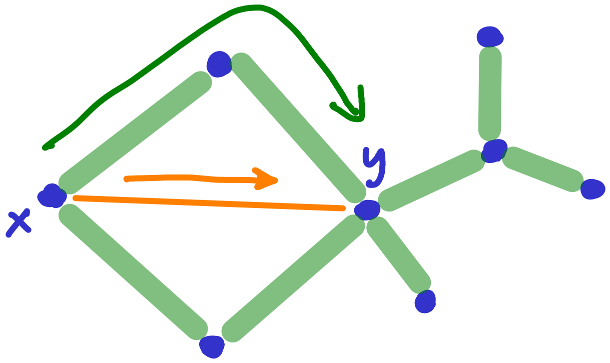
# Spanners




Objective

Given a set of vertices, form a graph (place edges) s.t. the spanning ratio (detour) for any pair of vertices is "low".

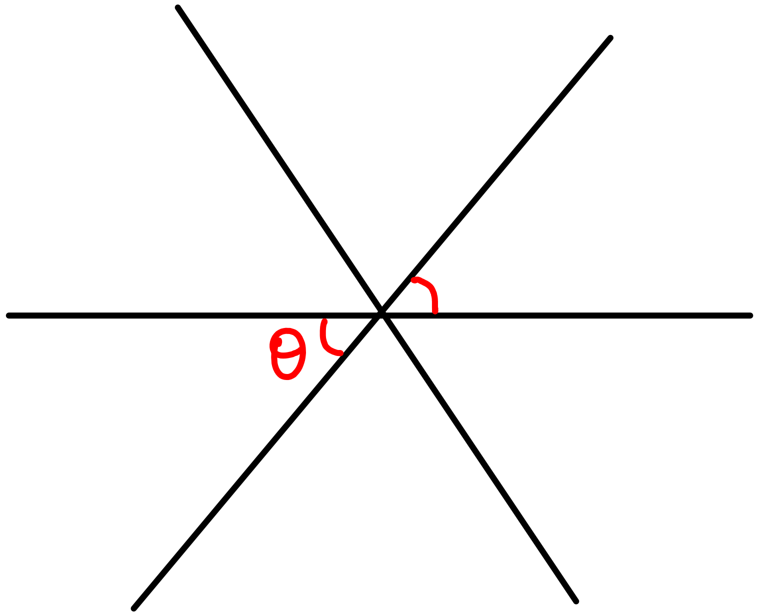
$$\text{ratio} = \frac{\text{distance via graph}}{\text{Euclidean dist.}}$$

## Other desirable properties

- use few edges
- minimize degree
- planarity
- minimize max. path length
- robustness: good connectivity

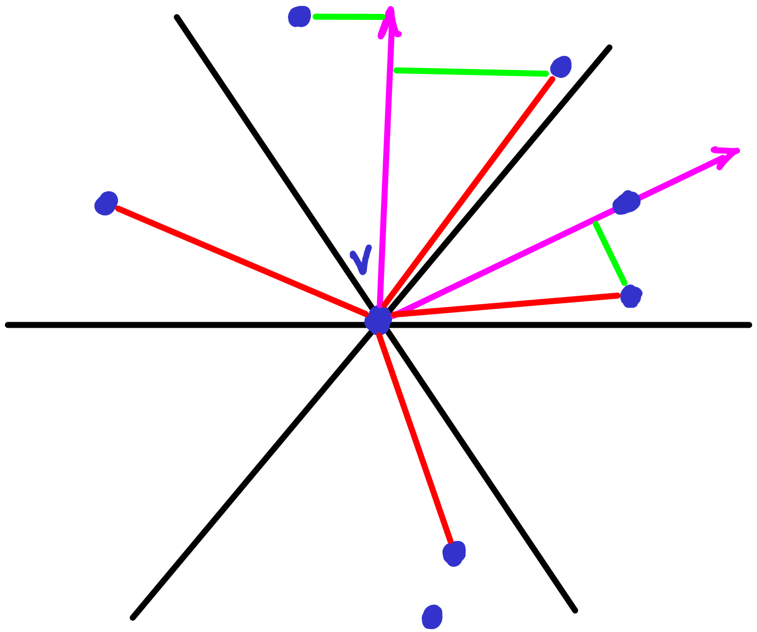
etc

# the $\Theta$-graph spanner




Form $K$ cones w/ angle $\theta = \frac{2\pi}{K}$

# the $\Theta$-graph spanner




Form $k$ cones w/ angle $\theta = \frac{2\pi}{k}$

For every vertex v

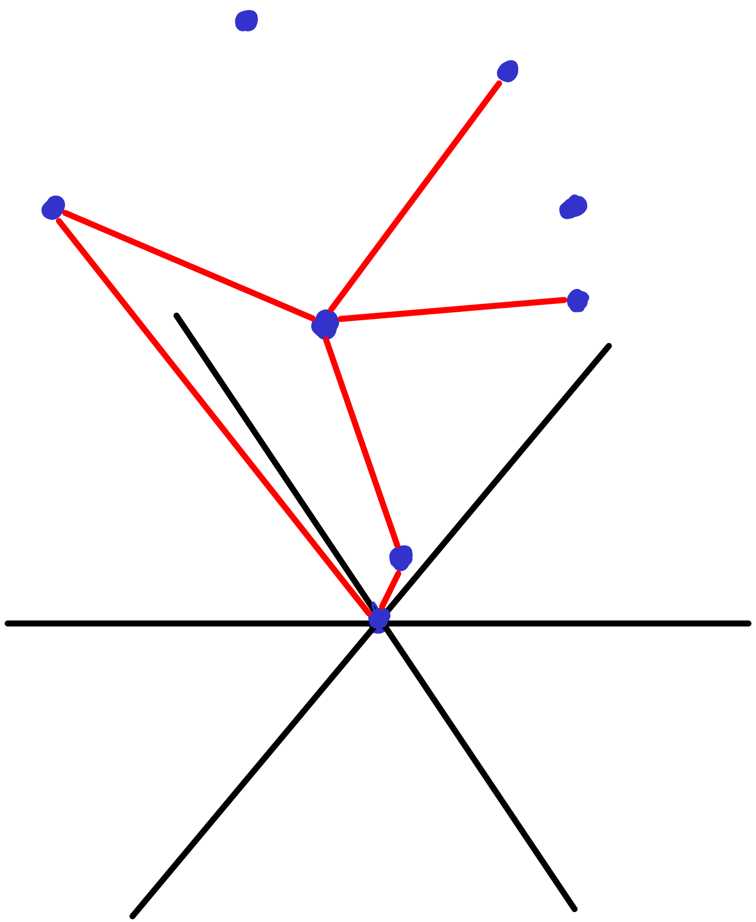
shift cone structure onto v

& connect v to 1 vertex in each cone (if non-empty)

Connection rule:

- project vertices onto cone bisector
- choose vertex with closest projection

# the $\Theta$-graph spanner



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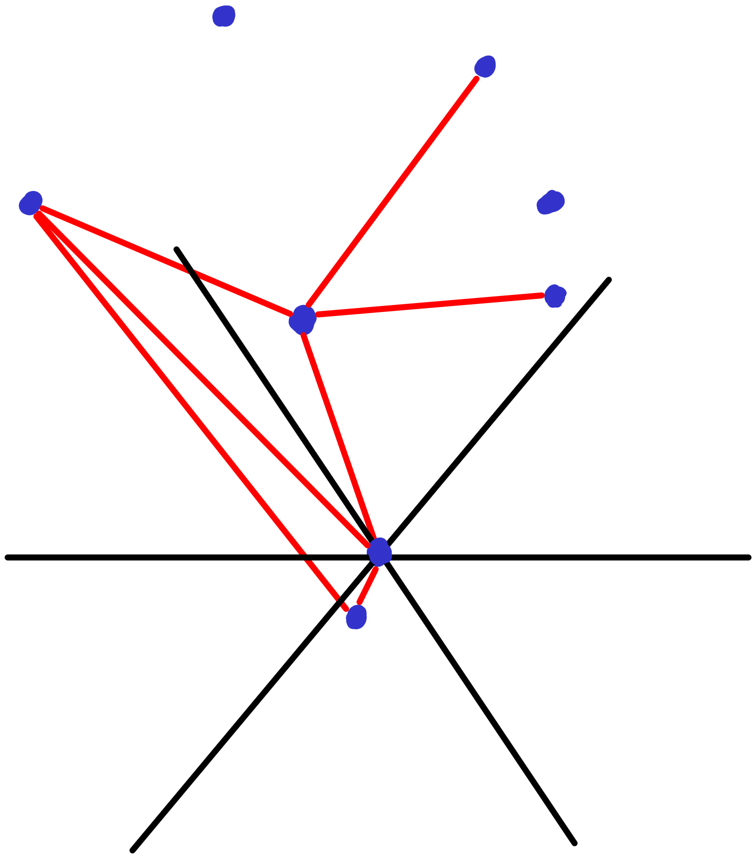
# the Θ-graph spanner



