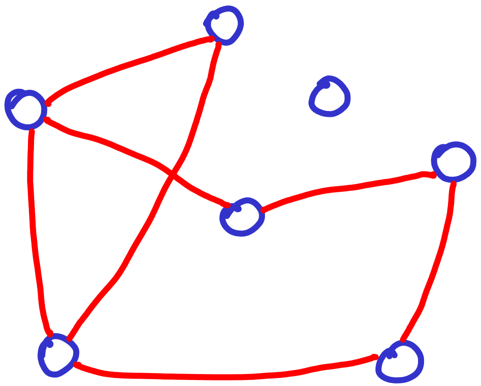
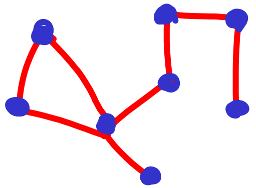
GRAPHS



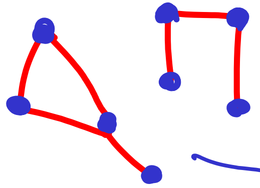
$G = \{V, E\}$

& vertices
edges

connected

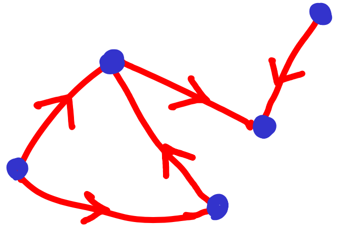


not connected

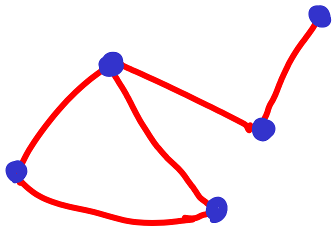


connected
components

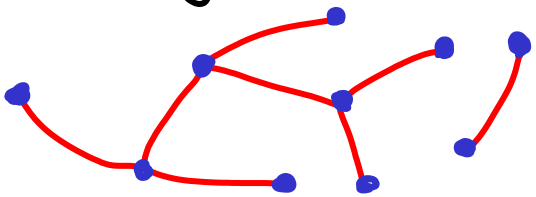
directed



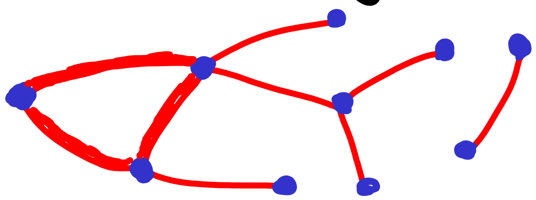
not directed

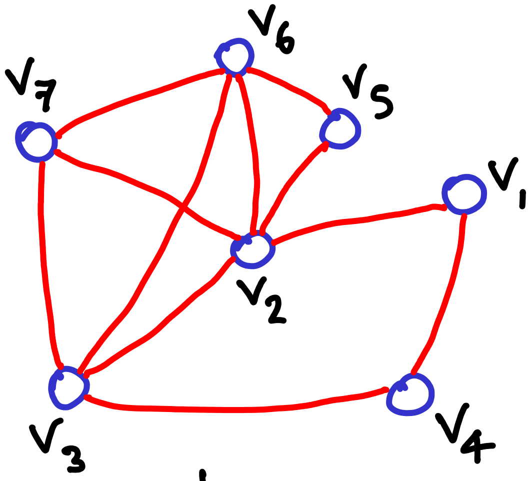


acyclic



not acyclic



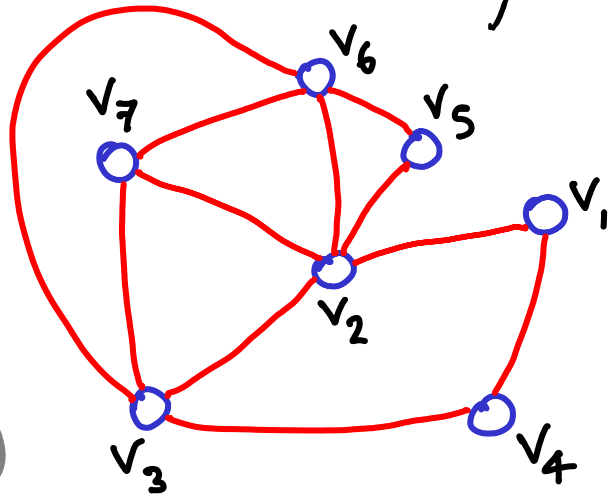


PLANAR GRAPH

...can redraw as a...

PLANE GRAPH

(no crossings)



$1 \rightarrow 2 \rightarrow 4$
$2 \rightarrow 1 \rightarrow 3 \rightarrow 7 \rightarrow 5$
$3 \rightarrow 2 \rightarrow 4 \rightarrow 7 \rightarrow 6$
$4 \rightarrow 1 \rightarrow 3$
$5 \rightarrow 2 \rightarrow 6$
$6 \rightarrow 2 \rightarrow 3 \rightarrow 7 \rightarrow 5$
$7 \rightarrow 2 \rightarrow 3 \rightarrow 6$

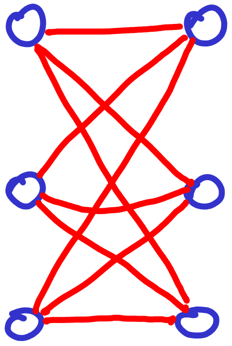
G "embedded" in the plane.

