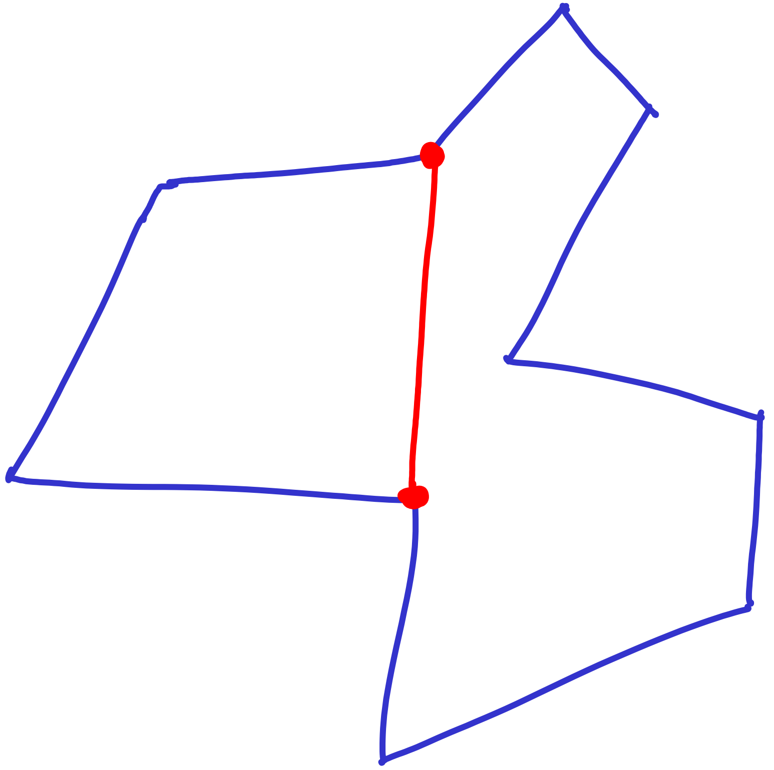
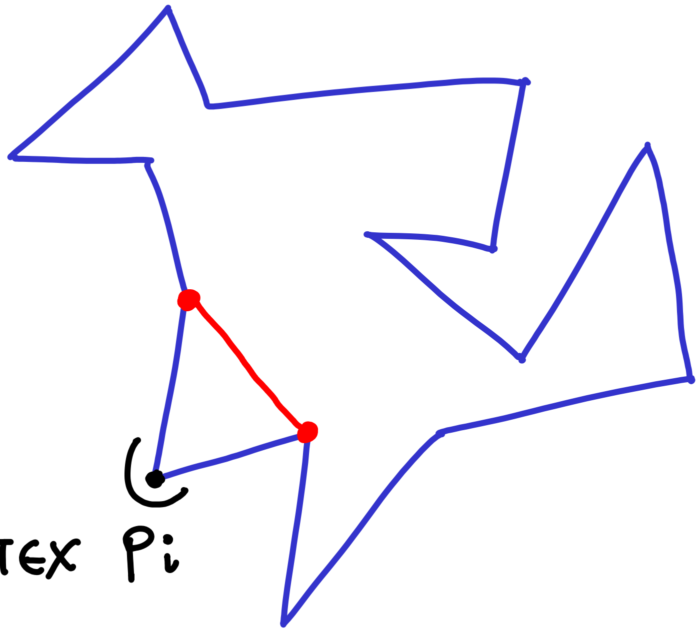


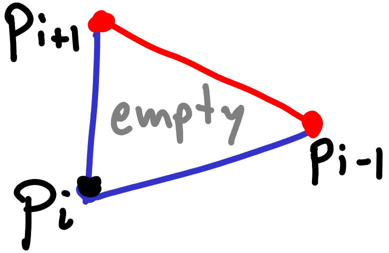
diagonal / chord

- vertex to vertex
- inside polygon
- splits polygon in 2

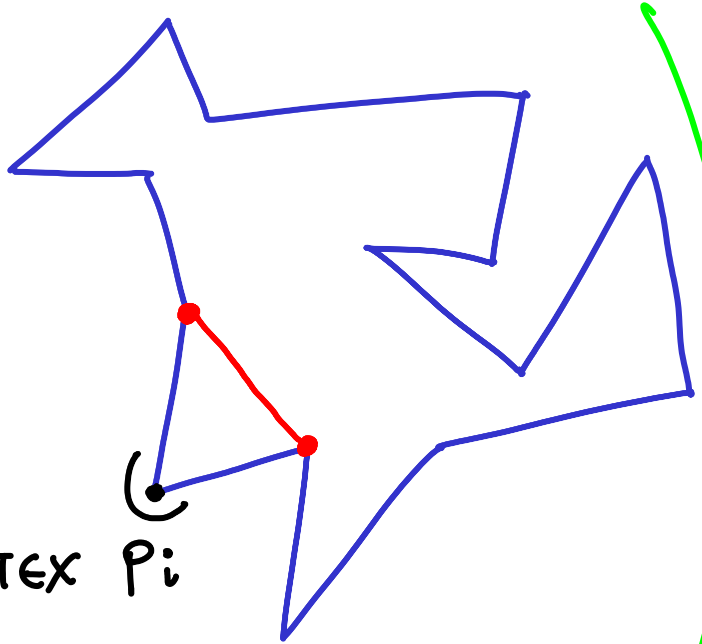
HOW WOULD YOU FIND A DIAGONAL ?



- PICK A CONVEX VERTEX P_i
- TRY $\overline{P_{i-1} P_{i+1}}$
- IF NOT INTERSECTED, DONE.



= **EAR**



- PICK A CONVEX VERTEX p_i
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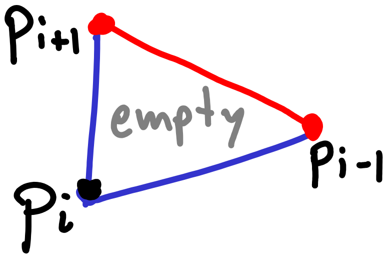
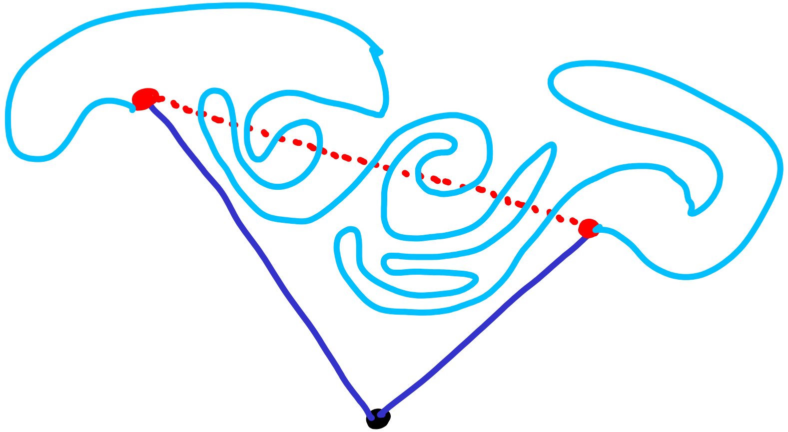
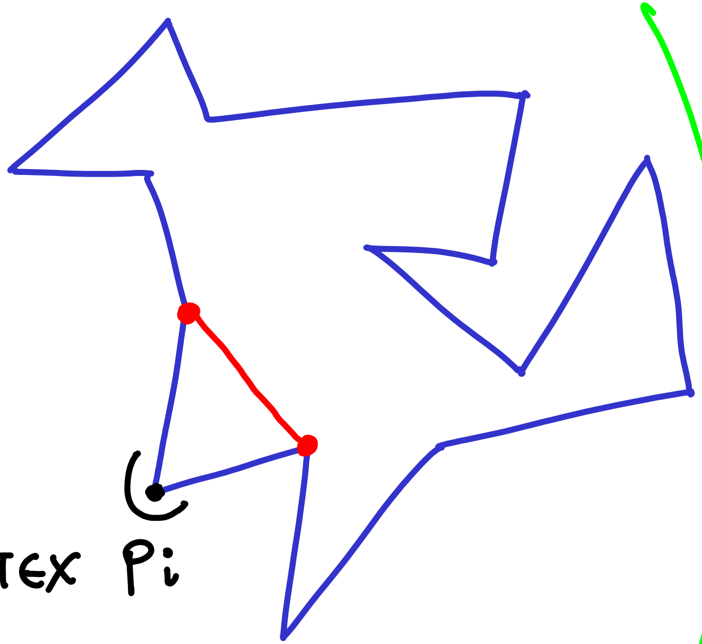


= **EAR**

What if the triangle is not empty?





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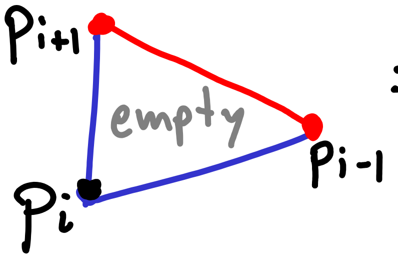

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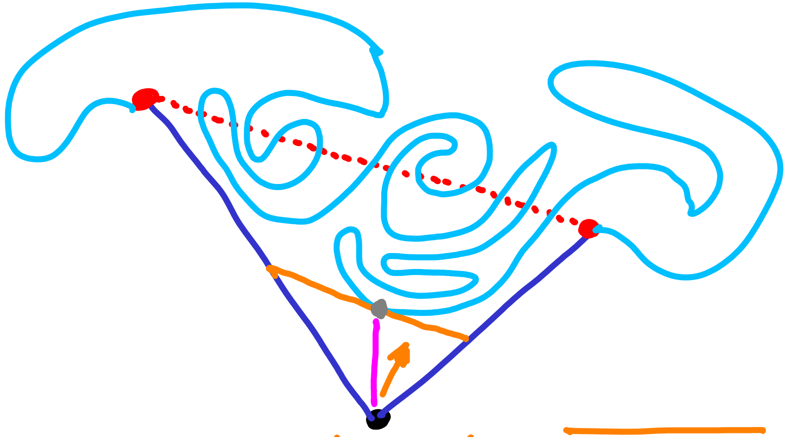

SWEEP parallel to $\overline{p_{i-1}\,p_{i+1}}$

Join to first vertex found. DONE

TIME : $O(n)$

(assumes some basic knowledge of comp.geom,
for instance that testing if a point is above a line, or
testing if two segments intersect, takes constant time.)

