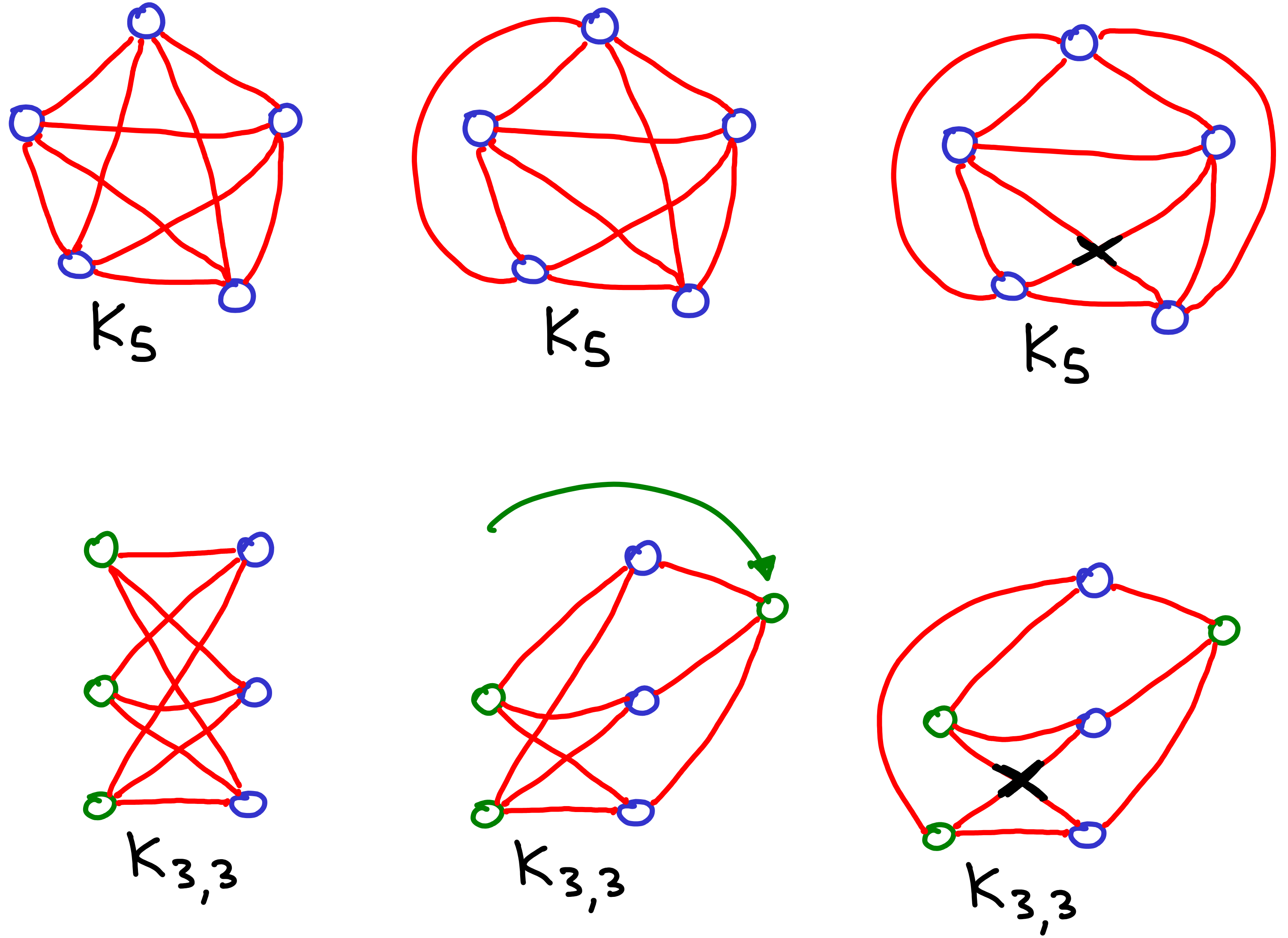


CROSSING NUMBER

cr(G)