↓

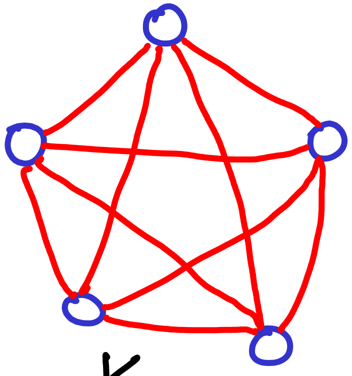
V & E have co-ordinates

Neighbors listed in cyclic order.

Non-planar graphs (can't redraw)



$K_{3,3}$



K_5

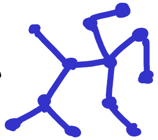
Euler formula $V - E + F = 2$

$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Euler formula $V-E+F=2$

$V-E+F=2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

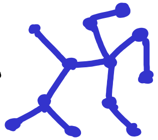
$F=1$:  : tree.

$E=V-1$

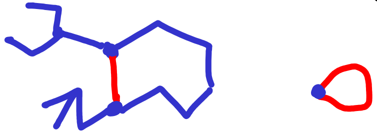
Euler formula $V - E + F = 2$

$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F = 1$:  : tree.
$E = V - 1$

$F > 1$:
Remove an edge between 2 faces.



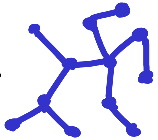
Remains connected.

$F \to F-1 \quad E \to E-1$

Euler formula $V - E + F = 2$

$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F = 1$:  : tree. $E = V - 1$

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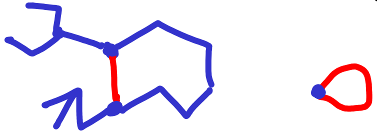
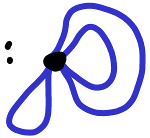


Remains connected.

$F \to F - 1$ $\quad$ $E \to E - 1$

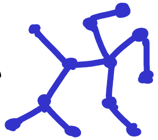
Induction on vertices

$V = 1$:  : only loops $F = E + 1$

Euler formula $V - E + F = 2$

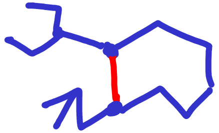
$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

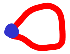
Induction on faces:

$F = 1$:  : tree. $E = V - 1$

$F > 1$:

Remove an edge between 2 faces.

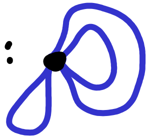




Remains connected.

$F \to F - 1 \qquad E \to E - 1$

Induction on vertices

$V = 1$:  : only loops $F = E + 1$

$V > 1$:

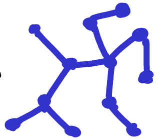
Contract edge $x \neq y$ 

$V \to V - 1 \qquad E \to E - 1$

Euler formula $V-E+F=2$

$V-E+F=2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F=1$:  : tree.
$E=V-1$

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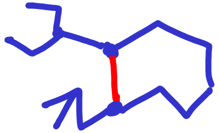
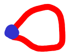
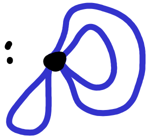
 

Remains connected.

$F\to F-1$ $\quad E\to E-1$

Induction on vertices

$V=1$:  : only loops
$F=E+1$

$V>1$:
Contract edge $x\neq y$ 

$V\to V-1$ $\quad E\to E-1$

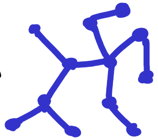
Induction on edges

$E=0$:  : one vertex
one face

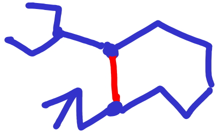
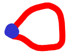
Euler formula $V - E + F = 2$

$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F = 1$:  : tree. $E = V - 1$

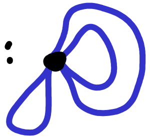
$F > 1$:
Remove an edge between 2 faces.

Remains connected.

$F \to F-1$ $\quad$ $E \to E-1$

Induction on vertices

$V = 1$:  : only loops $F = E + 1$

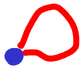
$V > 1$:
Contract edge $x \neq y$ 

$V \to V-1$ $\quad$ $E \to E-1$

Induction on edges

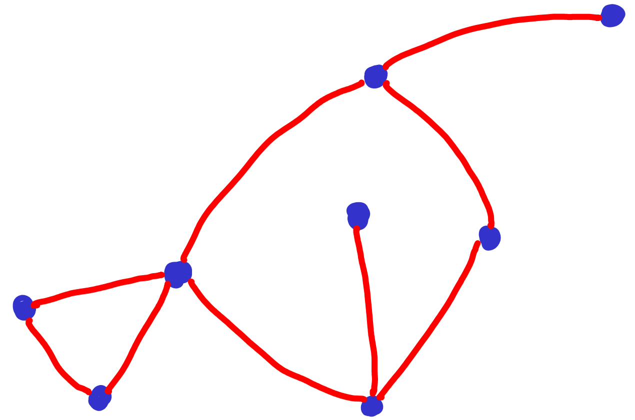
$E = 0$:  : one vertex one face

$E \geq 1$: if $x \neq y$  contract as before

else  remove as before

$E \to E-1$ & $F \to F-1$ OR $V \to V-1$

use the Euler formula $V - E + F = 2$
to show that a connected plane graph has $E \leq 3V - 6$