What if I prefer to start with a reflex vertex

(assuming non-convex polygon)

Can we still do it?

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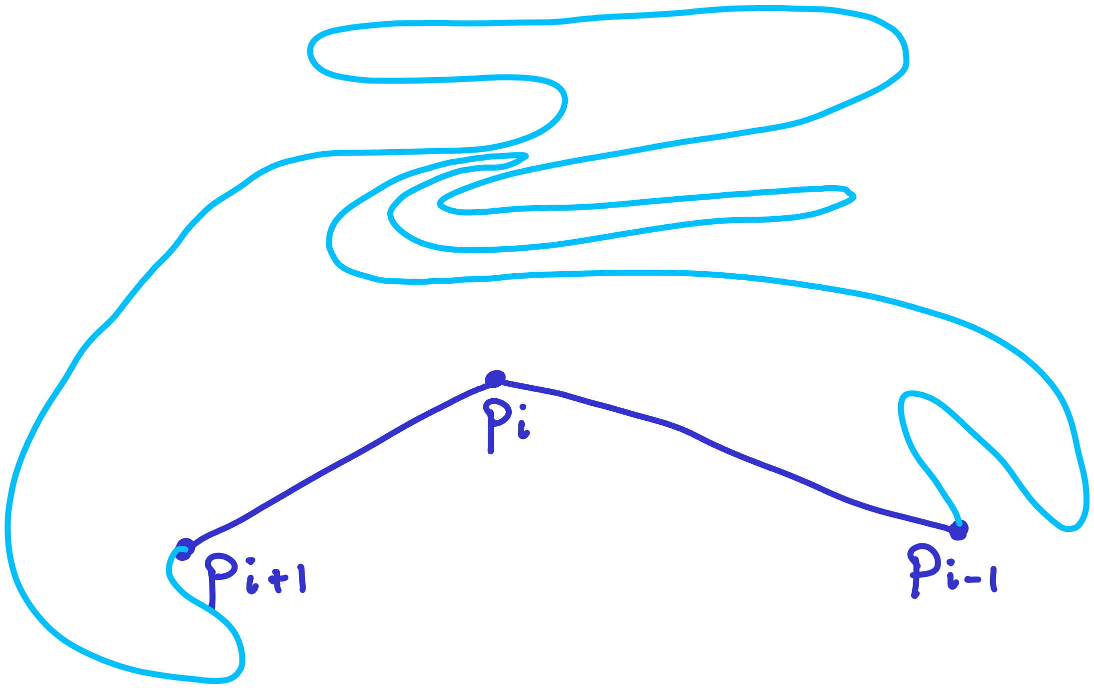

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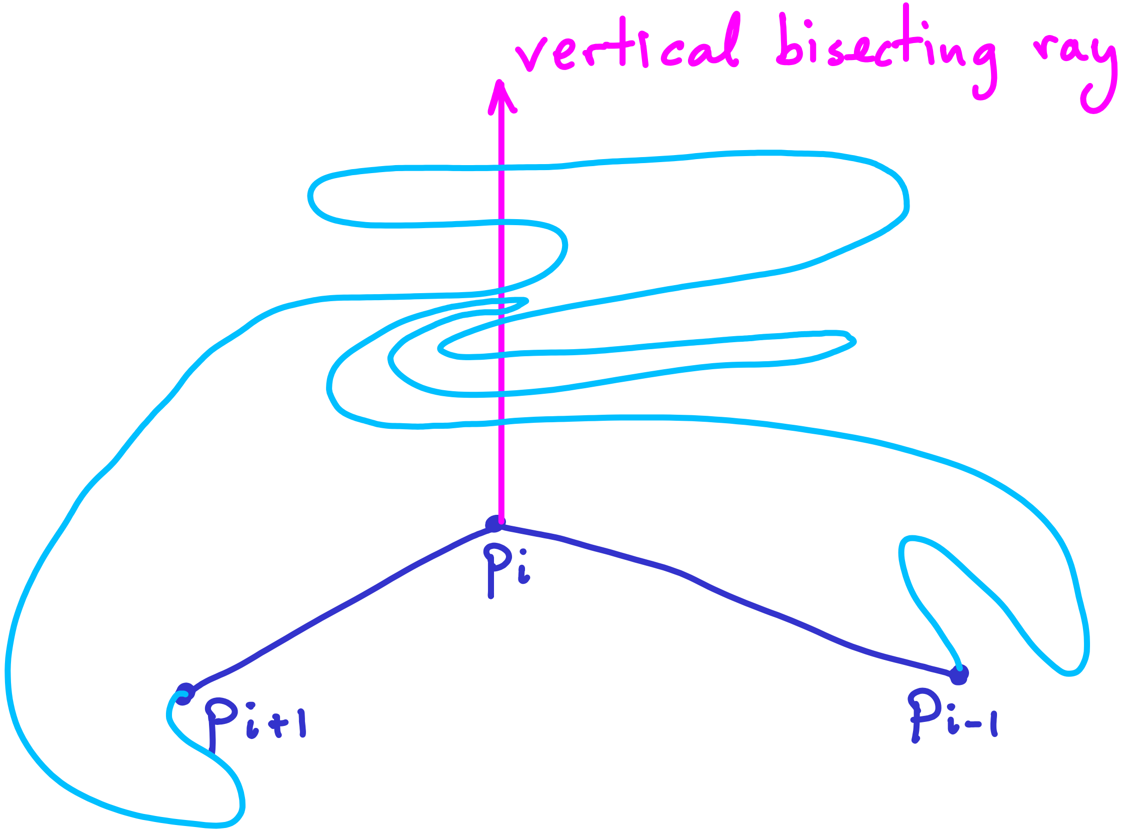

Assume bisecting ray is vertical

↳ WLOG, by rotation

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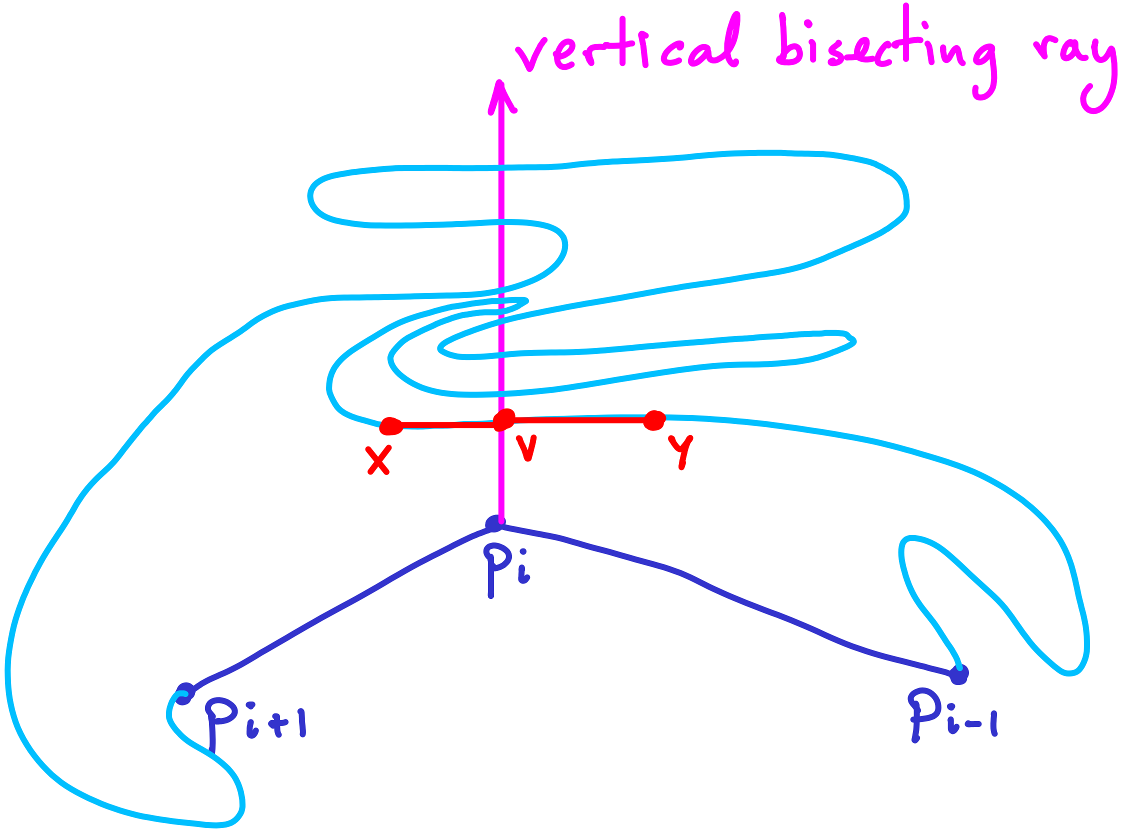