↓

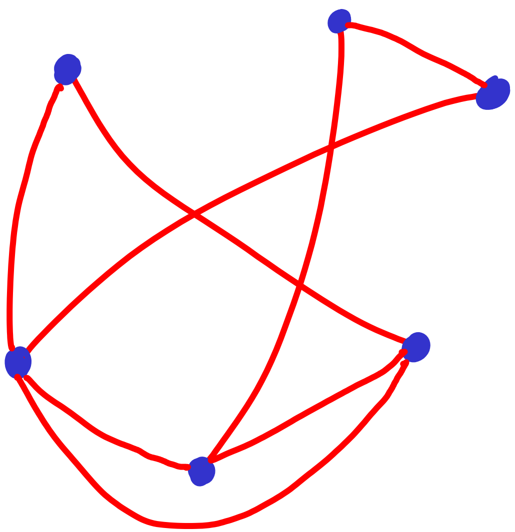
min # crossings possible for any embedding (drawing) of G

And remember, graph drawing is a huge field full of project possibilities. even "untangling" is a possible topic.

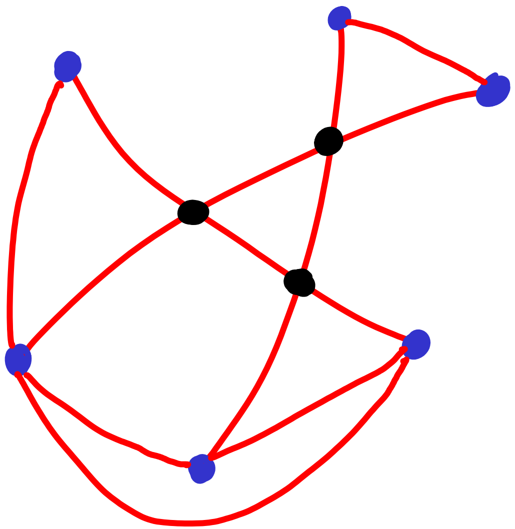
Given graph G & integer k, decide if $cr(G) \le k$. → NPC

(unless k=const)

Here is a lower bound on $cr(G)$:



$G=(V,E)$



make plane

$G'=(V',E')$

$$V' = V + \underline{\underline{cr(G)}}$$

↓

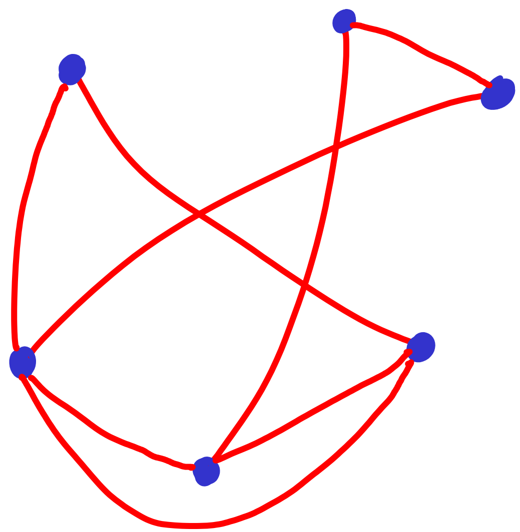
Note: you must start with the <u>best</u> drawing of G, unlike my example.

Here $V' > V + cr(G)$

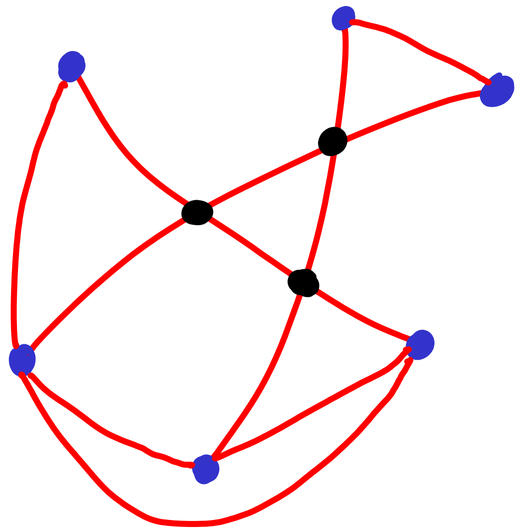
Given graph G & integer k, decide if $cr(G) \leq k$. → NPC