use the Euler formula $V - E + F = 2$

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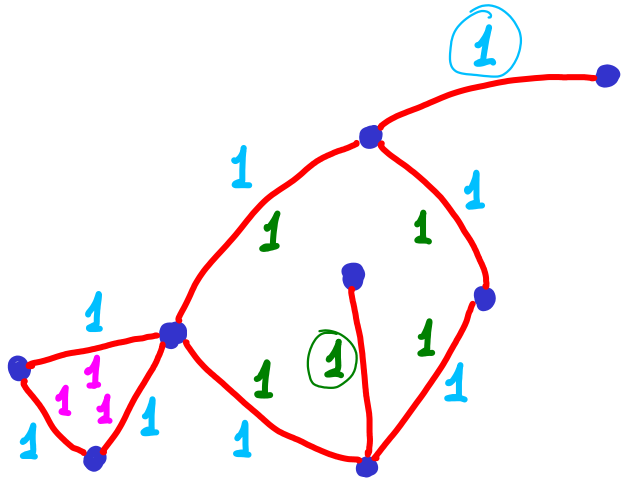
Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$



use the Euler formula $V - E + F = 2$

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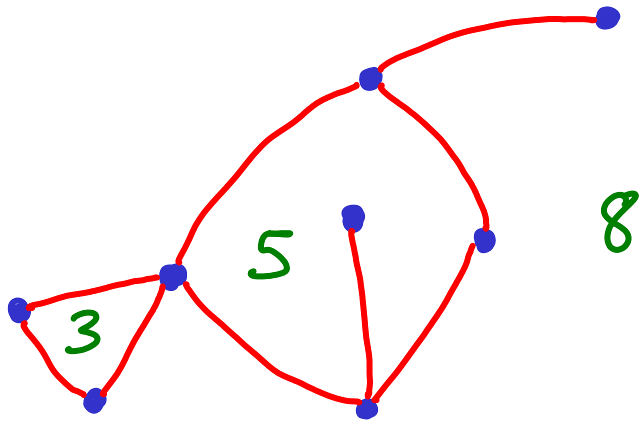
Every edge belongs to 1 or 2 faces $\displaystyle\sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V > 3$) $\displaystyle\sum_{\text{all faces}} e \geq 3F$



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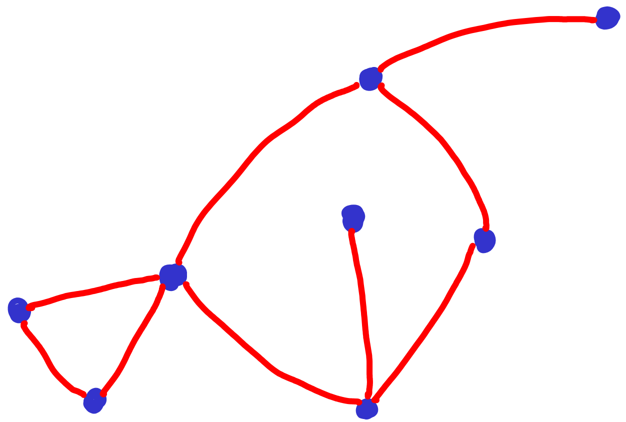
Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V>3$) $\sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

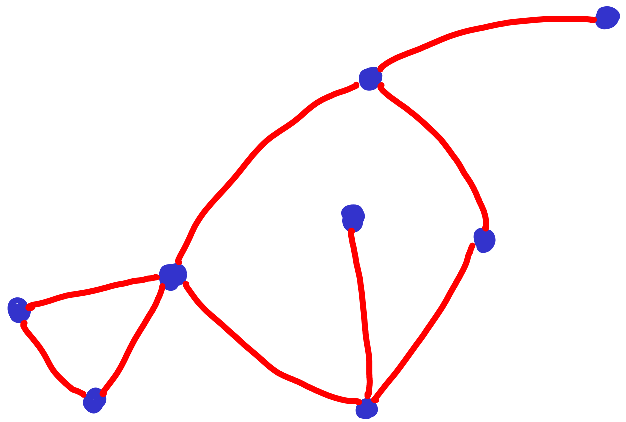


use the Euler formula $V - E + F = 2$

to show that a connected plane graph has $E \leq 3V - 6$

$$\left.\begin{array}{ll} \text{Every edge belongs to 1 or 2 faces} & \displaystyle\sum_{\text{all faces}} e \leq 2E \\ \text{Every face has } \geq 3 \text{ edges (for } V > 3) & \displaystyle\sum_{\text{all faces}} e \geq 3F \end{array}\right\} \quad 2E \geq 3F$$

$E - F = V - 2$



use the Euler formula $V - E + F = 2$

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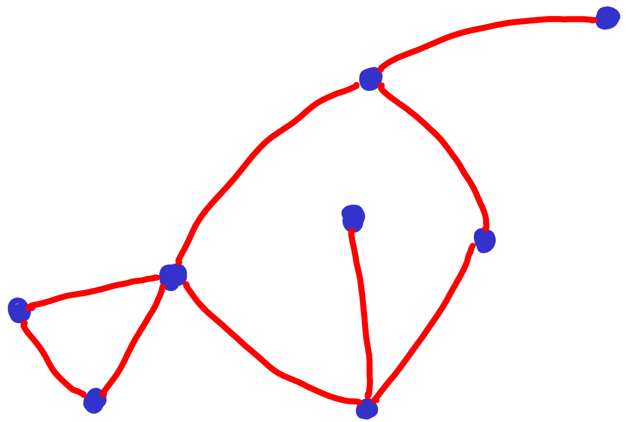
Every face has ≥ 3 edges (for $V > 3$) $\displaystyle\sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