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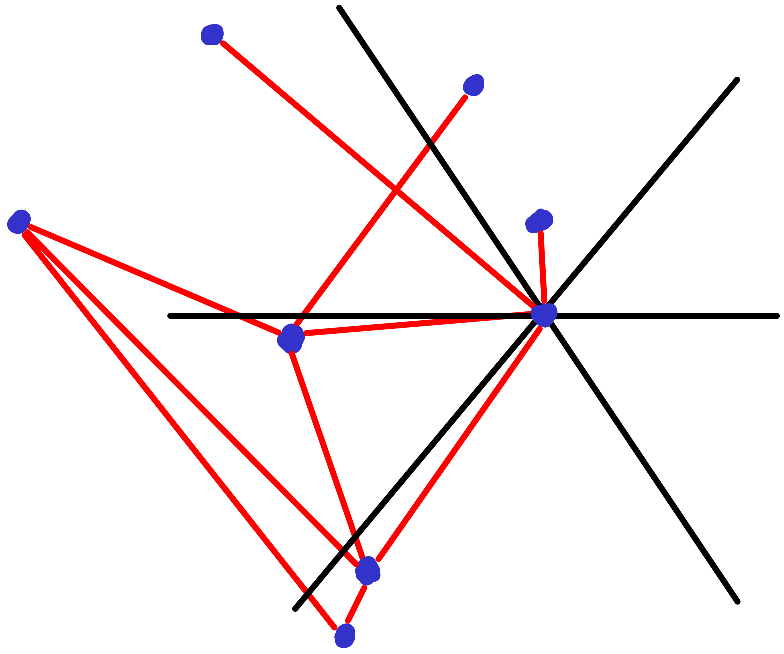

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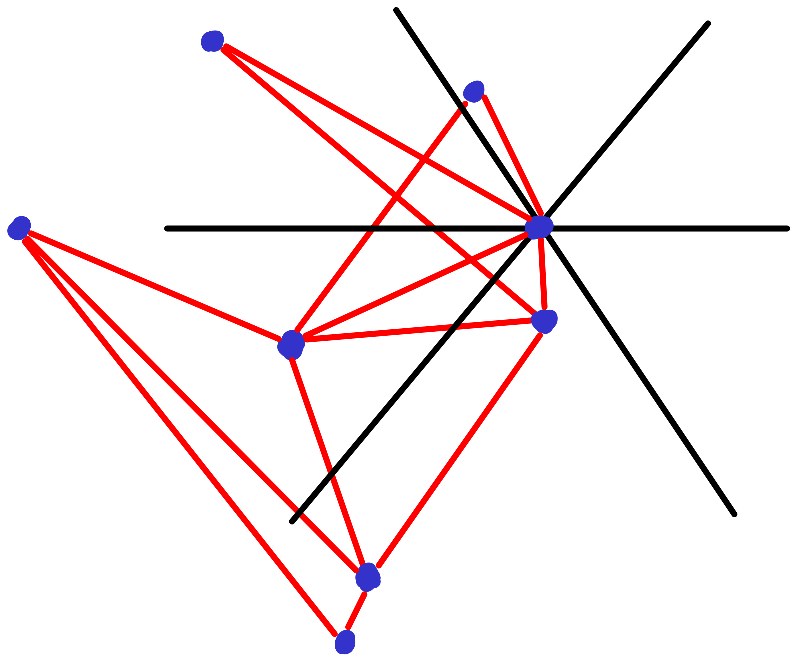

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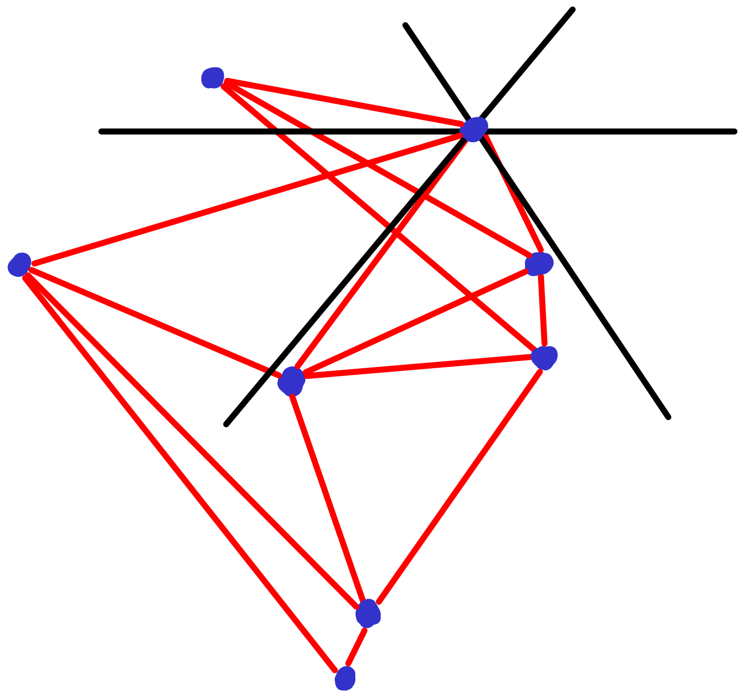

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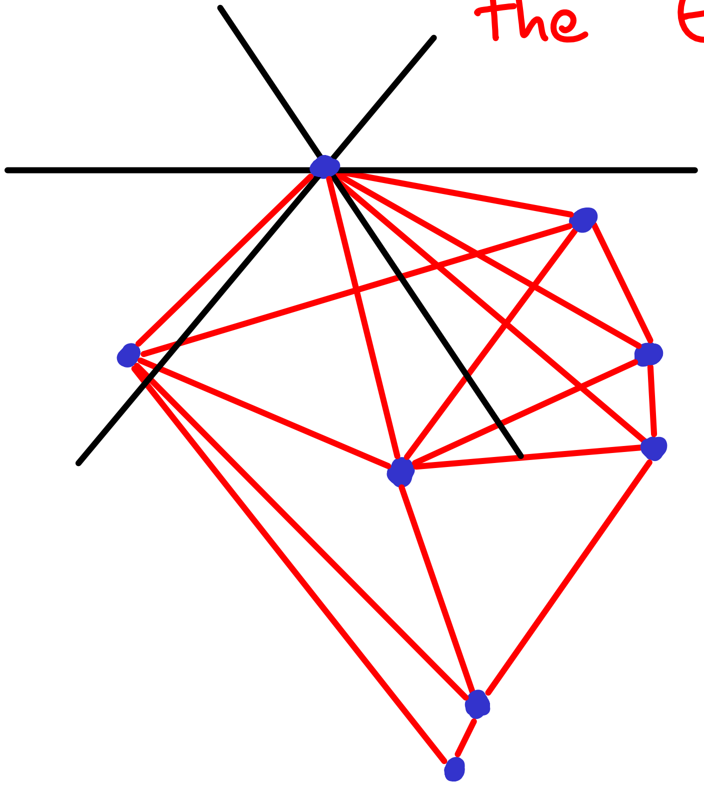

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For every vertex $v$

shift cone structure onto $v$

& <u>connect</u> $v$ to 1 vertex in each cone

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## Connection rule:

- <u>project</u> vertices onto cone bisector
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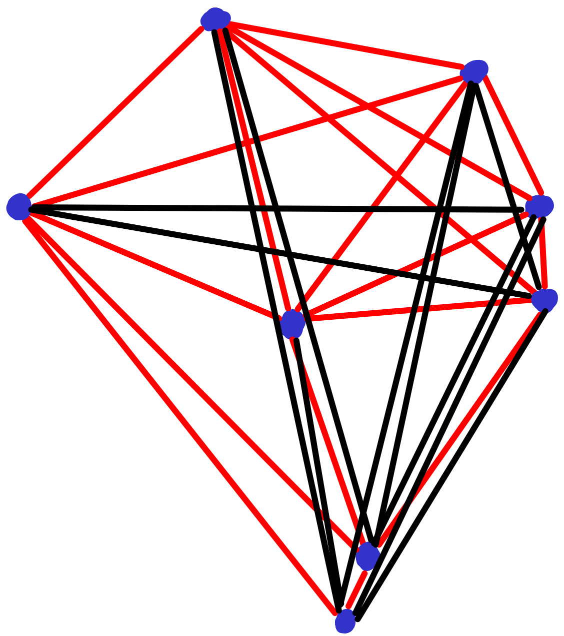


Form $k$ cones w/ angle $\theta = \frac{2\pi}{k}$

For every vertex $v$

shift cone structure onto $v$

& connect $v$ to 1 vertex in each cone (if non-empty)

Not so useful with $k \sim |E|$

But for $k = O(1)$ we keep

$\leq k \cdot V = O(V)$ edges

Connection rule:

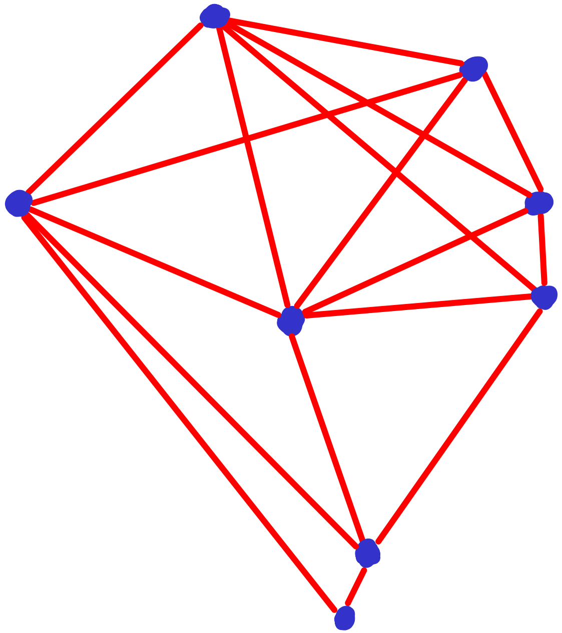
- project vertices onto cone bisector
- choose vertex with closest projection

$\Theta$-graph: good spanner? Depends on $\theta$.

connected for $\underbrace{\theta = 180°}_{\text{trivial}}$, $120°$?, $90°$?