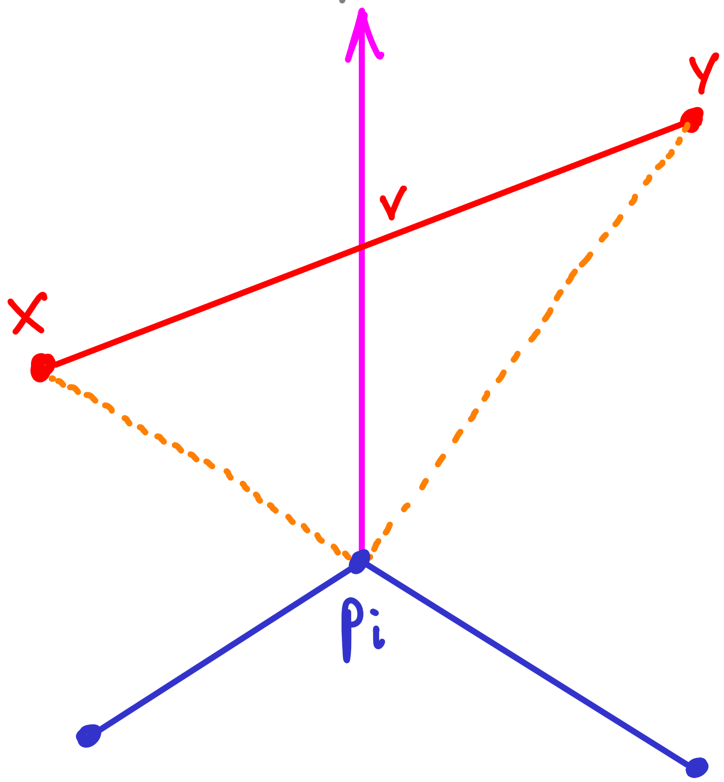
Assume bisecting ray is vertical

↳ WLOG, by rotation

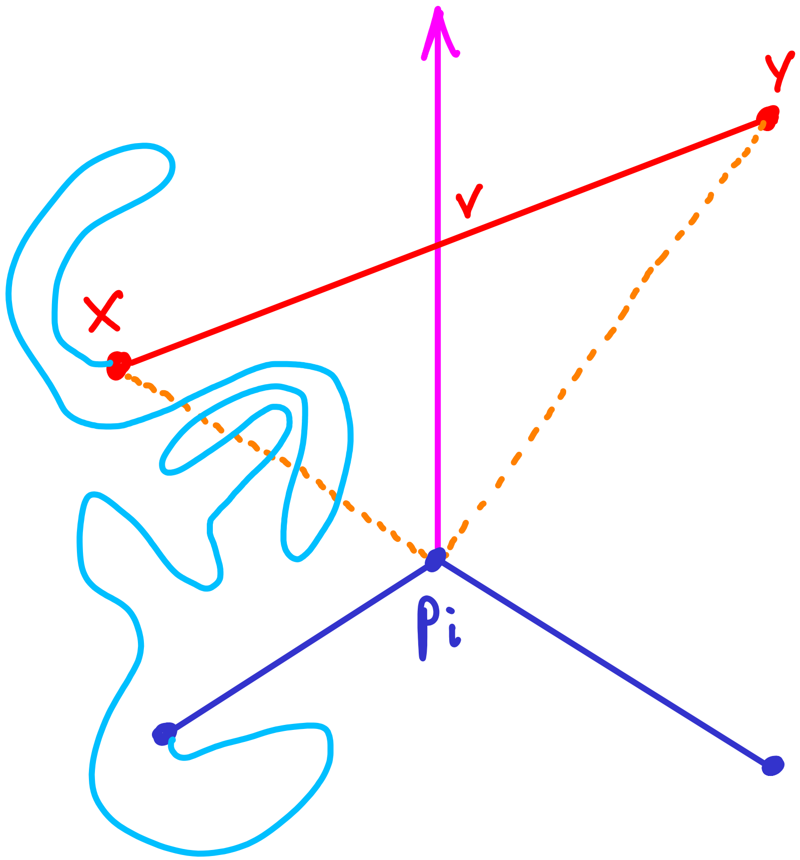
Let v be the first point hit, on segment $\overline{xy}$

If v is a vertex: done

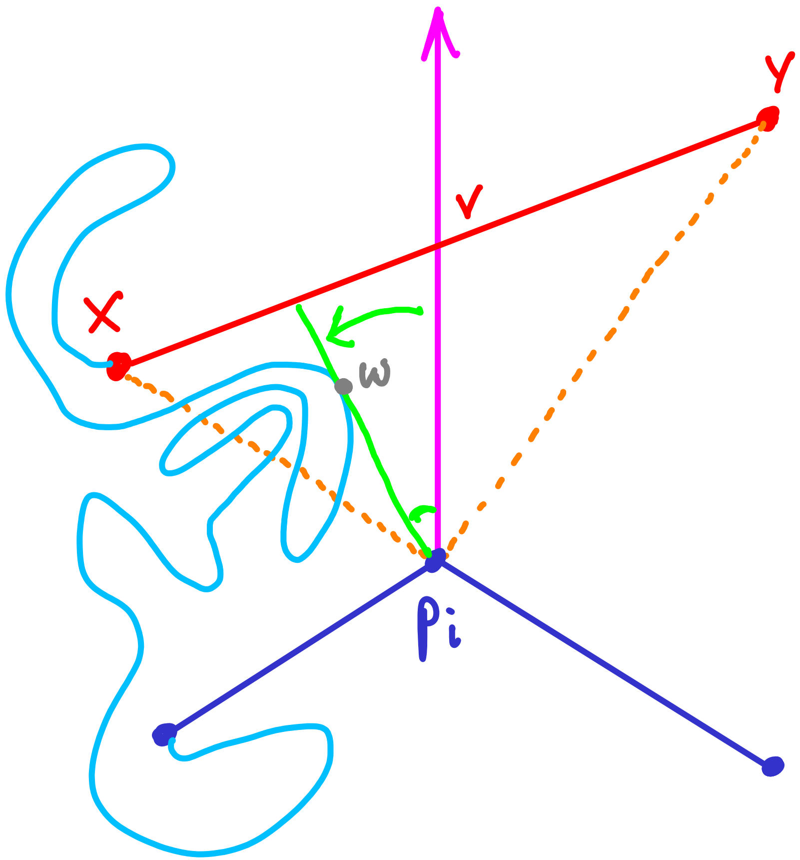
(v is not a vertex)



Try $\overline{p_i x}$.



Try $\overline{p_i x}$. If not a diagonal then polygon must cross it



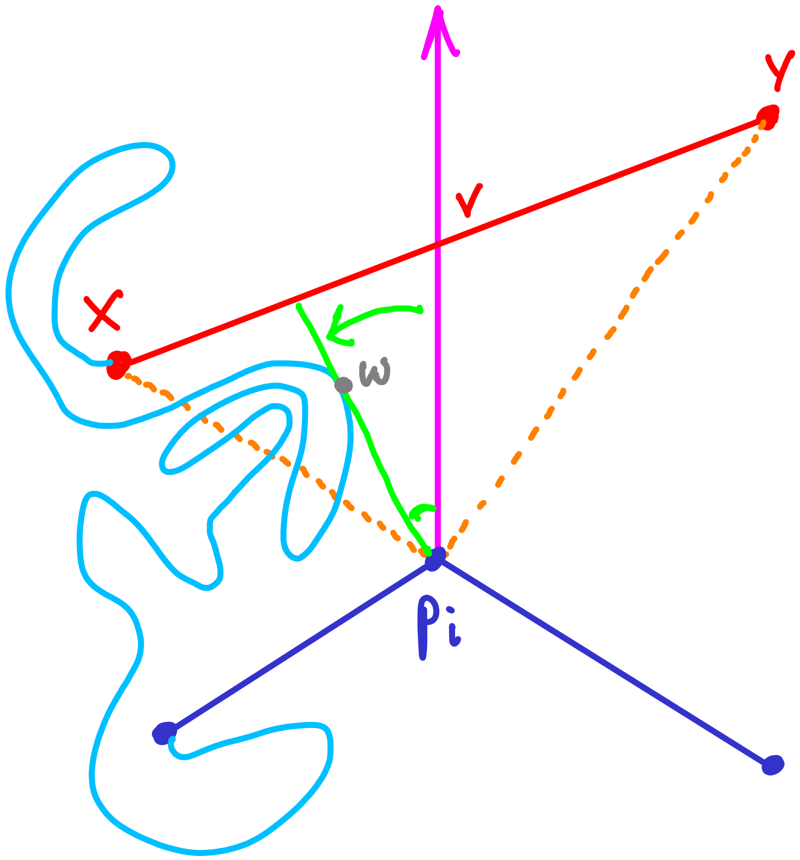
Try $\overline{p_i x}$. If not a diagonal then polygon must cross it

Then sweep inside $\Delta x v p_i$

[rotational fixed at p_i]

Because Δ not empty, will hit $\underline{\underline{w}}$.