$E - F = V - 2$

$E - \frac{2E}{3} \leq V - 2$

$E \leq 3V - 6$



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Every edge belongs to 1 or 2 faces $\quad \sum_{\text{all faces}} e \leq 2E$

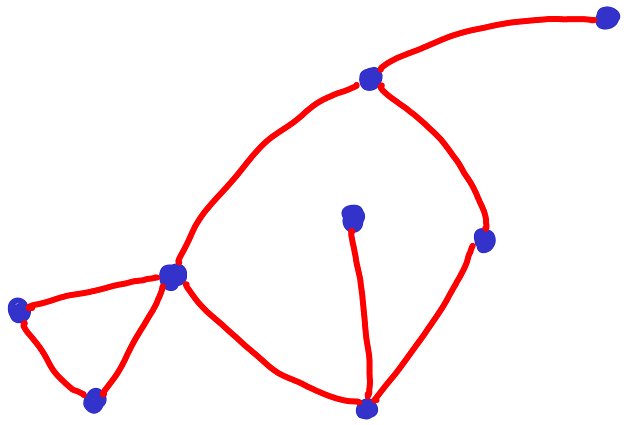
Every face has ≥ 3 edges (for $V > 3$) $\quad \sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

$V \leq 2$?

$$E - F = V - 2$$
$$E - \frac{2E}{3} \leq V - 2$$
$$E \leq 3V - 6$$

$$E - F = V - 2$$
$$\frac{3F}{2} - F \leq V - 2$$
$$F \leq 2V - 4$$



$E \leq 3V - 6$

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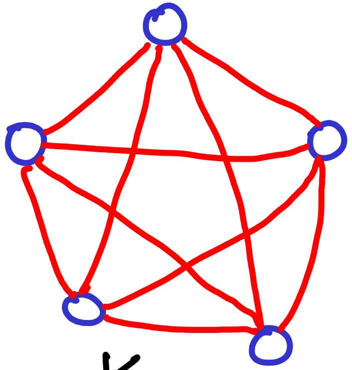
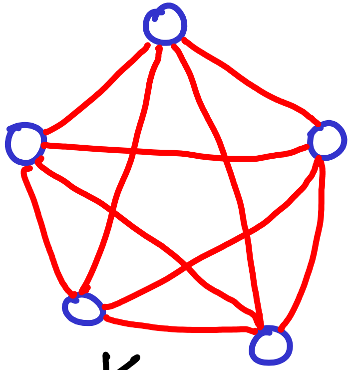


K_5

$E \leq 3V - 6$

$10 \leq 15 - 6$

!!!



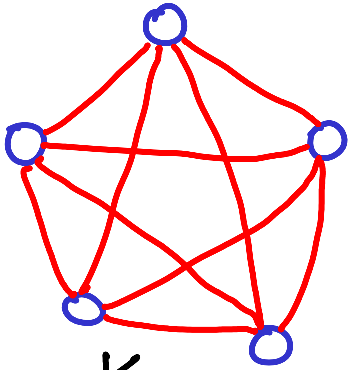
K_5

Not planar

$E \leq 3V - 6$

$10 \leq 15 - 6$

!!!



K_5

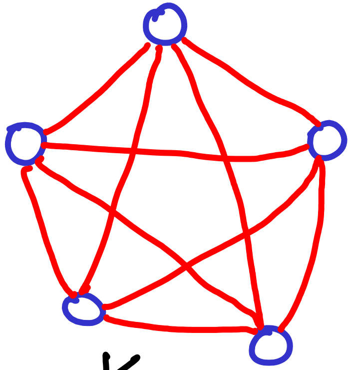
...OR...

$$\left[\begin{array}{ll} V - E + F = 2 & 2E \geq 3F \\ 5 - 10 + F = 2 & 2 \cdot 10 \not\geq 3 \cdot 7 \\ F = 7 & !!! \end{array}\right]$$

$E \leq 3V-6$

$10 \leq 15-6$

!!!

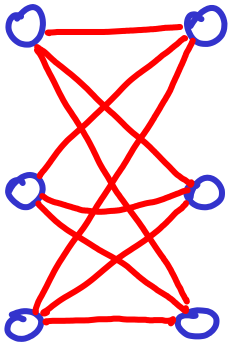


K_5

...OR...

$$\left[\begin{array}{ll} V-E+F=2 & 2E \geq 3F \\ 5-10+F=2 & 2\cdot 10 \geq 3\cdot 7 \\ F=7 & !!! \end{array}\right]$$

$E \leq 3V-6$

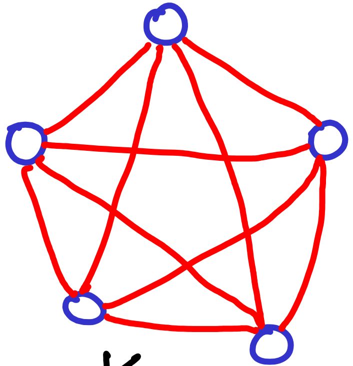


$K_{3,3}$

$E \leq 3V - 6$

$10 \leq 15 - 6$

!!!



K_5

...OR...

$V - E + F = 2$

$5 - 10 + F = 2$

$F = 7$

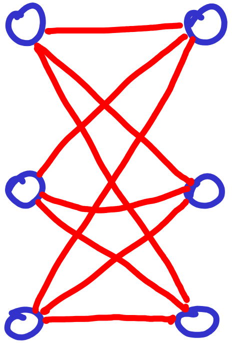
$2E \geqslant 3F$

$2 \cdot 10 \geqslant 3 \cdot 7$

!!!