(unless k=const)

Here is a lower bound on $cr(G)$:



$G = (V, E)$



make plane

$G' = (V', E')$

$$V' = V + cr(G)$$

For every crossing (•) we get 2 new edges

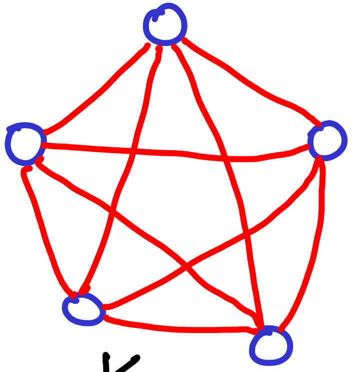
$$E' = E + 2 \cdot cr(G)$$

$$E' \leq 3V' - 6$$

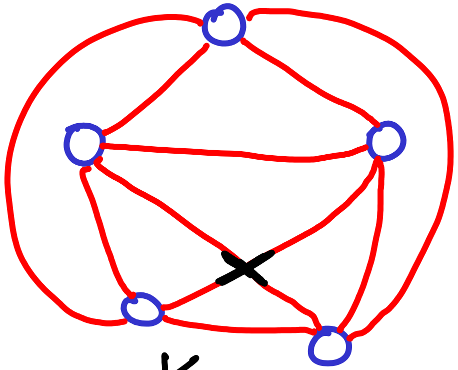
$$E + 2cr(G) \leq 3V + 3cr(G) - 6$$

$$cr(G) \geq E - 3V + 6$$

(for $V > 2$)