Here we focus on $k \geqslant 9$ i.e $\theta \leqslant \frac{2\pi}{9}$

even though the illustration has $k=6$



$\Theta$-graph: good spanner? Depends on $\theta$.

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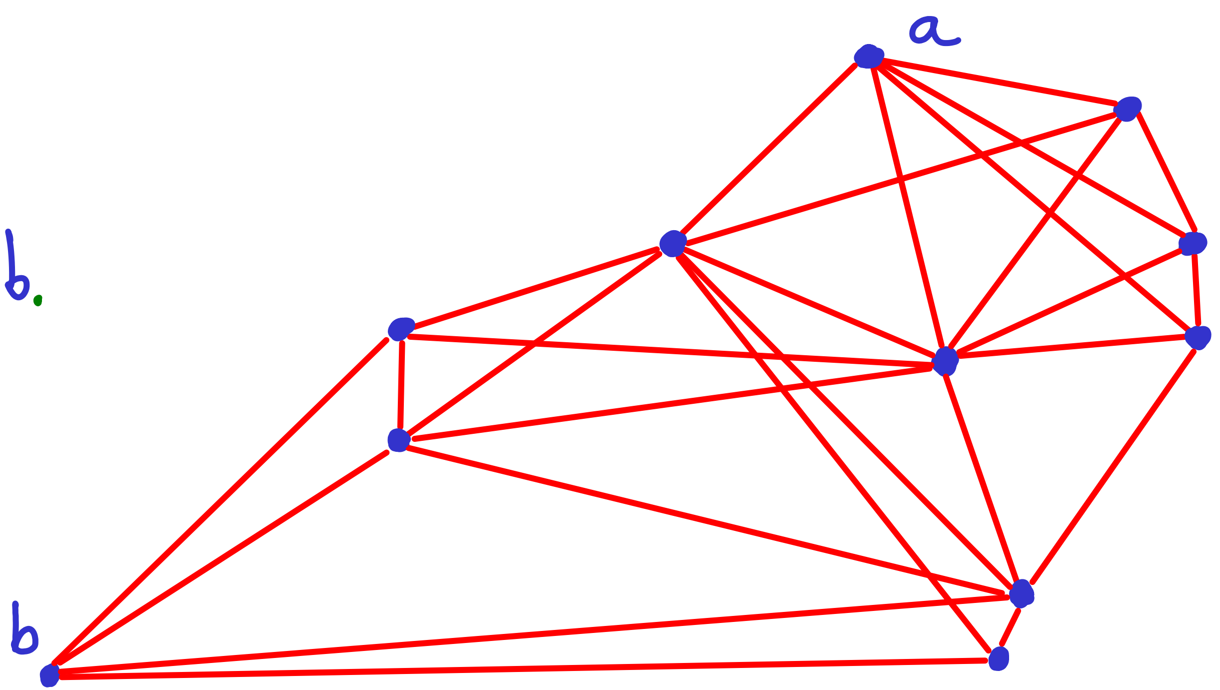
Now to define a sufficient path from any a to any b.



$\Theta$-graph: good spanner? Depends on $\theta$.

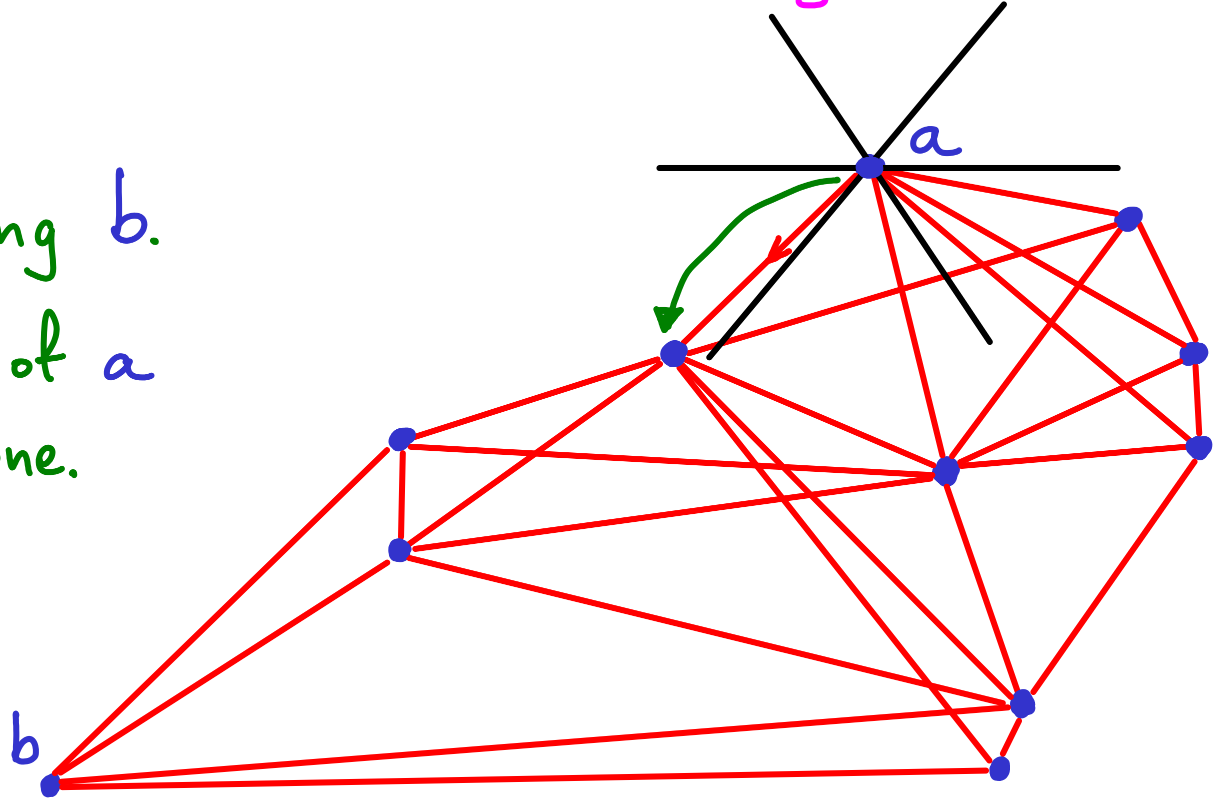
Here we focus on $k \geq 9$ i.e $\theta \leq \frac{2\pi}{9}$

even though the illustration has $k=6$

Starting at a

identify cone containing b.

Move to a neighbor of a in that cone.



$\Theta$-graph: good spanner? Depends on $\theta$.

Here we focus on $k \geqslant 9$ i.e $\theta \leqslant \frac{2\pi}{9}$

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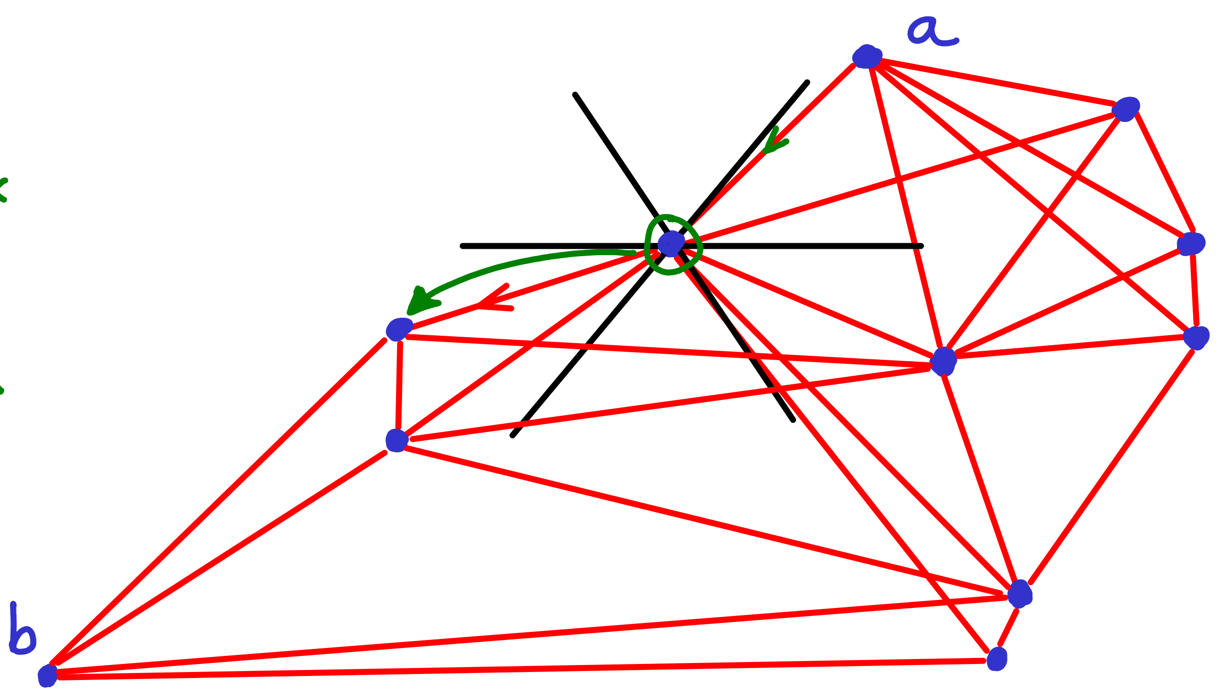
Repeat:

From current vertex move to a neighbor in cone containing b.

Any neighbor but why not the outgoing one



$\Theta$ - graph : good spanner ? Depends on $\theta$.

Here we focus on $k \geqslant 9$ i.e $\theta \leqslant \frac{2\pi}{9}$

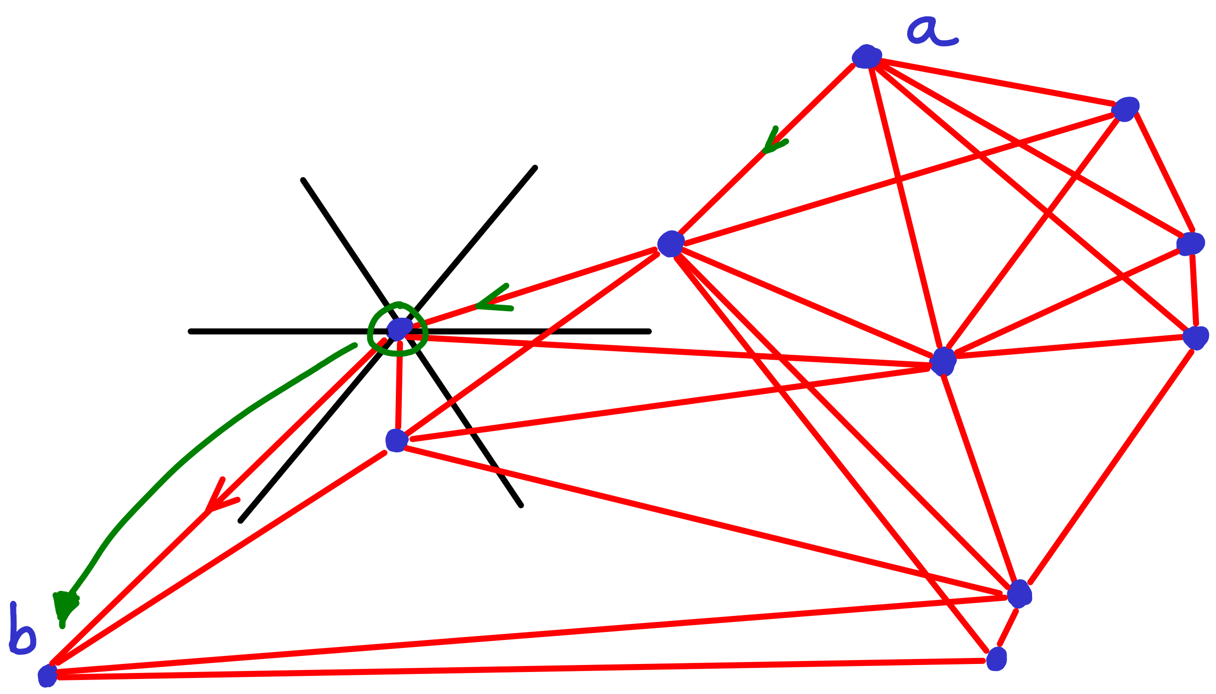
even though the illustration has $k=6$

Eventually (ideally) reach b.

$\downarrow$

can we cycle ?

let's see why not

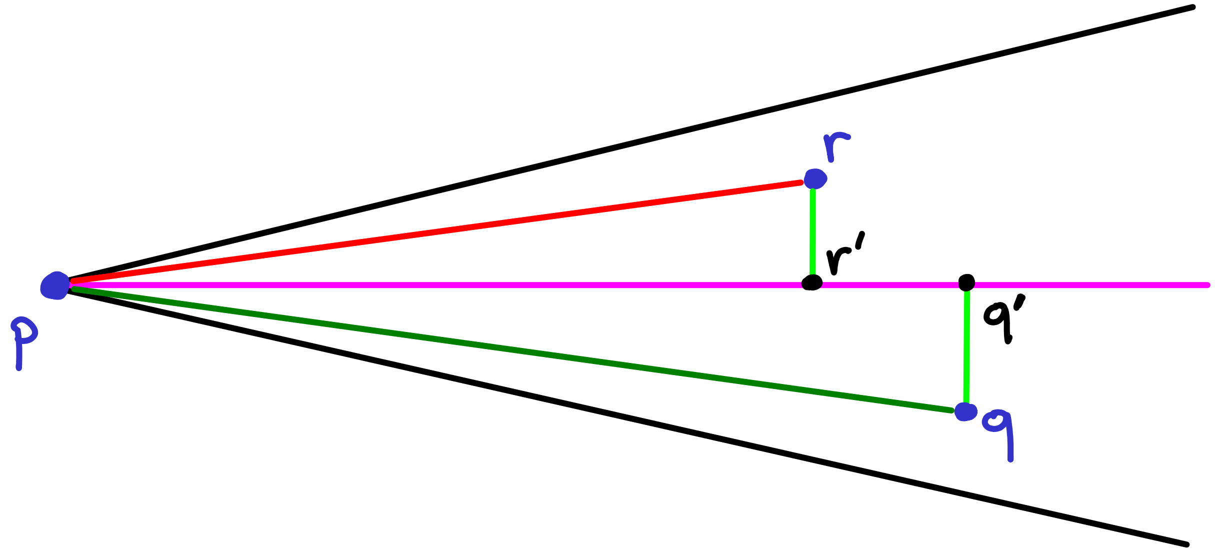


Given a cone at $p$: suppose $r$ projects no further than $q$

(i.e., $pq' \geq pr'$)

Claim: $pq \geq pr \cdot \cos\theta$

[note that pr can be greater than pq]



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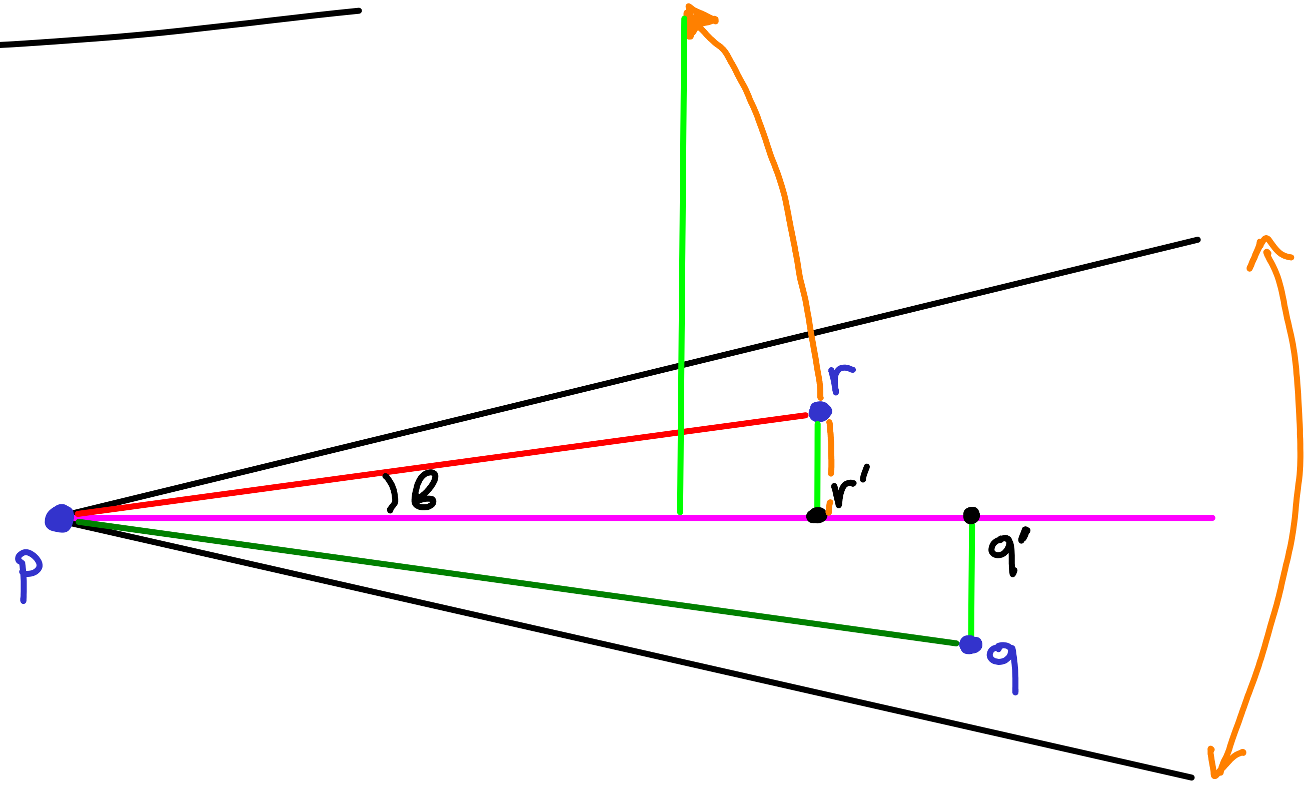
$pq \geqslant pq'$ // trivially