$E \leq 3V - 6$

$9 \leq 18 - 6$ ok!

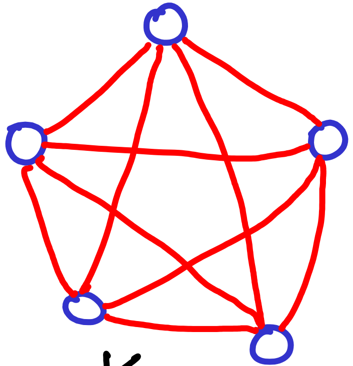


$K_{3,3}$

$E \leq 3V - 6$

$10 \leq 15 - 6$

!!!



...OR...

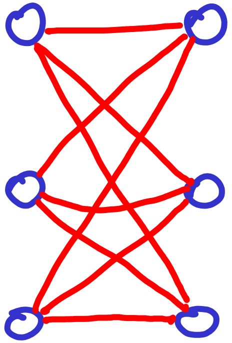
$V - E + F = 2$ $2E \geq 3F$

$5 - 10 + F = 2$ $2 \cdot 10 \geq 3 \cdot 7$

$F = 7$!!!

$E \leq 3V - 6$

$9 \leq 18 - 6$ OK!



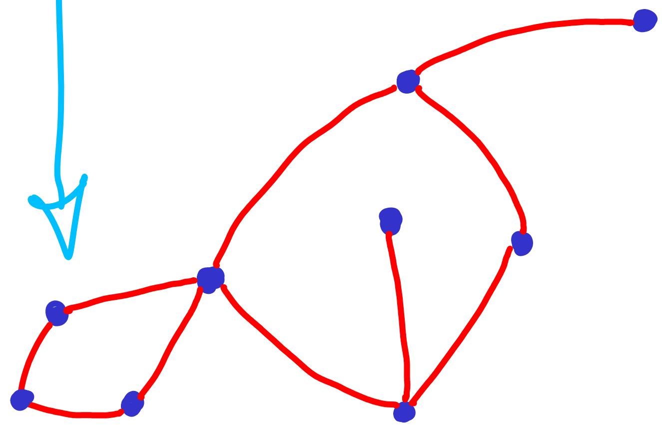
not iff

All planar graphs have $E \leq 3V-6$

Some non-planar graphs can too

$V - E + F = 2$

What if G has no triangles?



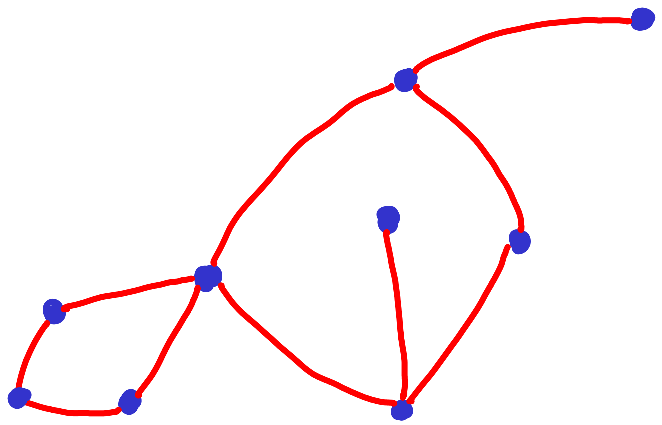
$V - E + F = 2$

What if G has no triangles?

Every edge belongs to 1 or 2 faces $\qquad \sum_{\text{all faces}} e \le 2E$

Every face has $\cancel{\geq 3}$ ≥ 4 edges (for $V > 4$) $\qquad \sum_{\text{all faces}} e \geq 4F$

$$\left.\begin{array}{l} \sum_{\text{all faces}} e \le 2E \\ \sum_{\text{all faces}} e \geq 4F \end{array}\right\} \quad E \geq 2F$$



$V - E + F = 2$

What if G has no triangles?

Every edge belongs to 1 or 2 faces $\quad \sum_{\text{all faces}} e \le 2E$

Every face has $\cancel{\geq 3}$ ≥ 4 edges (for $V > 4$) $\quad \sum_{\text{all faces}} e \geq 4F$

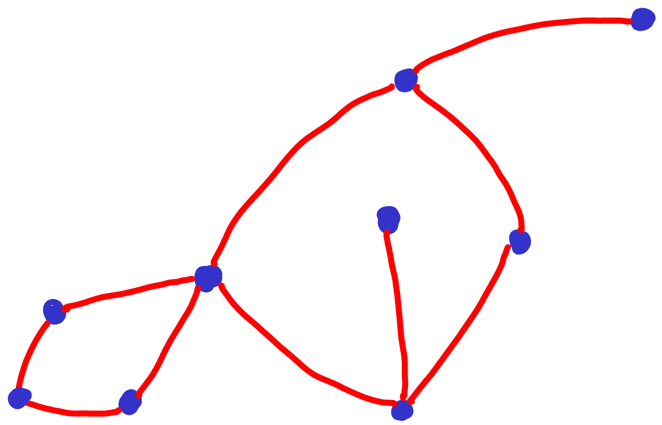
$E \geq 2F$

$E - F = V - 2$

$E - \frac{E}{2} \le V - 2$

$E \le 2V - 4$

instead of $\le 3V - 6$



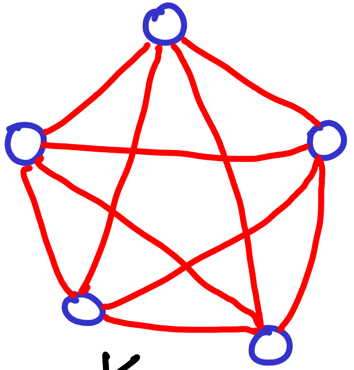
$E \leq 3V-6$

$10 \leq 15-6$

!!!