QED



Try $\overline{p_i x}$. If not a diagonal then polygon must cross it

Then sweep inside $\Delta x v p_i$

[rotational fixed at p_i]

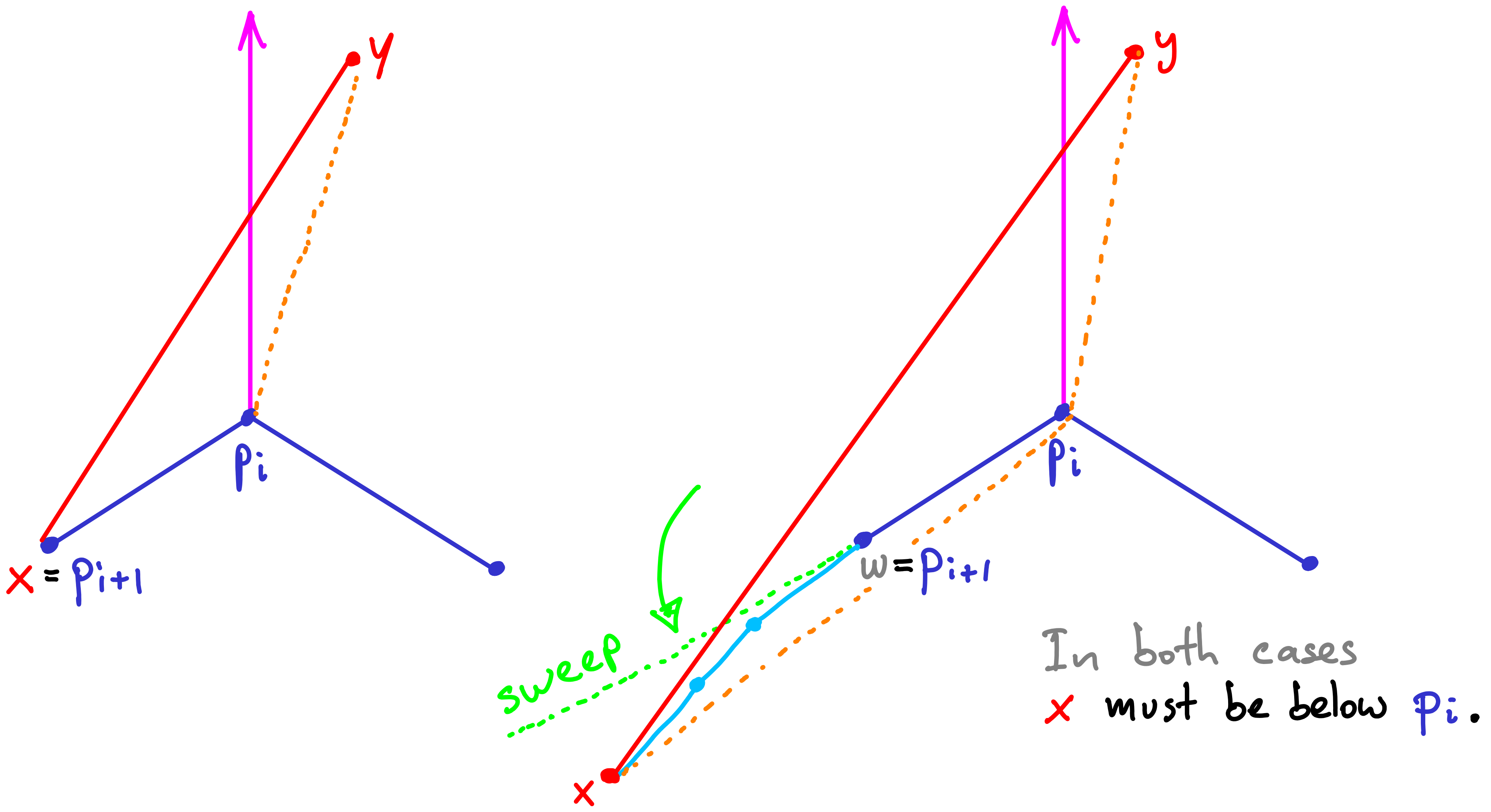
Because Δ not empty, will hit $\underline{\underline{w}}$.

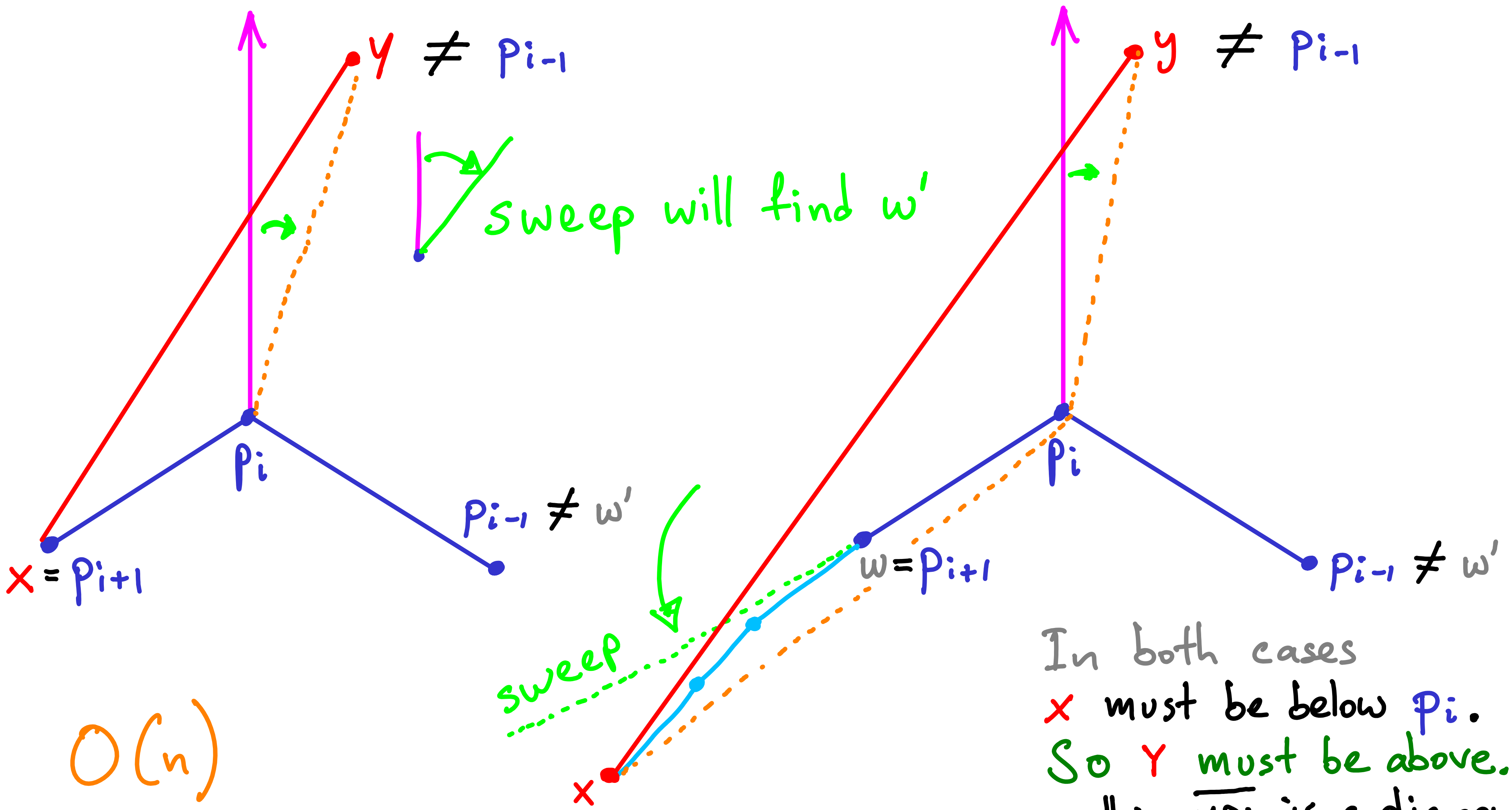
QED ??? NO

- What if $\overline{w p_i}$ is a polygon edge?

  ↳ $w = p_{i+1}$

- What if $x = p_{i+1}$?


y
p_i
x = p_{i+1}
sweep
y
p_i
w = p_{i+1}
x
In both cases
x must be below p_i.



$O(n)$

In both cases

x must be below p_i.

So Y must be above.

$\Rightarrow \overline{yp_i}$ is a diagonal

TRIANGULATION

- partition of polygon into triangles
- insertion of diagonals while possible

HOW DO WE KNOW IT EXISTS FOR EVERY n-gon?

TRIANGULATION

- partition of polygon into triangles
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HOW DO WE KNOW IT EXISTS FOR EVERY n-gon?

- We can find <u>one</u> diagonal
- It splits the polygon into 2 smaller ones
- Use induction

↳ Assume every (n-1)gon can be triangulated

Base case = $\triangle$

Algorithm?

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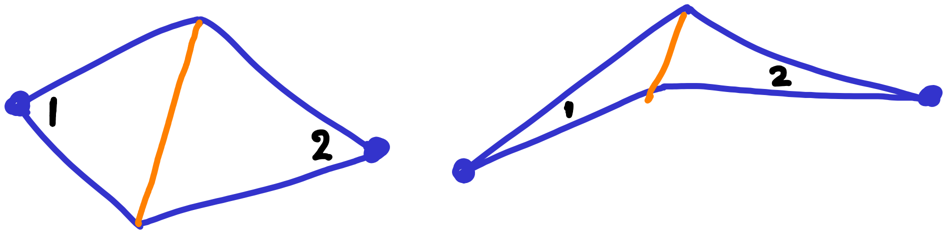
Algorithm? → $O(n)$ for diagonal ... could form EAR & (n-1)gon

$O(n^2)$

Meister's two-ear theorem

Every n-gon has ≥ 2 non-overlapping ears. $(n>3)$



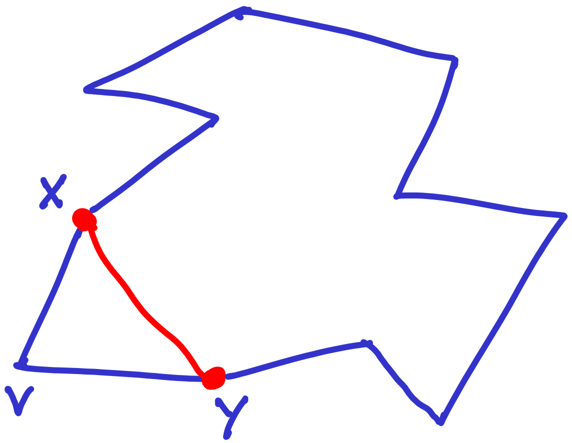
← Base case : $n=4$

Notice: adjacent (convex) vertices can't form a pair.

proof ?