$\geqslant pr'$ // by assumption

$pr' = pr \cdot \cos\beta$

$\geqslant pr \cdot \cos\theta$



Given a cone w/ $\theta \leq \frac{2\pi}{8}$ at $p$: suppose $r$ projects no further than $q$

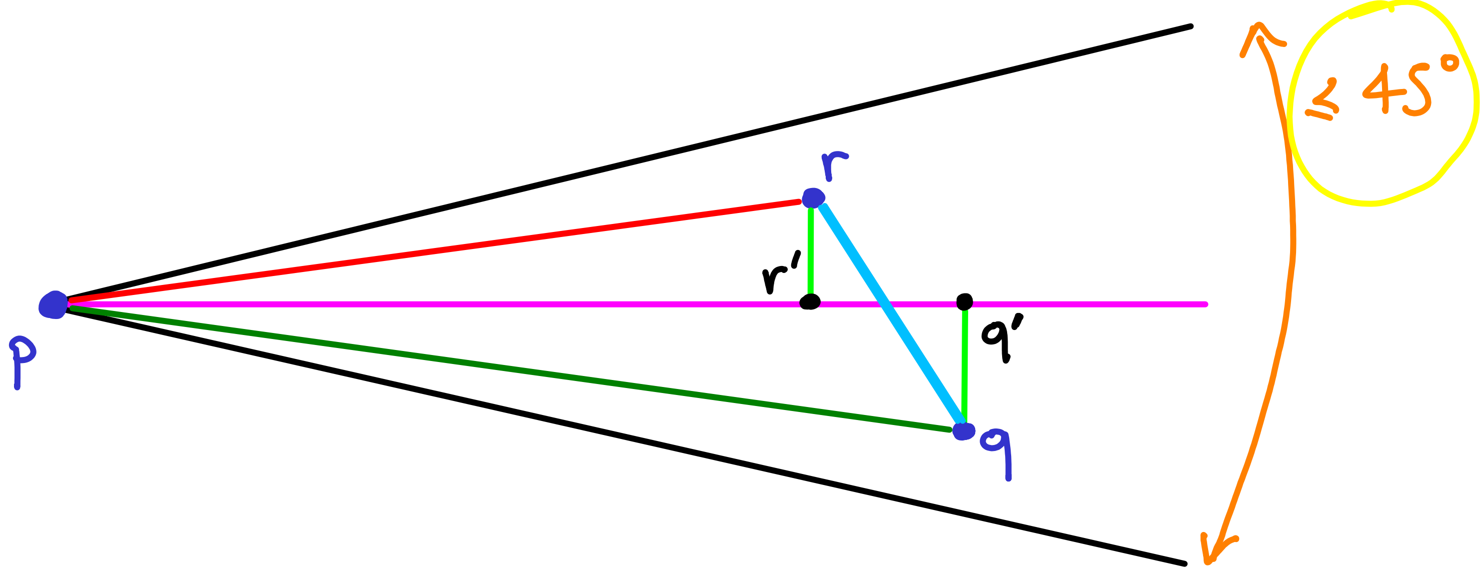
(i.e., $pq' \geq pr'$)

Proved : $pq \geq pr \cdot \cos\theta$

New claim : $rq \leq pq + (\sin\theta - \cos\theta) \cdot pr$

we are aiming for q
but we go to r
& we want to bound
the residual distance

If the residual gets
smaller, we won't cycle.



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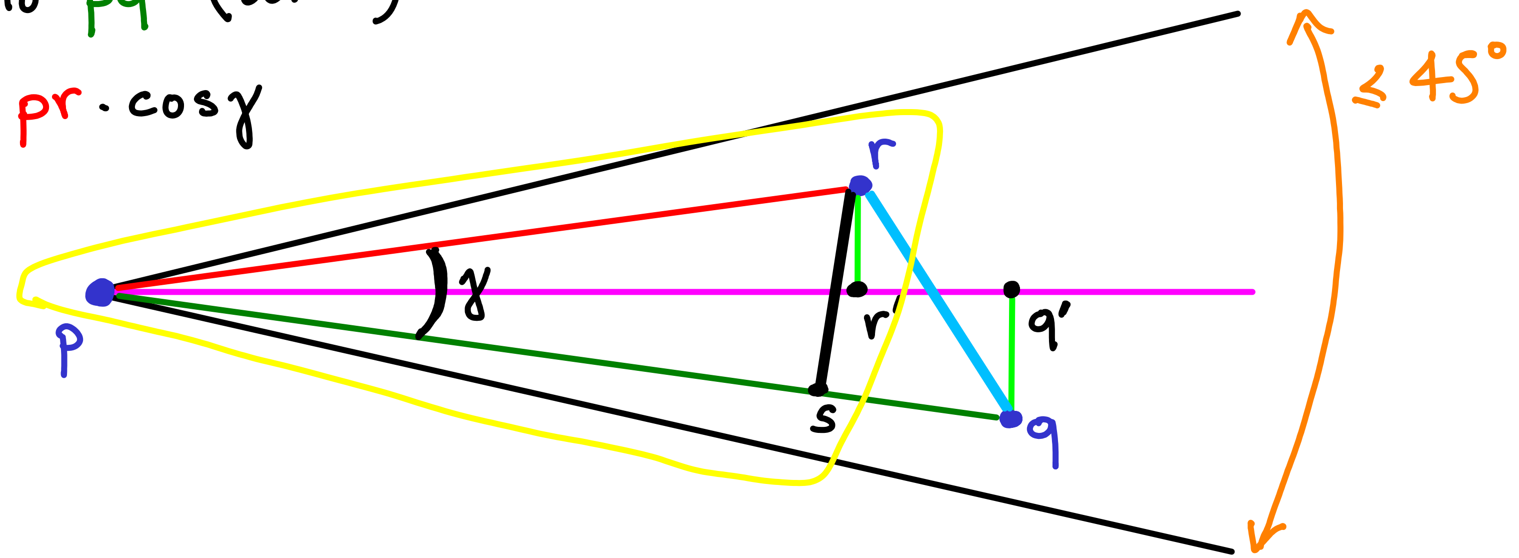
Suppose $r$ projects onto $pq$ (at $s$)

$rs = pr \cdot \sin\gamma$ $\qquad$ $ps = pr \cdot \cos\gamma$



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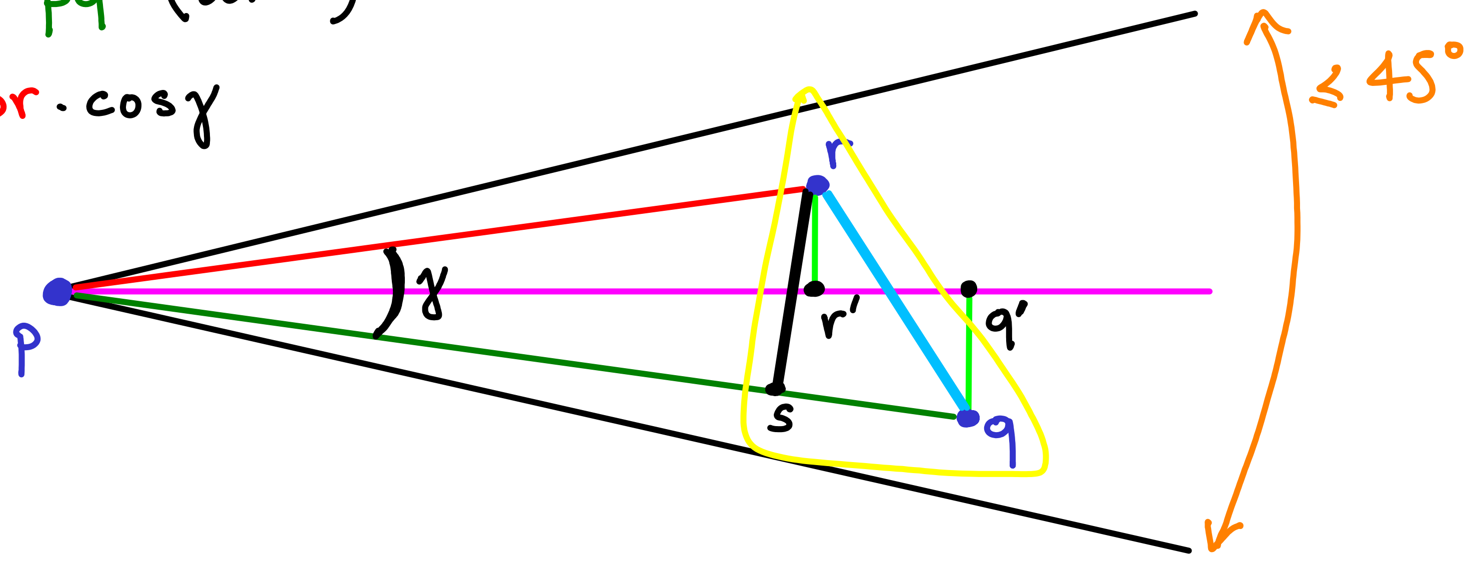
$rq \leq rs + sq$ // triangle