...OR...

$$\left[\begin{array}{ll} V-E+F=2 & 2E \geq 3F \\ 5-10+F=2 & 2\cdot 10 \geq 3\cdot 7 \\ F=7 & !!! \end{array}\right]$$



NOT PLANAR

for triangle free: $E \leq 2V-4$

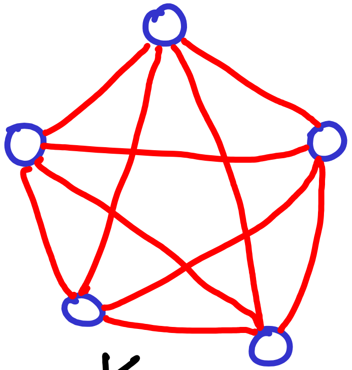
$E \leq 3V - 6$

$10 \leq 15 - 6$

!!!

...OR...

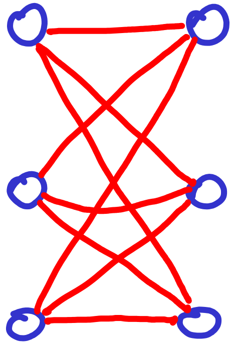
$$\left[\begin{array}{ll} V - E + F = 2 & 2E \geq 3F \\ 5 - 10 + F = 2 & 2 \cdot 10 \geq 3 \cdot 7 \\ F = 7 & !!! \end{array}\right]$$



K_5

NOT PLANAR

for triangle free: $E \leq 2V - 4$



$K_{3,3}$

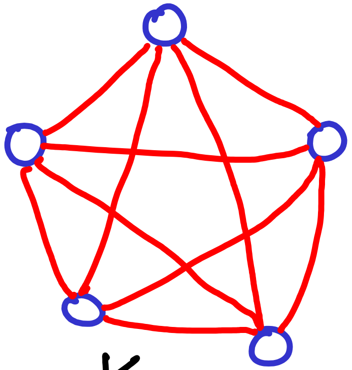
$E \leq 3V-6$

$10 \leq 15-6$

!!!

...OR...

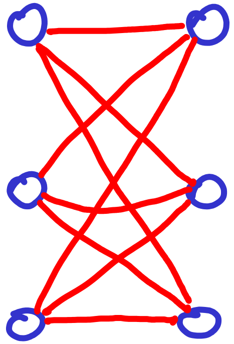
$$\left[\begin{array}{ll} V-E+F=2 & 2E \geqslant 3F \\ 5-10+F=2 & 2\cdot 10 \not\geqslant 3\cdot 7 \\ F=7 & !!! \end{array}\right]$$