K_5

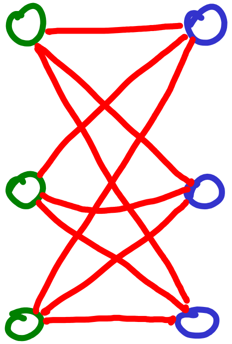


K_5

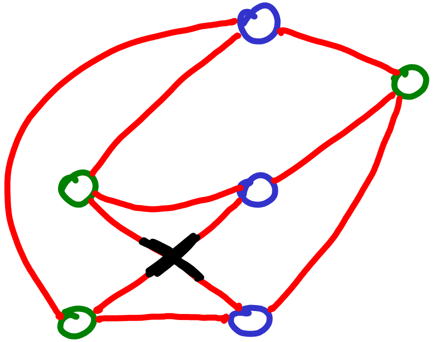
$$\boxed{cr(G) \geq E - 3V + 6}$$

$\geq 10 - 15 + 6$

≥ 1



$K_{3,3}$



$K_{3,3}$

$\geq 9 - 18 + 6$

≥ -3

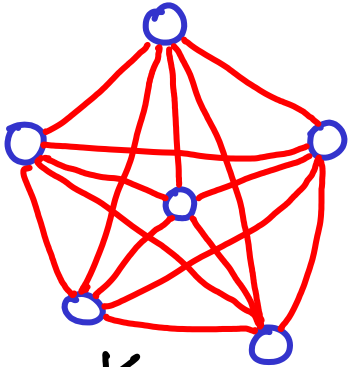
inconclusive

$cr(G) \geq E - 3V + 6$

For K_n : $cr(K_n) \geq \binom{V}{2} - 3V + 6$

$= \frac{1}{2}V(V-1) - 3V + 6$

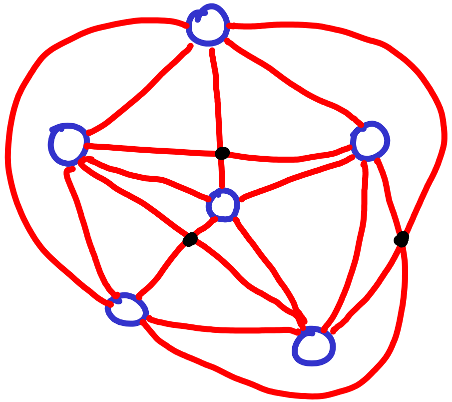
$= \frac{1}{2}V^2 - 3.5V + 6$



K_6

$cr(K_6) \geq \frac{1}{2}36 - 21 + 6$

$= 3$



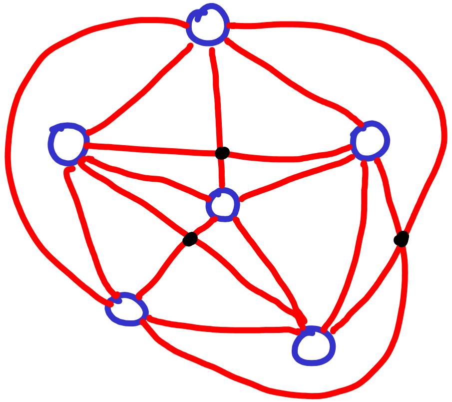
Also, trivially, $cr(K_n) \leq \binom{\binom{V}{2}}{2} = O(V^4)$

so $cr(G) = \Omega(V^2)$

$= O(V^4)$

A better bound

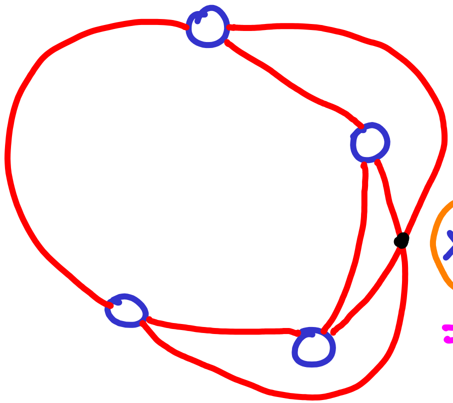
[Leighton 1983]



Given a graph G, $\exists$ a drawing w/ $cr(G)$ crossings

Suppose you also have a parameter $0 < p \leq 1$

G_p : subgraph of G :
- for each $v \in V$, keep v with probability p
- for each $e \in E$ keep e iff both endpoints survive



$v_p = 4$

$e_p = 5$

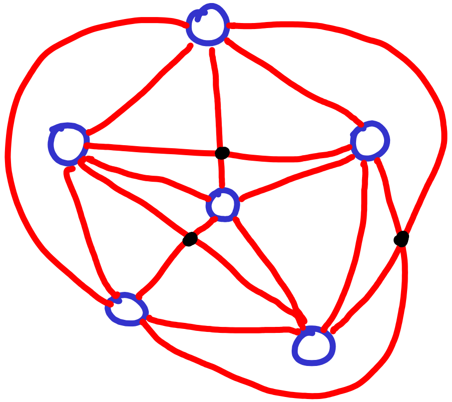
$x_p \geq cr(G_p)$

$=1$

actual # crossings in G_p

A better bound

[Leighton 1983]



K_6

Given a graph G, $\exists$ a drawing w/ $cr(G)$ crossings

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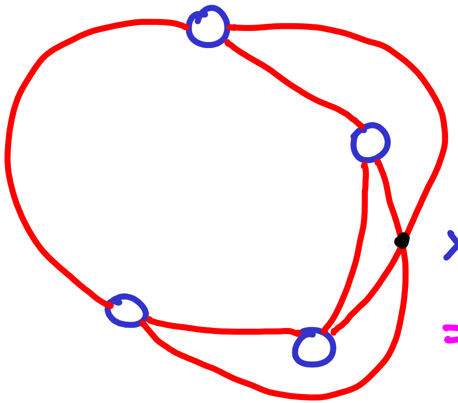


$x_p \geqslant cr(G_p)$

$= 1$

$v_p = 4$

$e_p = 5$

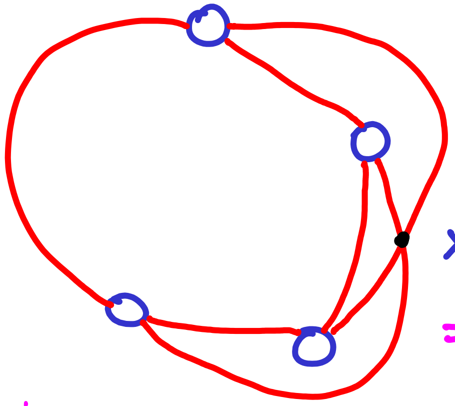
We have proved $cr(G) \geqslant E - 3V + 6$ (for $V \geqslant 3$)