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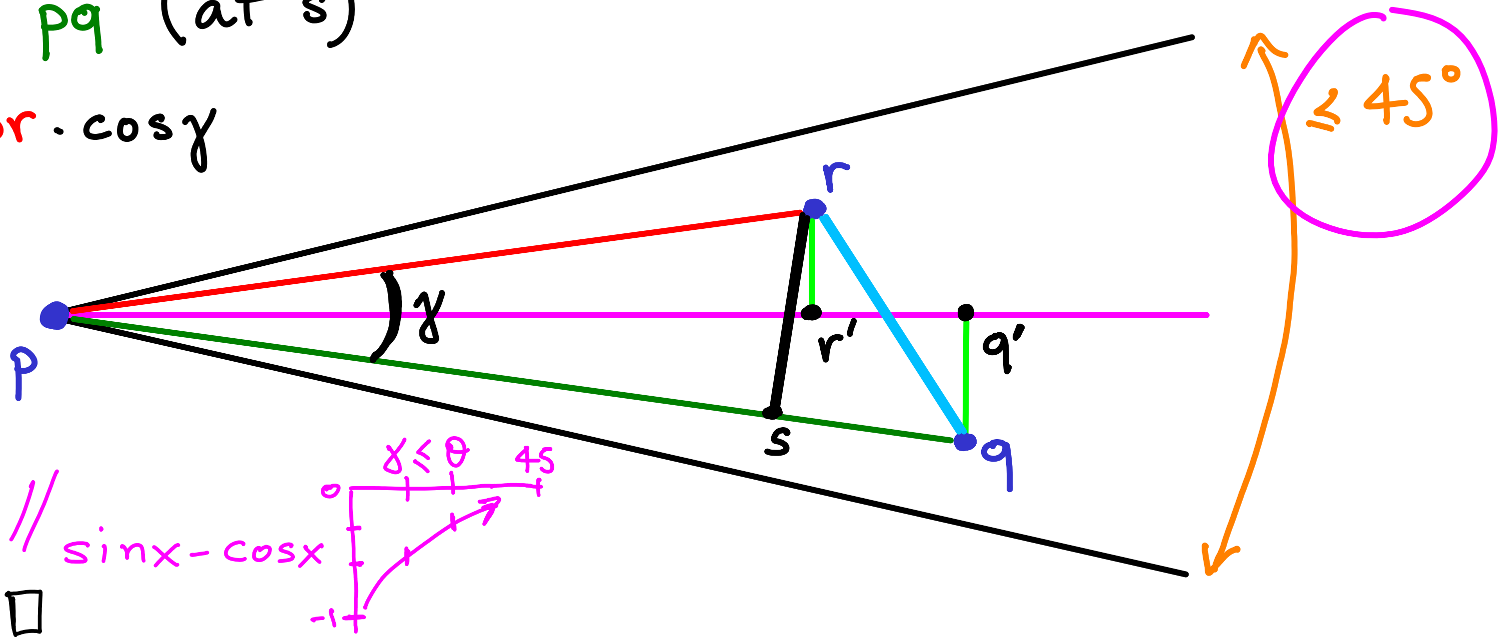
Suppose $r$ projects onto $pq$ (at $s$)

$rs = pr \cdot \sin\gamma$ $\quad\quad$ $ps = pr \cdot \cos\gamma$

$$\begin{aligned} rq &\leq rs + sq \\ &= rs + pq - ps \\ &= pr \cdot (\sin\gamma - \cos\gamma) + pq \\ &\leq pr \cdot (\sin\theta - \cos\theta) + pq \end{aligned}$$

$\square$



Given a cone w/ $\theta \leq \frac{2\pi}{8}$ at $p$: suppose $r$ projects no further than $q$

Proved : $pq \geq pr \cdot \cos\theta$ (i.e., $pq' \geq pr'$)

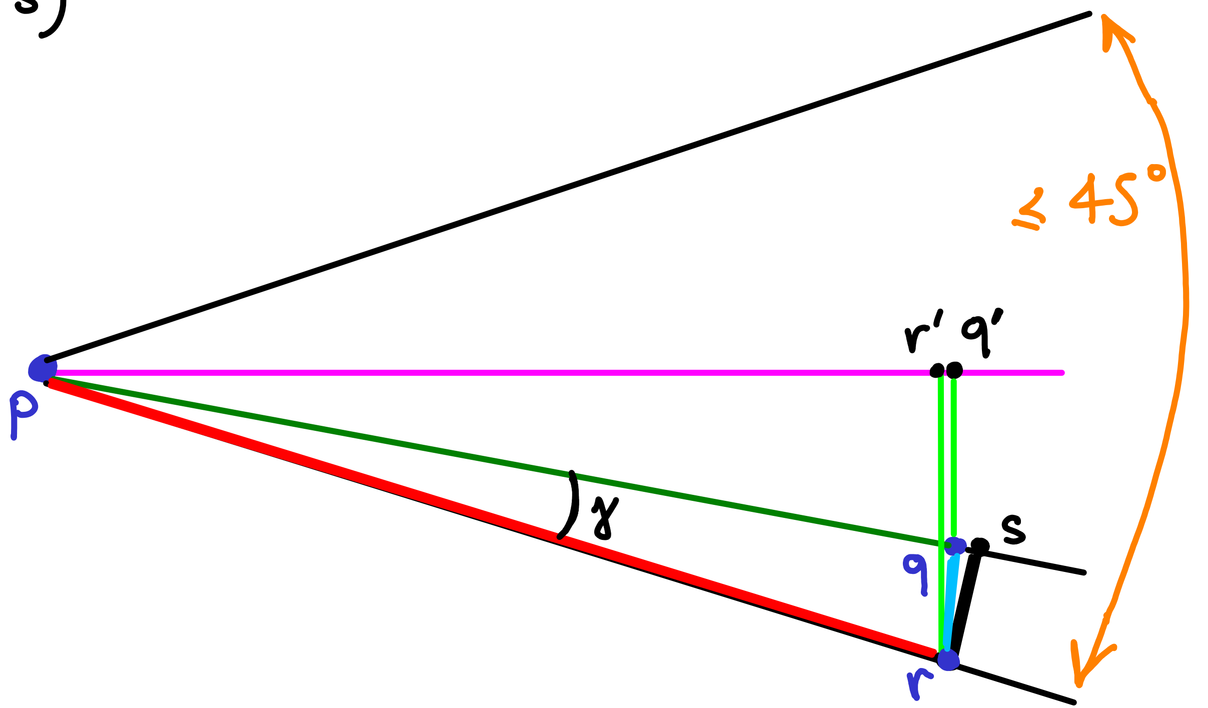
New claim : $rq \leq pq + (\sin\theta - \cos\theta) \cdot pr$

Suppose $r$ projects OFF $pq$ (at $s$)