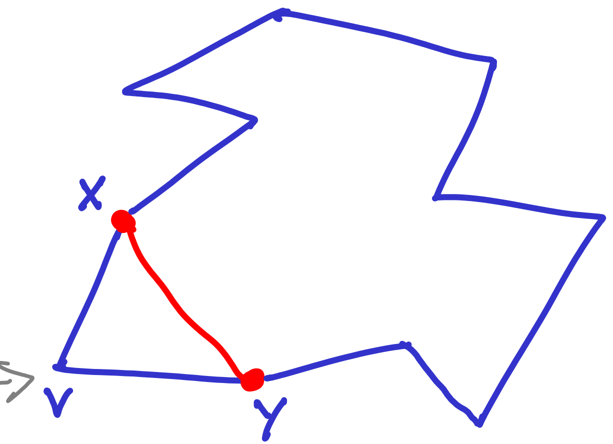
Proof

- Find a convex vertex. If it's an ear, cut it off.

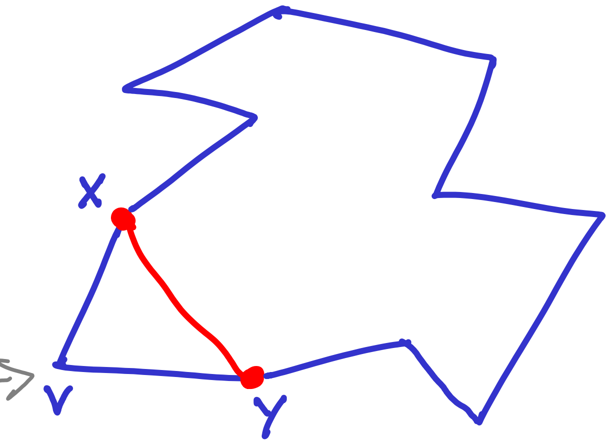


Proof

- Find a convex vertex.
  If it's an ear, cut it off.
  Apply induction.
  On sub-polygon, there are ≥2 ears. They can't be at x AND y.
  So one ear must not overlap v.



Proof



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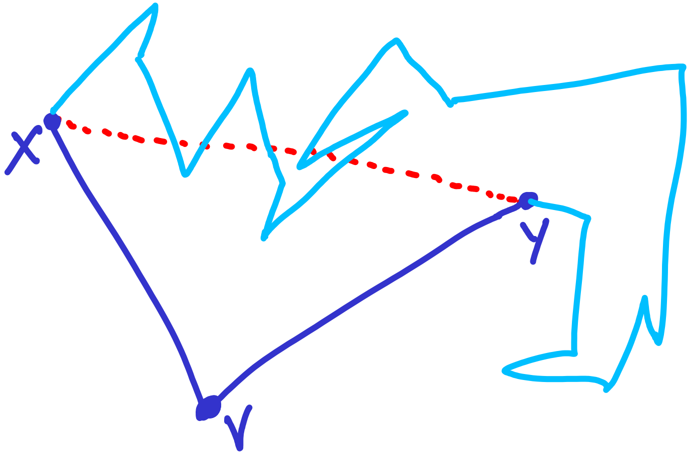
Else: v is not an ear.



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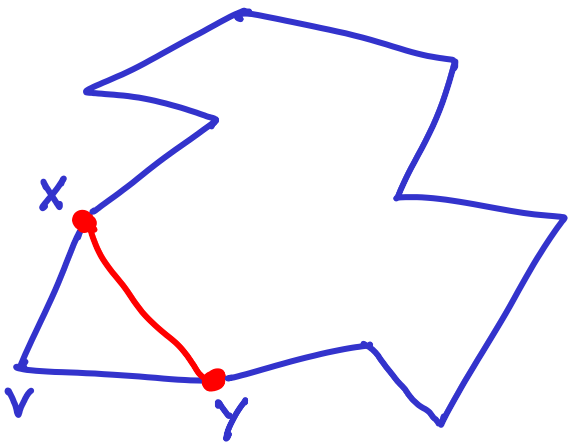
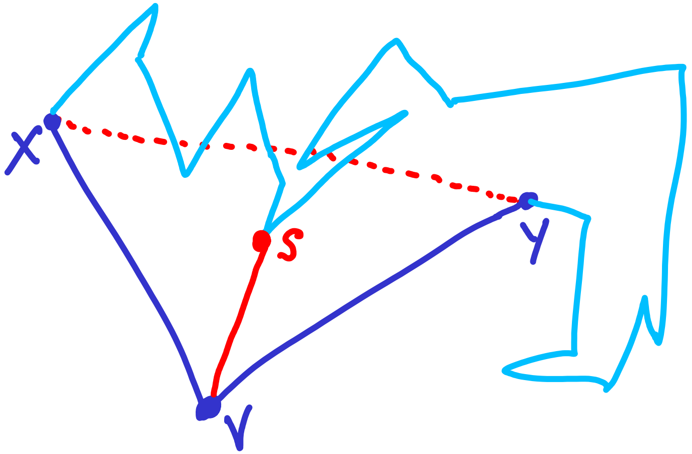


Else: v is not an ear. Then find diagonal $\overline{vs}$ as shown before.



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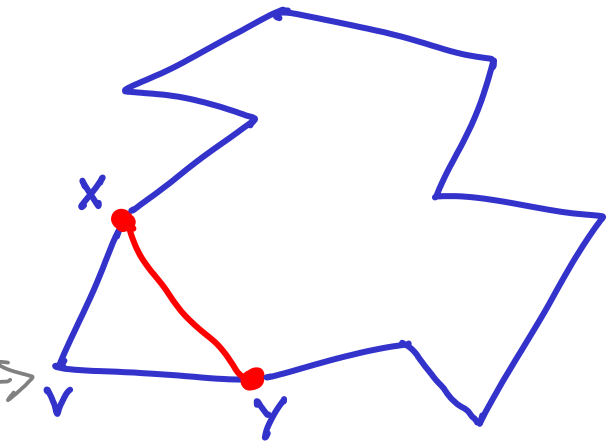

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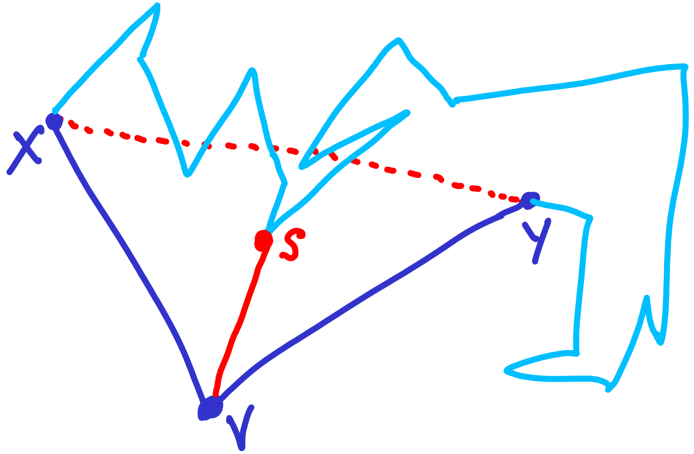


↳ By induction, both sub-polygons have 2 ears

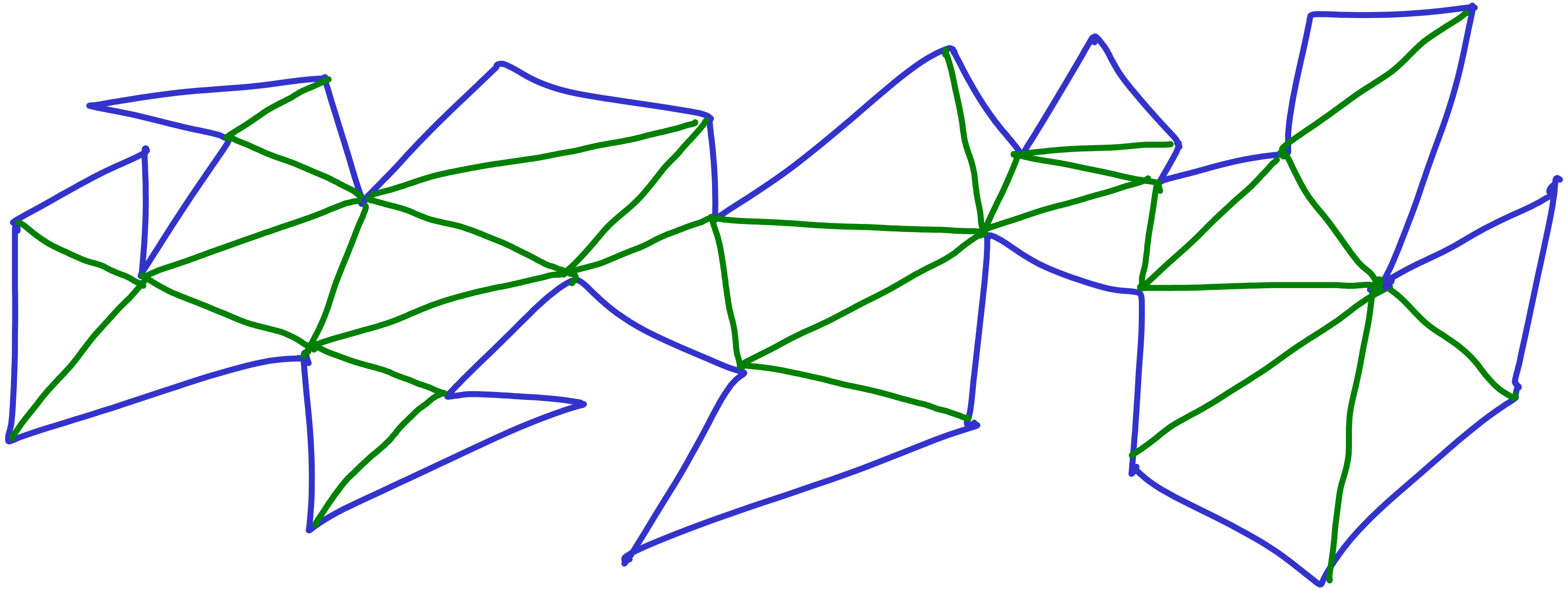
As above, not both can be on $\overline{vs}$.