we can relax this:

$$cr(G_p) - e_p + 3v_p \geqslant 0 \quad (\text{for } V > 0)$$

$$x_p - e_p + 3v_p \geqslant 0$$

$x_p - e_p + 3V_p \geq 0$ These are random variables



$V_p = 4$

$e_p = 5$

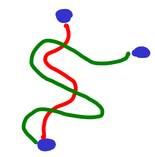
$x_p \geq cr(G_p)$
$= 1$

$E[x_p - e_p + 3V_p] \geq 0$

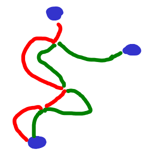
$E[x_p] - E[e_p] + E[3V_p] \geq 0$

$E[V_p] = p \cdot V$ // every vertex appears w/ prob p

$E[e_p] = p^2 \cdot E$ // for every edge, both endpoints must survive



$E[x_p] = p^4 \cdot cr(G)$ // any crossing in G will survive iff its 2 edges survive
= iff 4 endpoints survive



$p^4 \cdot cr(G) - p^2 \cdot E + 3p \cdot V \geq 0$ $\Rightarrow cr(G) \geq \dfrac{p^2 E - 3pV}{p^4}$ (for any p)

Choose $p = \dfrac{4V}{E}$ $\Rightarrow cr(G) \geq \dfrac{1}{64} \cdot \dfrac{E^3}{V^2}$

Assumption? $E \geq 4V$
$(p \leq 1)$

$$cr(G) \geq \frac{1}{64} \cdot \frac{E^3}{V^2}$$

$$cr(K_V) = \cancel{\Omega(V^2)} \geq \frac{1}{64} \frac{(V^2)^3}{V^2} = \Omega(V^4)$$

$$= O(V^4)$$

$$p = \frac{4V}{E} = \frac{4V}{\binom{V}{2}} = \frac{4V}{\frac{1}{2}V(V-1)} = \frac{8}{V-1} \leftarrow \text{for } K_V$$