$rs = pr \cdot \sin\gamma$ $ps = pr \cdot \cos\gamma$

$$\begin{aligned} rq &\leq rs + sq \\ &= rs + ps - pq \\ &= pr \cdot (\sin\gamma + \cos\gamma) - pq \end{aligned}$$

like before but signs changed



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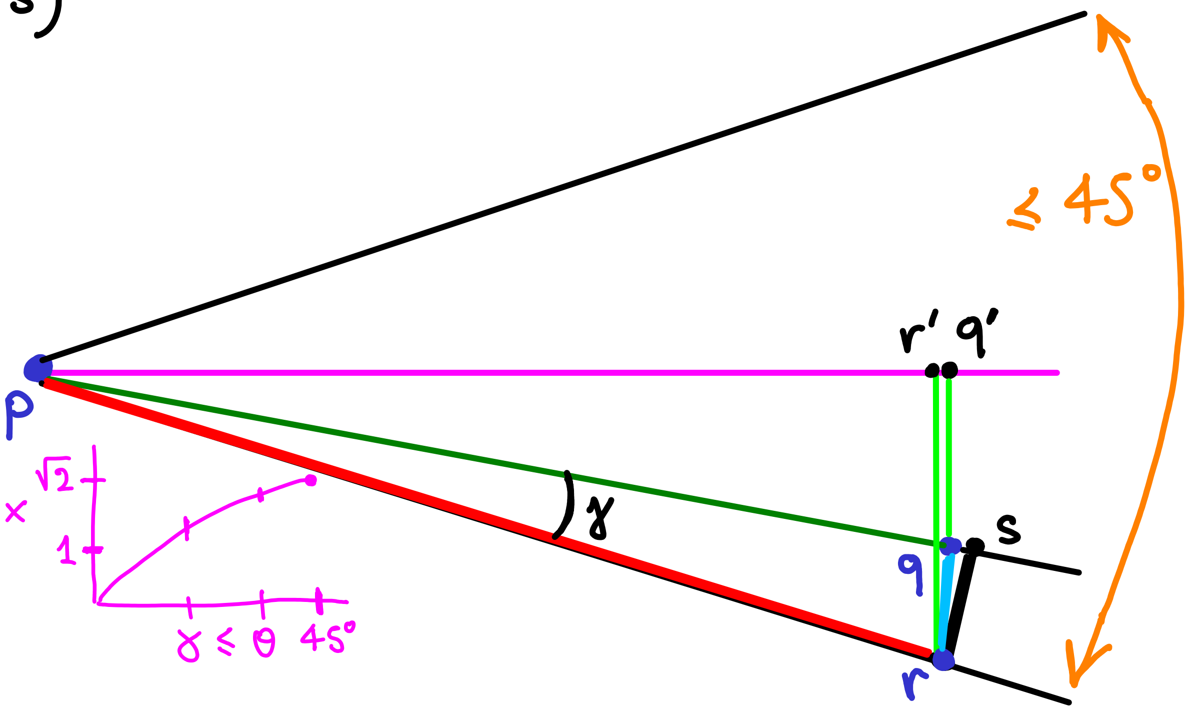
$$\begin{aligned} rq &\leq rs + sq \\ &= rs + ps - pq \\ &= pr \cdot (\sin\gamma + \cos\gamma) - pq \\ &\leq pr \cdot (\sin\theta + \cos\theta) - pq \quad // \sin x + \cos x \end{aligned}$$



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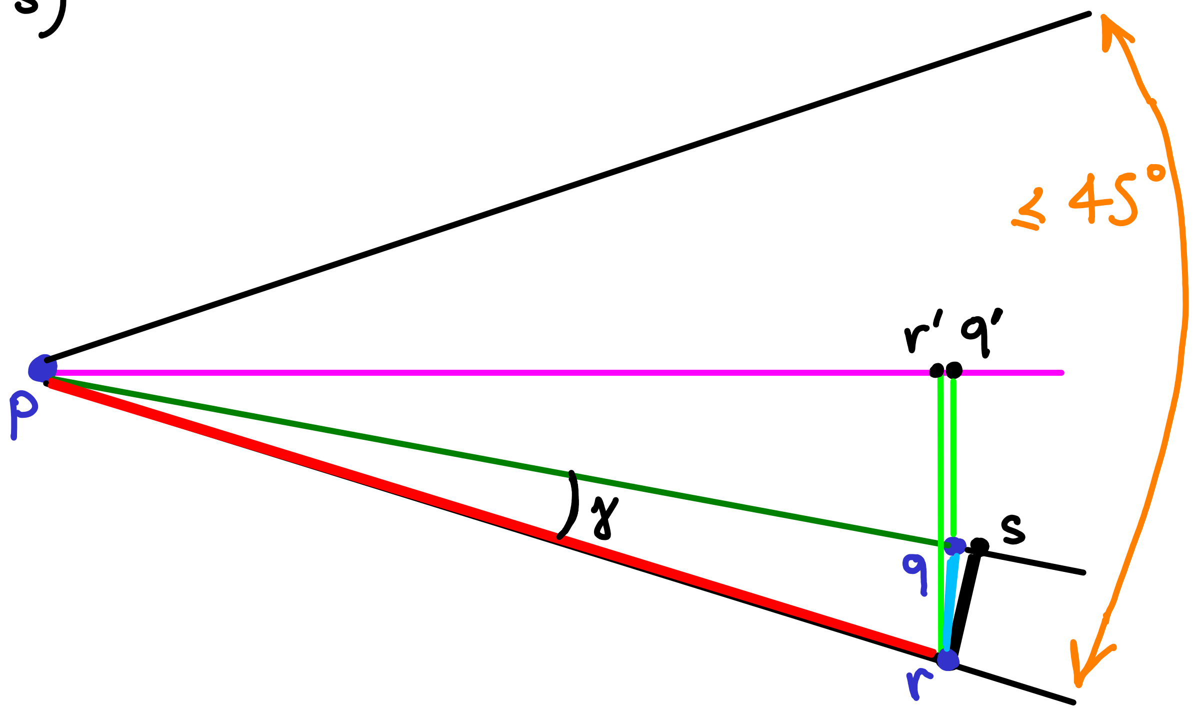
Suppose $r$ projects OFF $pq$ (at $s$)

$rs = pr \cdot \sin\gamma \qquad ps = pr \cdot \cos\gamma$

$$\begin{aligned} rq &\leq rs + sq \\ &= rs + ps - pq \\ &= pr \cdot (\sin\gamma + \cos\gamma) - pq \\ &\leq pr \cdot (\sin\theta + \cos\theta) - pq \\ &\leq pq + pr\sin\theta - pq \quad * \\ &\leq pq + pr\sin\theta - (pr\cos\theta) \quad * \qquad \square \end{aligned}$$



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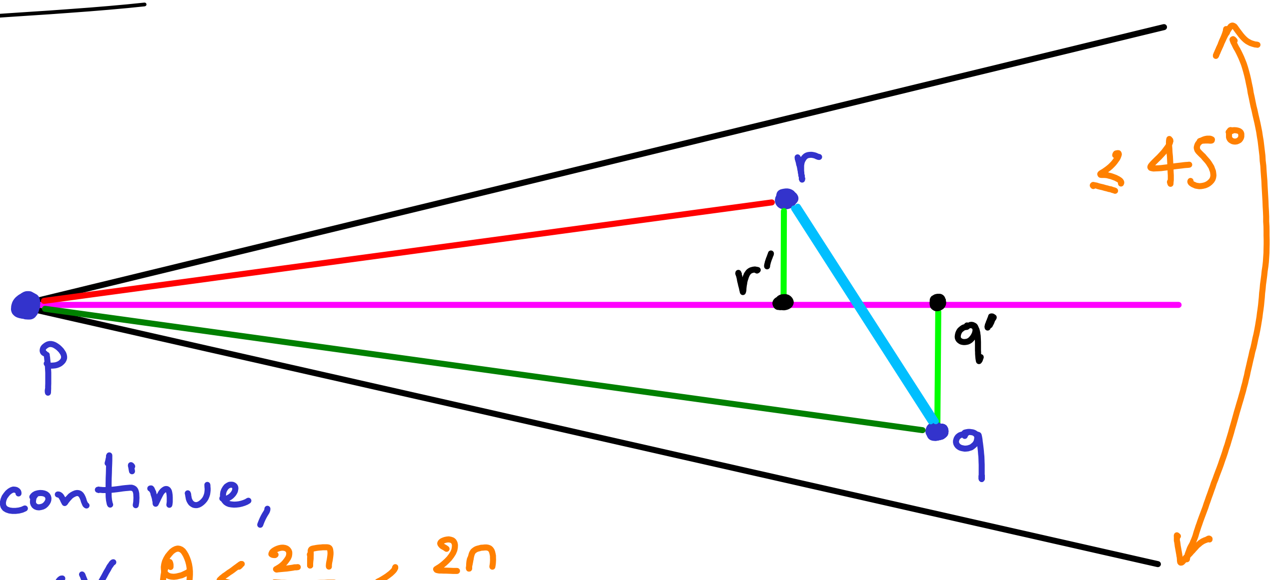
Proved: $rq \le pq + (\sin\theta - \cos\theta) \cdot pr$

Now, put it into context.




We are constructing a path from some vertex to q.

Let p be on that path. To continue, p looks for q in some cone, w/ $\theta \le \frac{2\pi}{9} < \frac{2\pi}{8}$

If p links to q, great. Otherwise it must link to some r.