On each side take the ear not at v or s.

QED.

Faster proof of 2-ear thm.

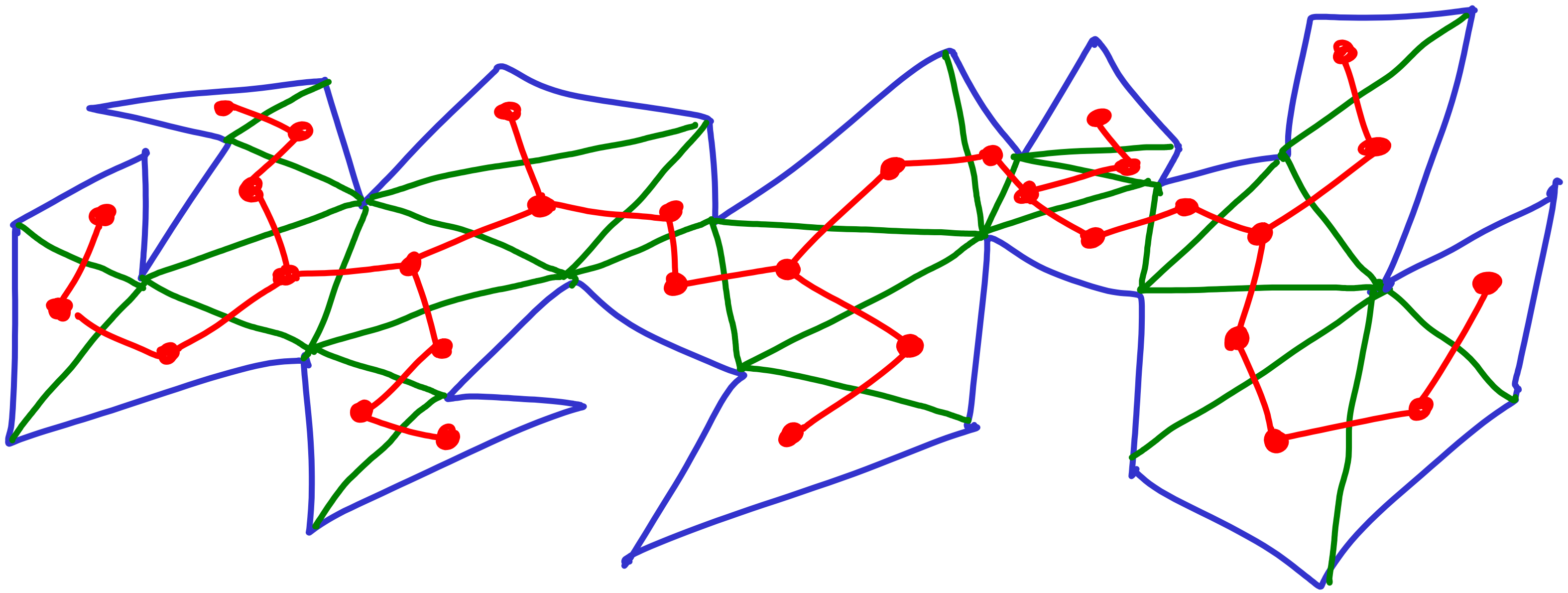
Every polygon has a triangulation.



Faster proof of 2-ear thm.

Every polygon has a triangulation.

- Every triangulation has a dual tree.

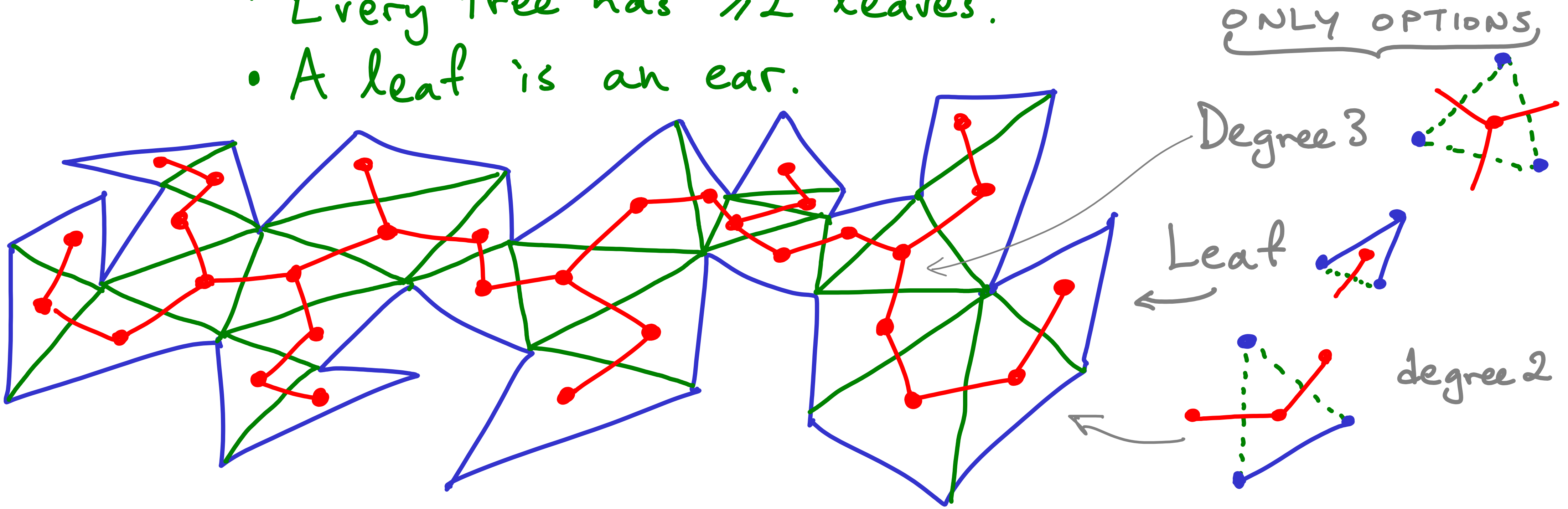


Faster proof of 2-ear thm.

Every polygon has a triangulation.

Every triangulation has a <u>dual tree</u>.

- Every tree has ≥ 2 leaves.
- A leaf is an ear.



The "fast" proof assumes triangulations exist.

↳ requires a proof like what we started with (existence of diagonal)

The first proof by induction relies on nothing

HOW LONG DOES IT TAKE TO FIND ONE EAR ?

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↳ just finding diagonal and searching one side : $O(n^2)$

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↳ A little better:

- List all convex vertices
- Try each one : $O(n)$

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still $O(n^2)$ { • List all convex vertices
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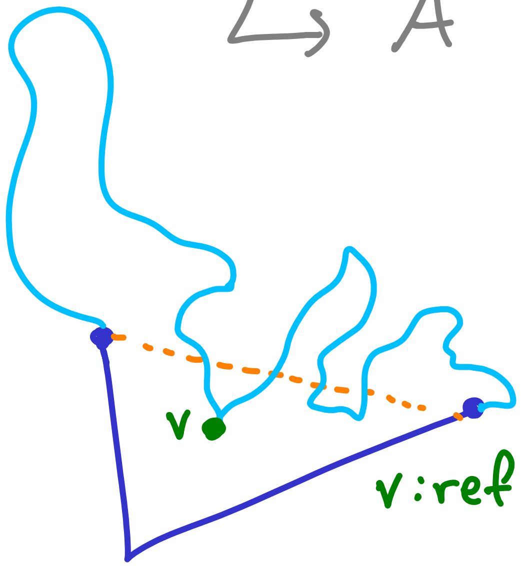

still $O(n^2)$ {
- List all convex vertices
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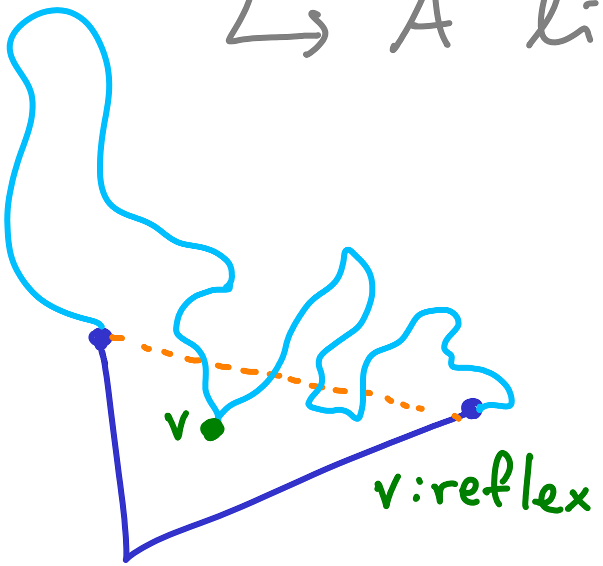


still $O(n^2)$ {
- List all convex vertices
- Try each one : $O(n)$
  ↳ by looking for reflex vertices inside $O(k)$

$\underbrace{}_{O(nk)}$ if k reflex

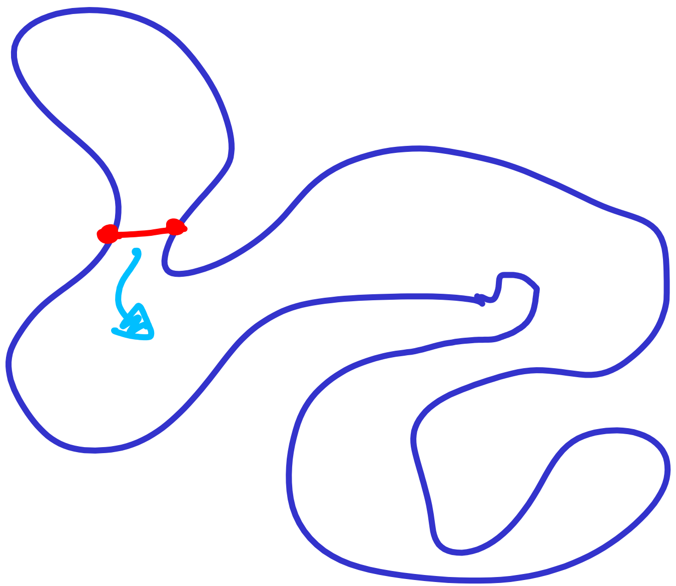
If you have k convex, also $O(nk)$

if k=Theta(n), still O(n^2)