$$E[X_p] - E[e_p] + E[3V_p] \geq 0$$

$$E[V_p] = p \cdot V \simeq 8$$

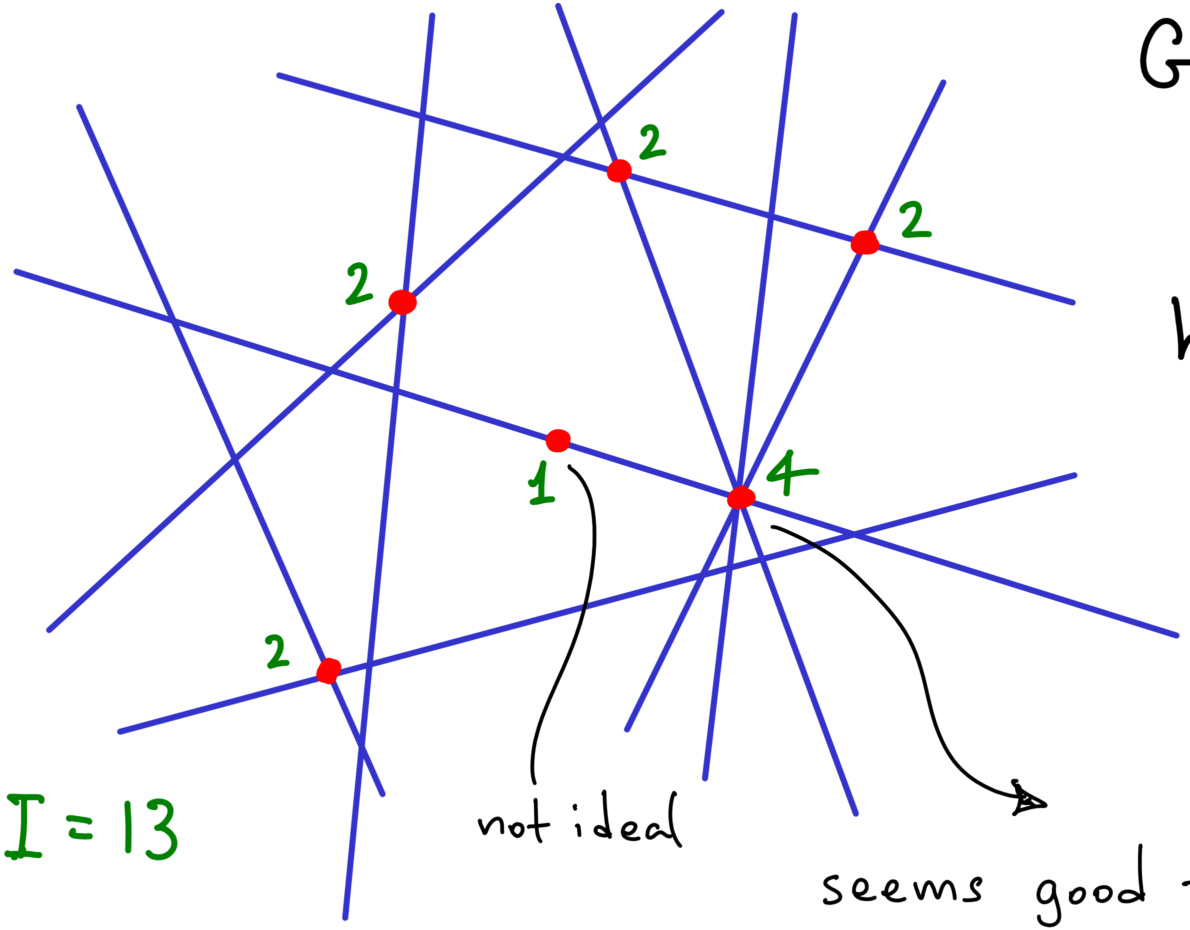
$$E[e_p] = p^2 \cdot E \simeq 64$$

$$E[X_p] = p^4 \cdot cr(K_V) \simeq \frac{4096}{V^4} \cdot cr(K_V)$$

$$\frac{4096}{V^4} \cdot cr(K_V) \geq 56$$

$$cr(K_V) \geq \frac{7}{512} \cdot V^4$$

An application



$I = 13$

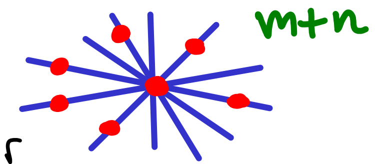
Given n points
and m lines