K_5

NOT PLANAR

$V=6,\ E=9$

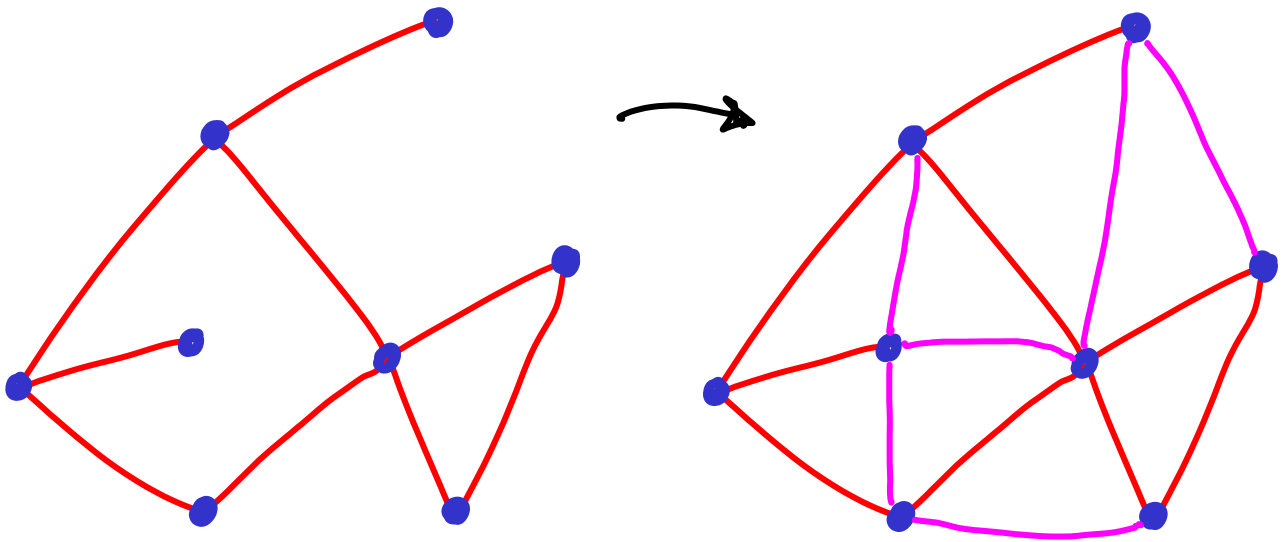


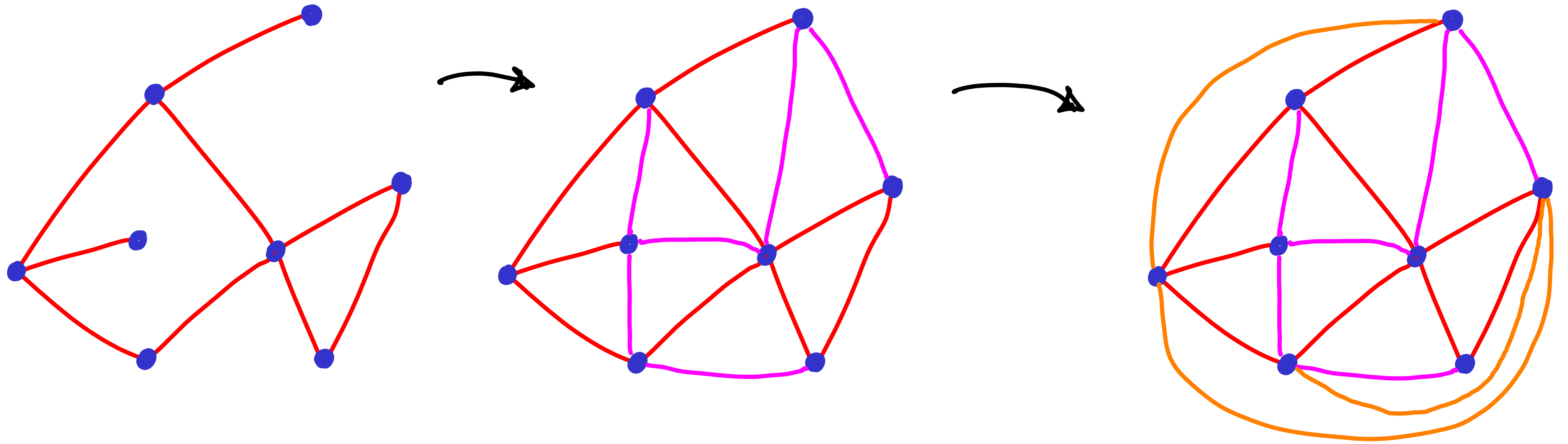
$K_{3,3}$

for triangle free: $E \leq 2V-4$

$9 \leq 2\cdot 6-4$

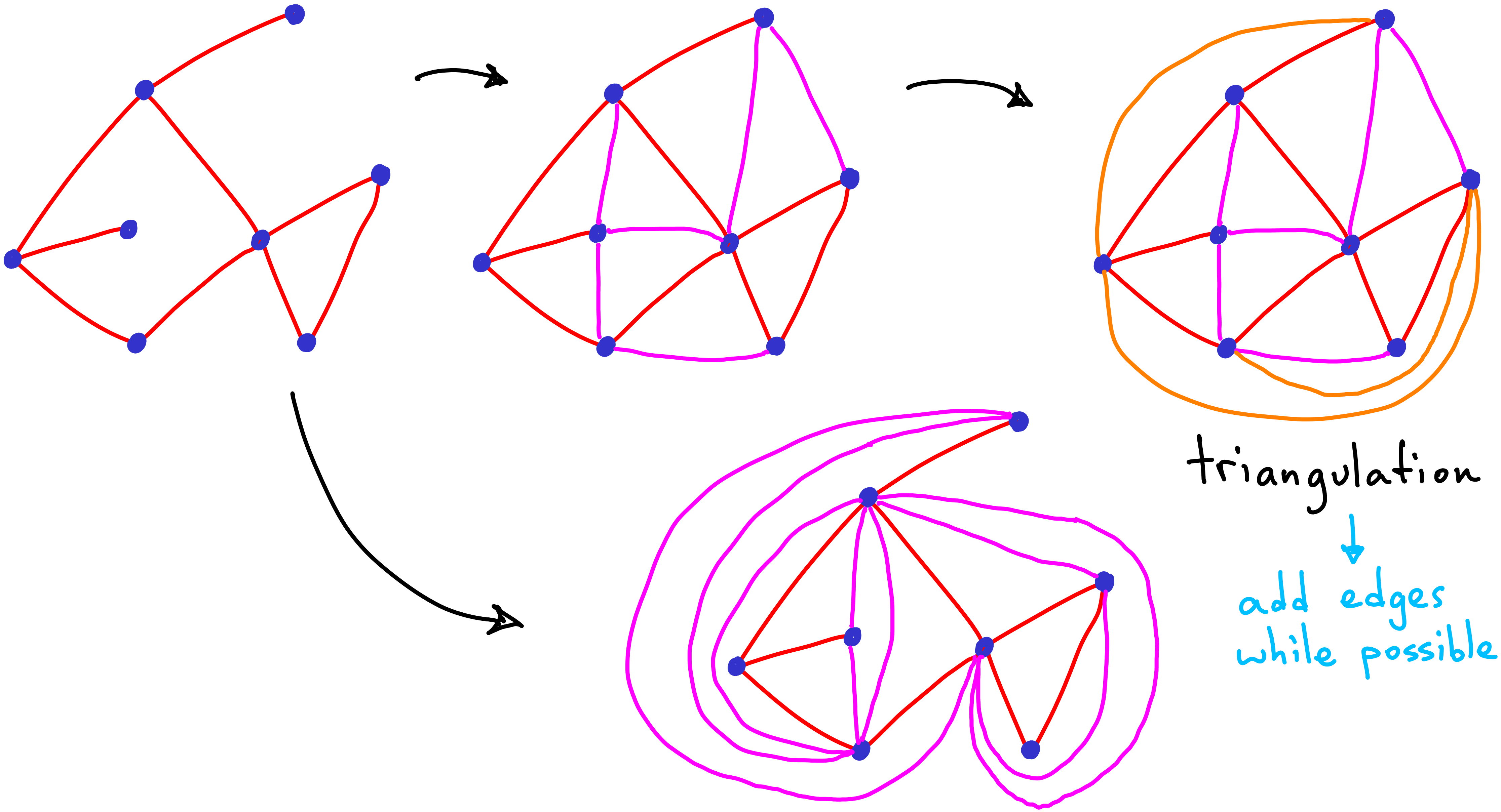
!!!