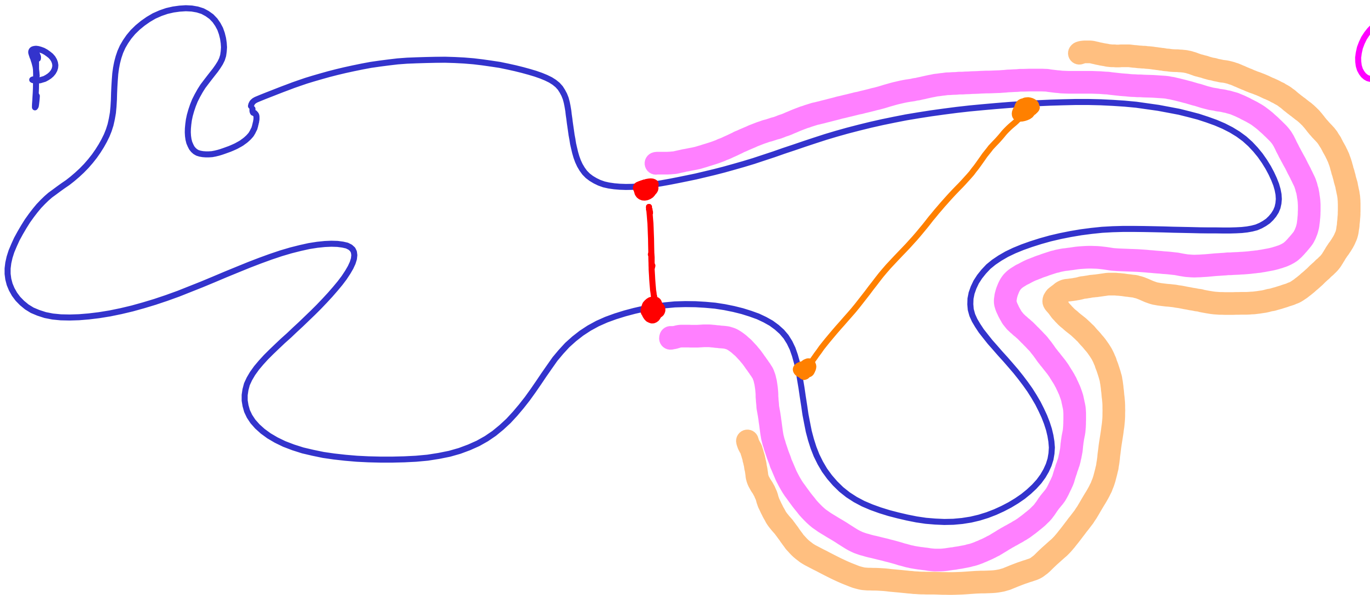
HOW LONG DOES IT TAKE TO FIND ONE EAR?

↳ just finding diagonal and searching one side: $O(n^2)$

What about searching both sides?

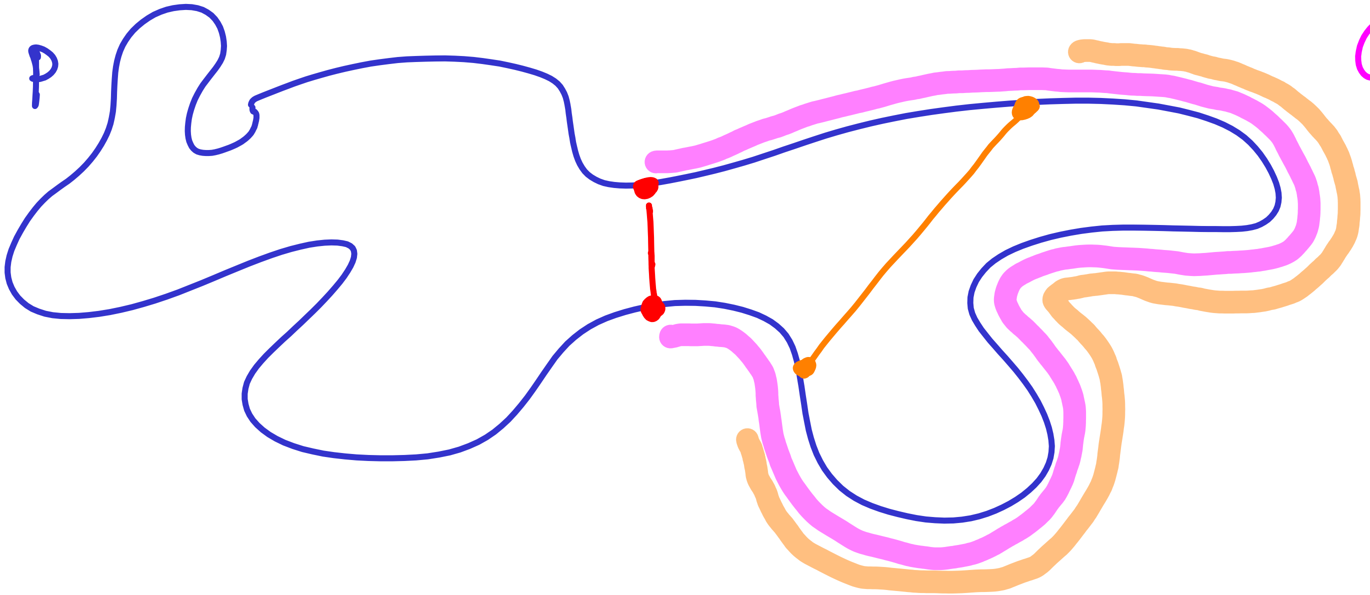


You would have to make unbalanced cuts and go the wrong way each time to get $O(n^2)$



$\underbrace{\text{GOOD SUB-POLYGON}}(P)$

- subset
- has only one new edge



GOOD SUB-POLYGON(P)

- subset
- has only one new edge

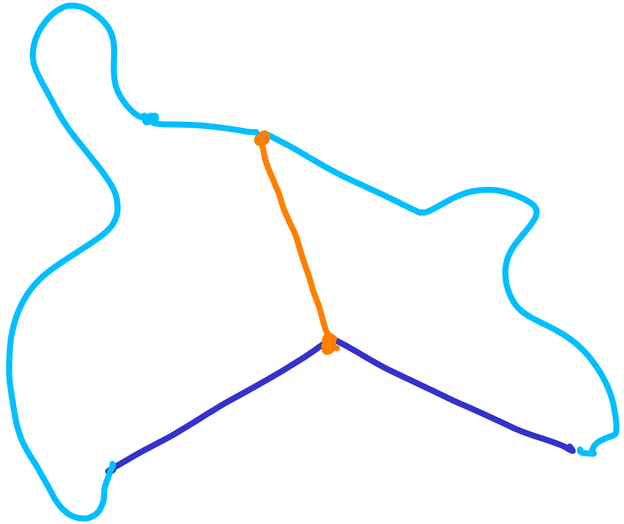
Every time you partition a GSP with a diagonal, one of the two sides is still a GSP (of the original)

FINDING an EAR in LINEAR TIME

- Pick <u>any</u> vertex : test if its an ear $O(n)$ by sweep

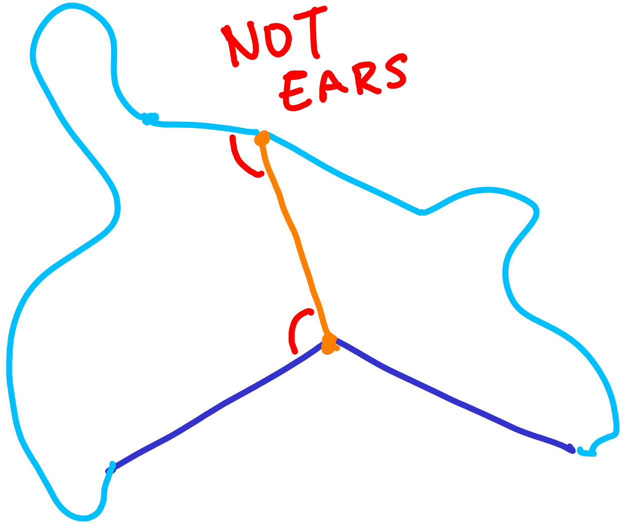
FINDING an EAR in LINEAR TIME

- Pick any vertex : test if its an ear   O(n) by sweep
  - If not → form a diagonal   O(n)
    → choose one side, arbitrarily
    → find ear on that side



