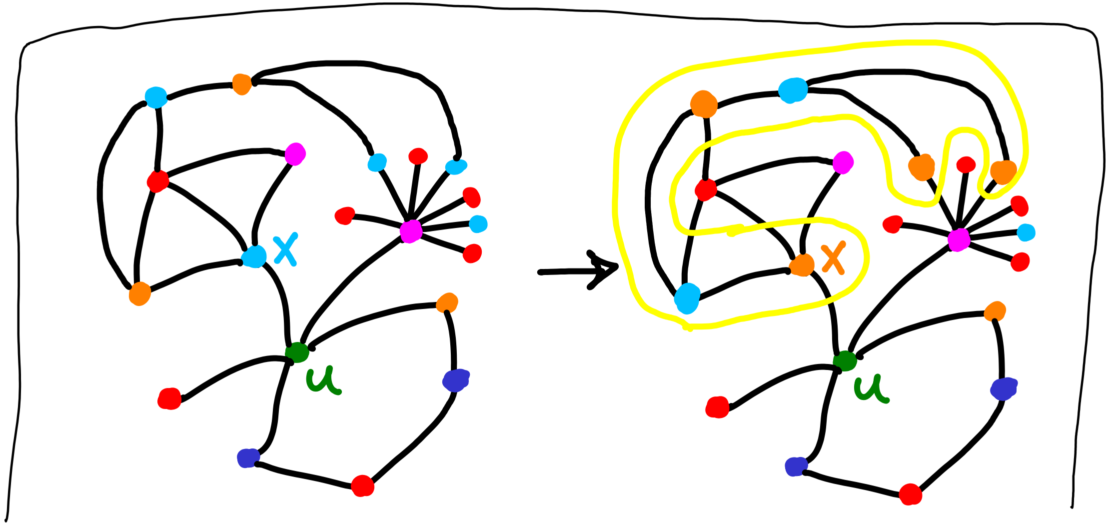

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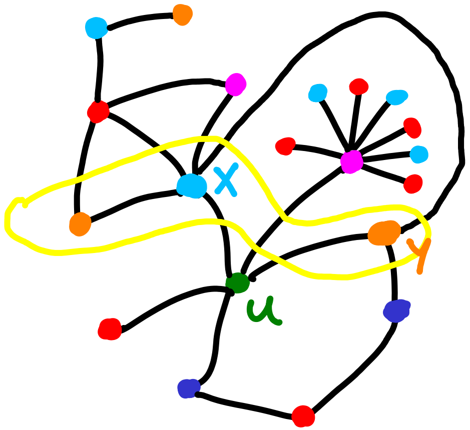
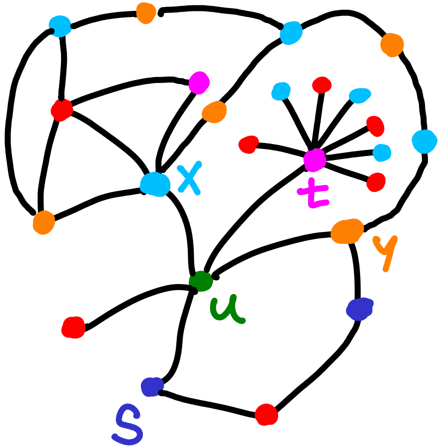
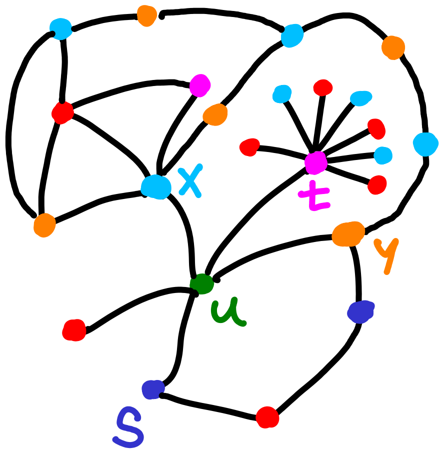




The only bad case involves a path from x to y

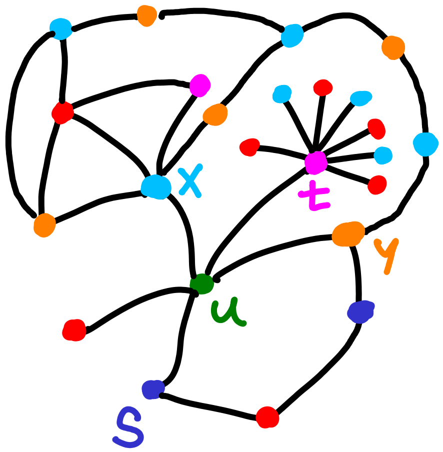
that alternates $x \bullet \bullet \bullet \bullet \ldots y$



The only bad case involves a path from x to y that alternates x ● ● ● ●... y

Together with u the path forms a cycle surrounding the ● neighbor of u.

So ... ?



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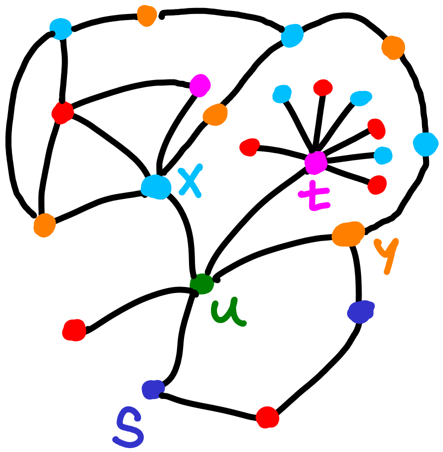
Restart the entire procedure using s & t instead of x & y.



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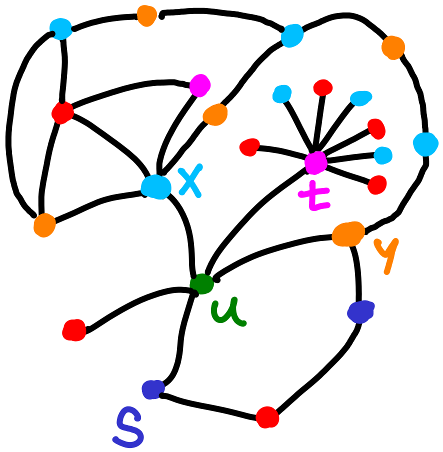

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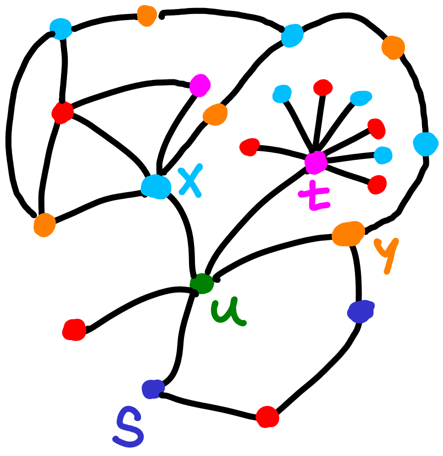

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Impossible: This is a plane drawing

□

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3-coloring:
- clearly not always possible
- if triangle-free then 3-colorable (in fact if ≤ 3 triangles)