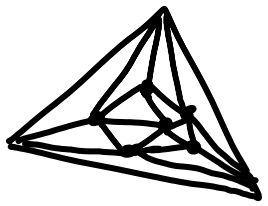


triangulation

add edges while possible


triangulation
add edges
while possible

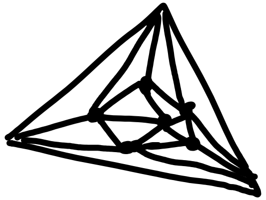
$$e = 3n - 6$$



Why?

$e = 3n - 6$

Why?



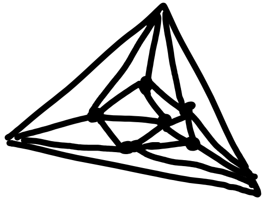
$V - E + F = 2$

previous statement:

$$\left.\begin{array}{ll} \text{Every edge belongs to 1 or 2 faces} & \displaystyle\sum_{\text{all faces}} e \leq 2E \\ \text{Every face has} \geq 3 \text{ edges (for } V > 3) & \displaystyle\sum_{\text{all faces}} e \geq 3F \end{array}\right\} 2E \geq 3F$$

$\underline{\underline{e = 3n - 6}}$

Why?



$V - E + F = 2$

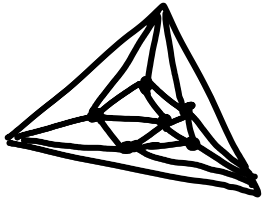
Every edge belongs to ~~1 or~~ 2 faces $\qquad \sum_{\text{all faces}} e \overset{=}{\cancel{\leq}} 2E$

Every face has $\overset{=}{\cancel{\geq}} 3$ edges (for $V > 3$) $\qquad \sum_{\text{all faces}} e \overset{=}{\cancel{\geq}} 3F$

$2E \overset{=}{\cancel{\geq}} 3F$

$\underline{\underline{e = 3n - 6}}$

Why?



$V - E + F = 2$

Every edge belongs to ~~1 or~~ 2 faces $\quad \sum_{\text{all faces}} e \overset{=}{\cancel{\leq}} 2E$

Every face has $\overset{=}{\cancel{\geq}}$ 3 edges (for $V > 3$) $\quad \sum_{\text{all faces}} e \overset{=}{\cancel{\geq}} 3F$

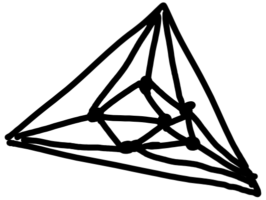
$$2E \overset{=}{\cancel{\geq}} 3F$$

$$E - F = V - 2$$

$$E - \frac{2E}{3} = V - 2$$

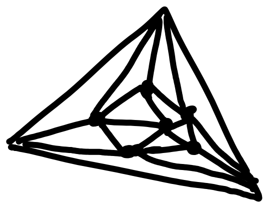
$$E = 3V - 6$$

$e = 3n - 6$



What is the average degree of a triangulation?

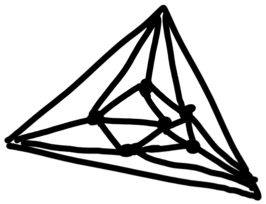
$e = 3n - 6$



What is the average degree of a triangulation?

$$\frac{1}{n} \cdot \sum_{i=1}^{n} d(v_i)$$

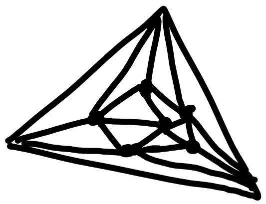
$e = 3n - 6$



What is the average degree of a triangulation?

$$\frac{1}{n} \cdot \sum_{i=1}^{n} d(v_i) = \frac{1}{n} \cdot 2e$$

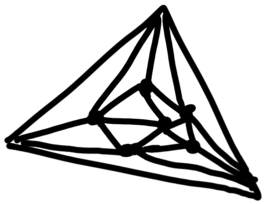
$e = 3n - 6$



What is the average degree of a triangulation?

$$\frac{1}{n} \cdot \sum_{i=1}^{n} d(v_i) = \frac{1}{n} \cdot 2e = \frac{6n-12}{n} < 6$$

$\underline{\underline{e = 3n - 6}}$



What is the average degree of a triangulation?

$$\frac{1}{n} \cdot \sum_{i=1}^{n} d(v_i) = \frac{1}{n} \cdot 2e = \frac{6n-12}{n} \underline{\underline{< 6}}$$

↳ Every triangulation has a vertex w/ degree ≤ 5

(but might only have 12)

↳ Immediately applies to any planar graph