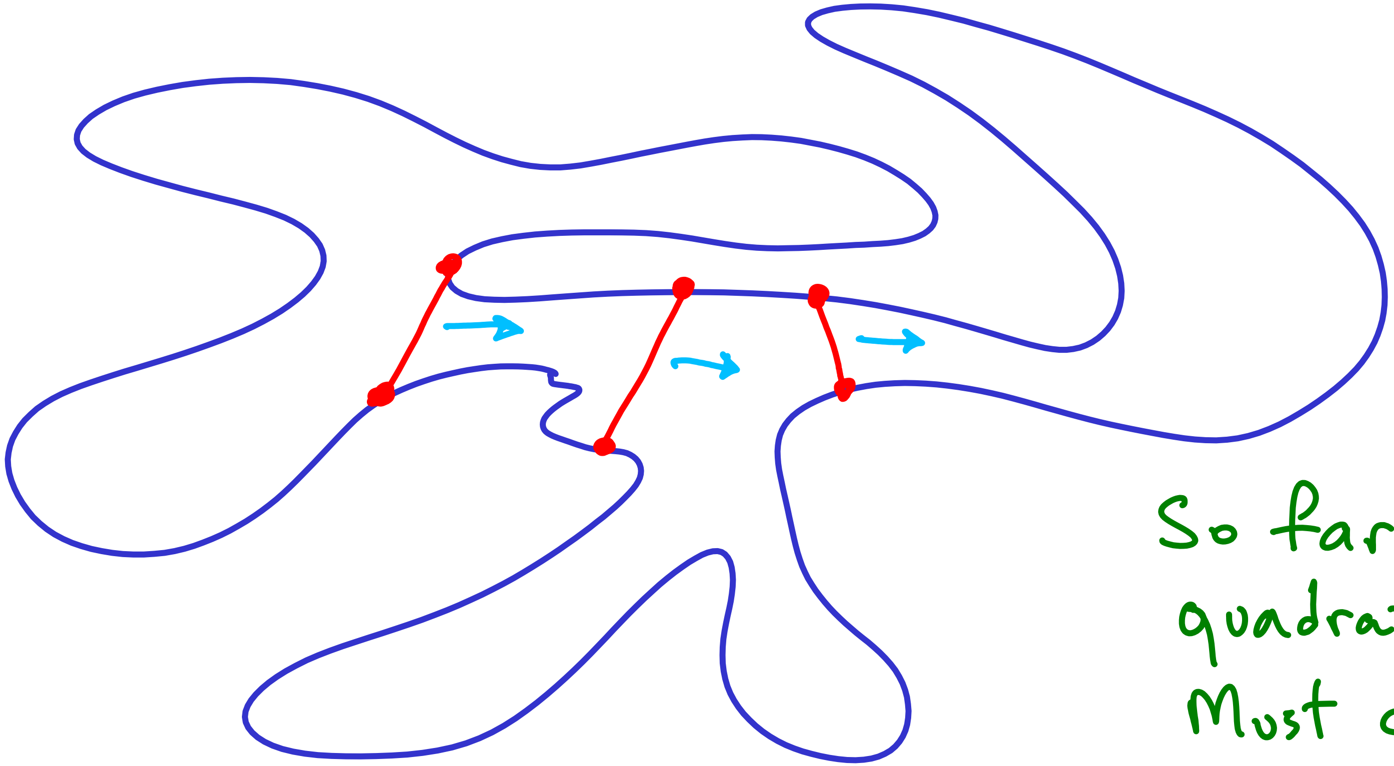
FINDING an EAR in LINEAR TIME

- Pick <u>any</u> vertex : test if its an ear   O(n) by sweep
    - If not → form a diagonal   O(n)
      → choose one side, arbitrarily
      → find ear on that side
        ↳ find a particular ear
        (can't involve diagonal)



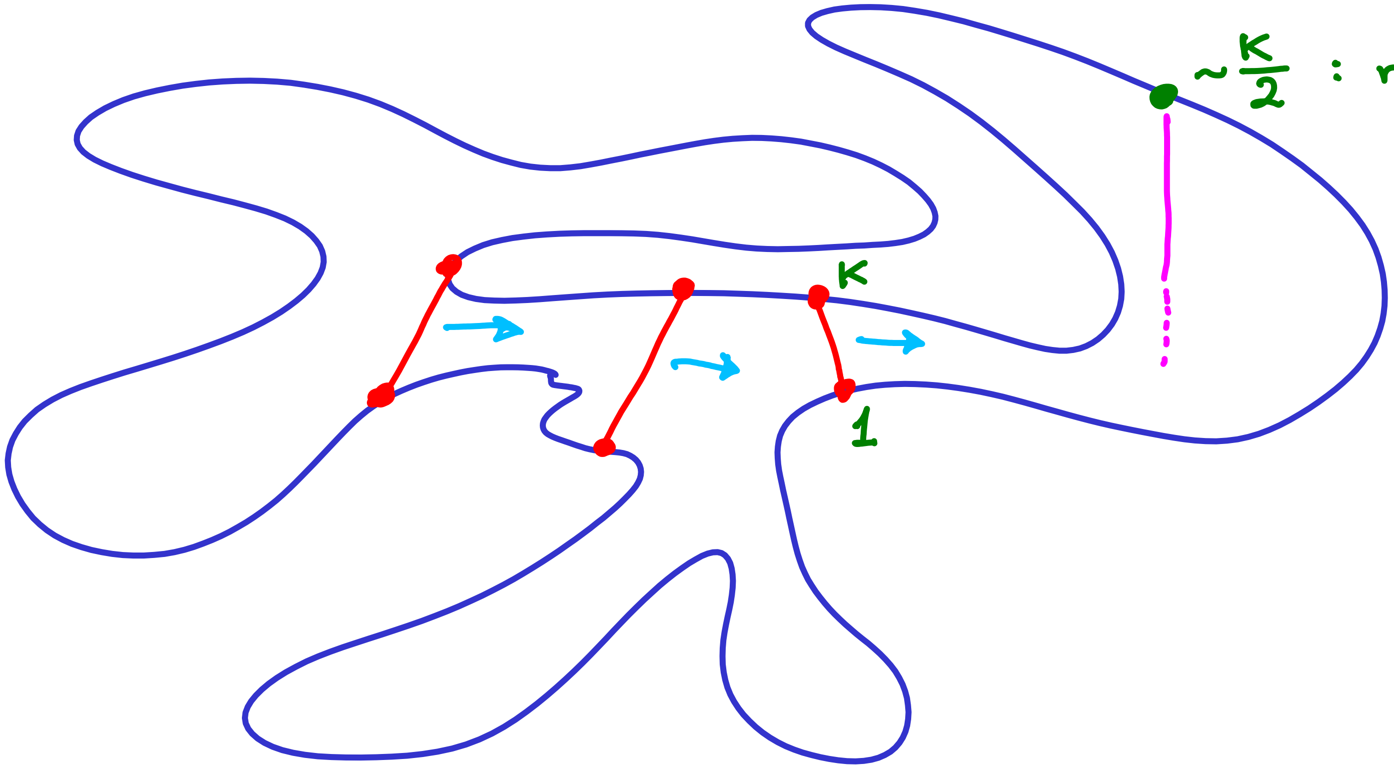
We have seen that when we cut a polygon with a diagonal, one of the 2 ears of each subpolygon is not on the diagonal.

So our first choice of sub-polygon was arbitrary
but we will choose the GSP side each time



So far this is still quadratic.
Must choose diagonal carefully

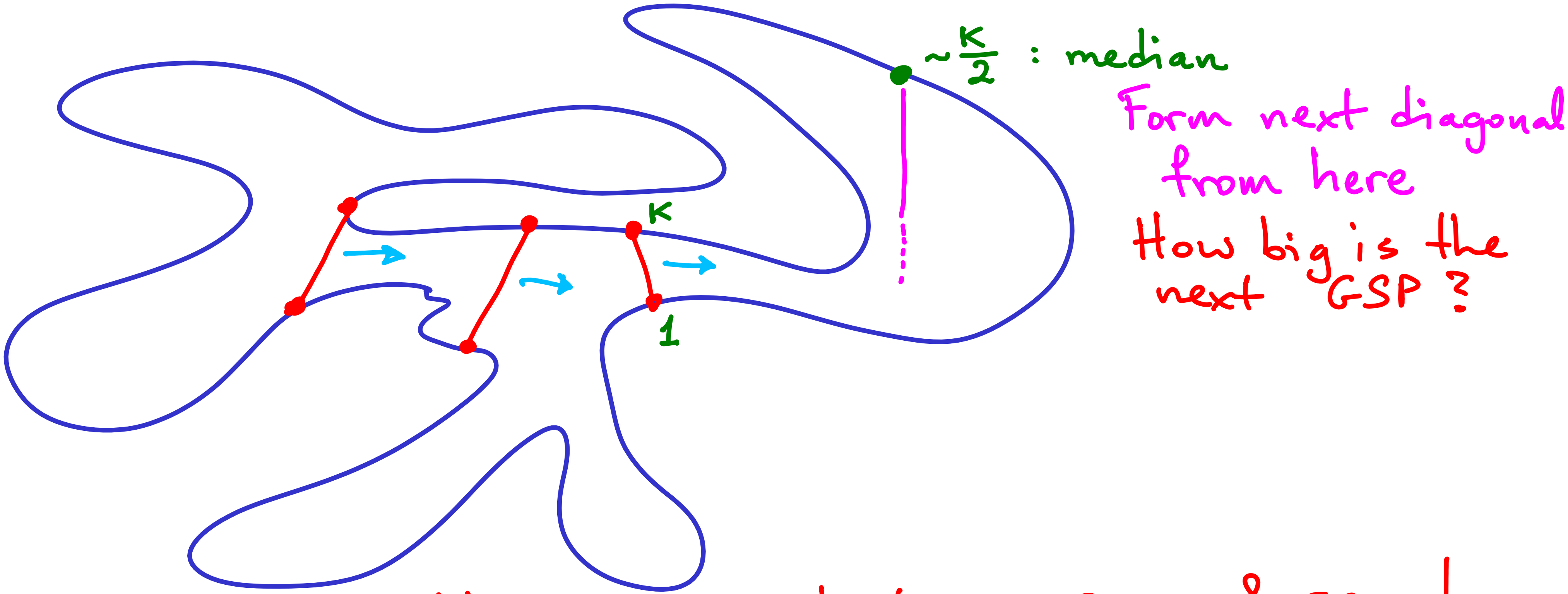
So our first choice of sub-polygon was arbitrary but we will choose the GSP side each time



Form next diagonal from here

How big is the next GSP ?

So our first choice of sub-polygon was arbitrary but we will choose the GSP side each time



Divide the problem size each time : prune & search