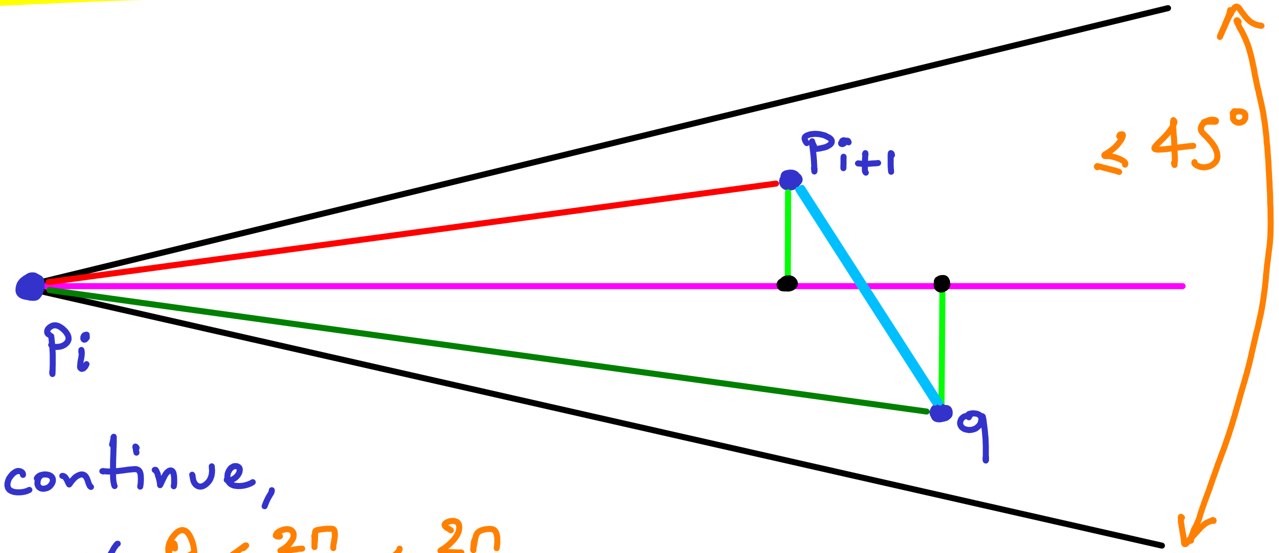
Our algorithm will move to r.

Given a cone w/ $\theta \leq \frac{2\pi}{8}$ at $p_i$: suppose $p_{i+1}$ projects no further than $q$

Proved : $p_{i+1}q \leq p_i q + (\sin\theta - \cos\theta) \cdot p_i p_{i+1}$

Now, put it into context.

We are constructing a path from some vertex to q.

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If p links to q, great. Otherwise it must link to some r.

Our algorithm will move to r. Label $p \rightarrow p_i$ & $r \rightarrow p_{i+1}$

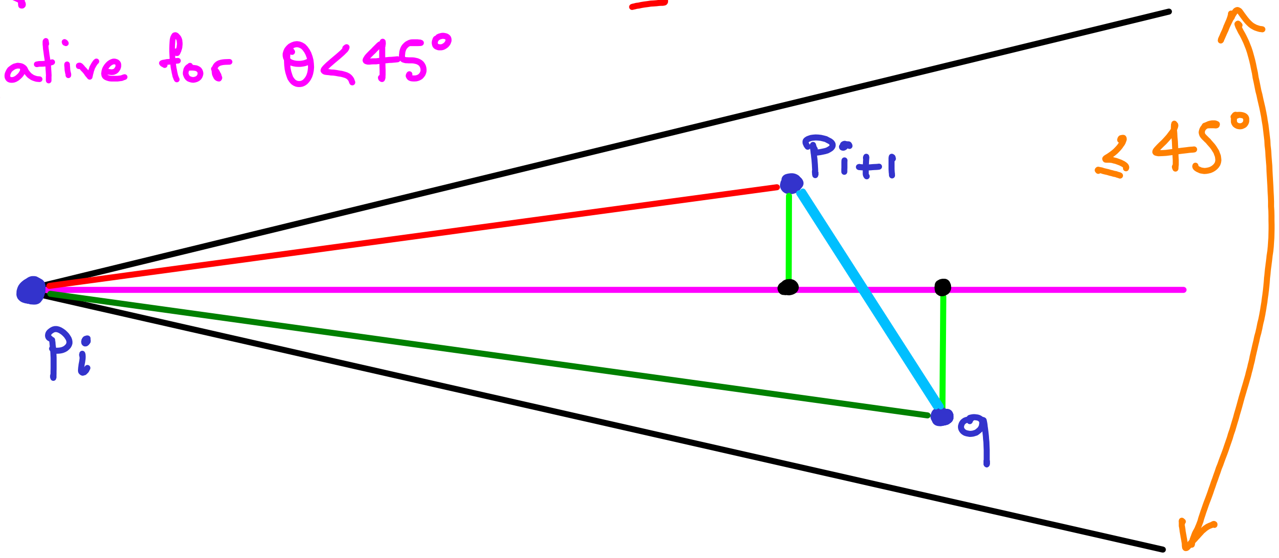
Given a cone w/ $\theta \leq \frac{2\pi}{8}$ at $p_i$: suppose $p_{i+1}$ projects no further than $q$

Proved : $\left[ p_{i+1}q \leq p_i q + \underbrace{(\sin\theta - \cos\theta)}_{\text{negative for } \theta < 45^\circ} \cdot p_i p_{i+1} < p_i q \right]$

$p_{i+1}q < p_i q \rightarrow$ no cycles

We have a spanner!

(we will reach q)



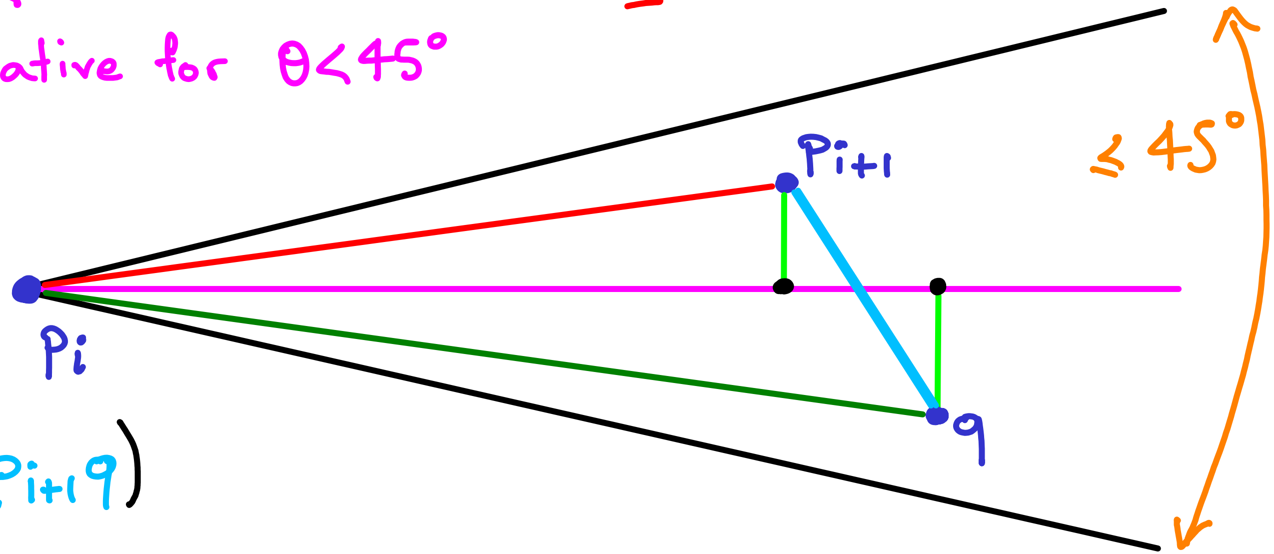

Derive upper bound on total path?

Given a cone w/ $\theta \leq \frac{2\pi}{8}$ at $p_i$: suppose $p_{i+1}$ projects no further than $q$

Proved : $\left[ p_{i+1}q \leq p_iq + \underbrace{(\sin\theta - \cos\theta)}_{\text{negative for } \theta < 45^\circ} \cdot p_ip_{i+1} < p_iq \right]$

$p_{i+1}q < p_iq \rightarrow$ no cycles

We have a spanner!




$$p_ip_{i+1} \leq \frac{1}{\cos\theta - \sin\theta}\left(p_iq - p_{i+1}q\right)$$

$$\text{Total path} = \sum_{i=0}^{m-1} p_ip_{i+1} \leq \frac{1}{\cos\theta - \sin\theta}\sum_{i=0}^{m-1}\left(p_iq - p_{i+1}q\right) = \frac{1}{\cos\theta - \sin\theta}\cdot\left(p_0q - \cancel{p_mq}\right)$$

$$= \frac{1}{\cos\theta - \sin\theta}\cdot aq$$

$\underset{p_0}{a} \rightarrow \underset{p_m}{q}$

# Summary

For $\theta < 45°$, i.e. w/ $k \geq 9$ cones,

a $\theta$-graph is a $t$-spanner, where $t \leq \dfrac{1}{\cos\theta - \sin\theta}$

Also, it uses $\leq k \cdot V$ edges

| $k$ | $\theta$ | $t$ |
|---|---|---|
| 9 | 40° | 8.1 |
| 12 | 30° | 2.7 |
| 24 | 15° | 1.4 |
| 120 | 3° | 1.06 |

Construction of $\theta$-graph
Getting bounded degree
: see links