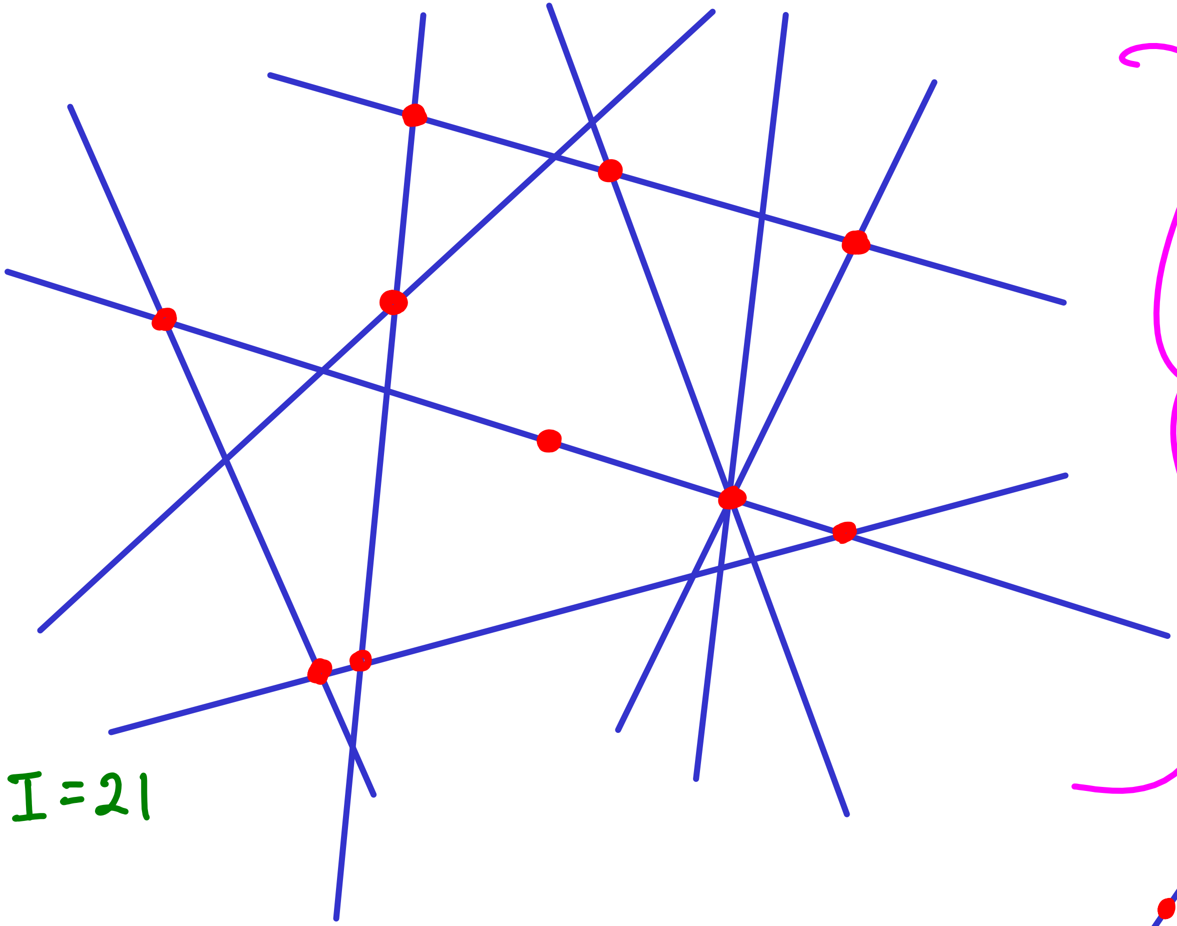
how many pairs touch? I

How many could possibly touch?

↪ easy : $I \leq n \cdot m$

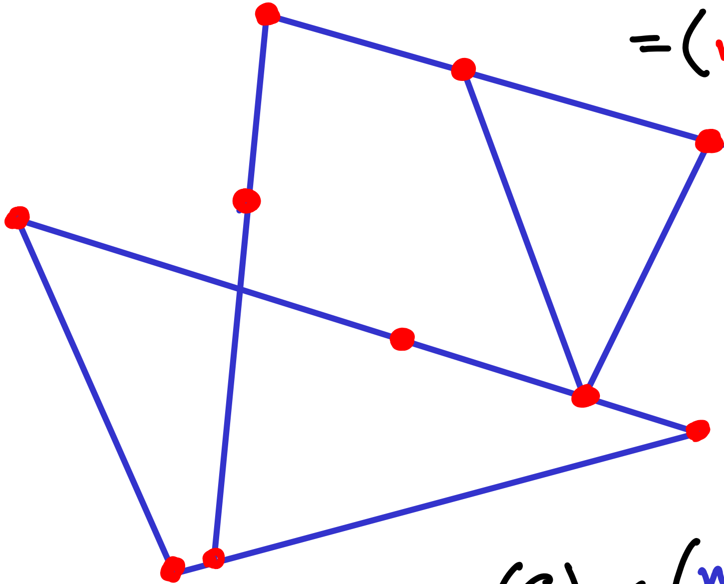
seems good to place lines together
but then you lose on other points





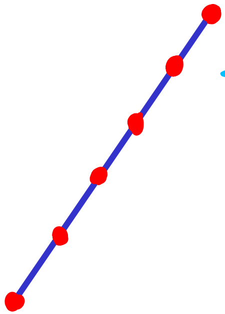
$I=21$

$$G=(V,E)$$
$$=(n,E)$$



$$cr(G) \le \binom{m}{2}$$

$$E \ge I - m$$



for each line
every vertex counts
incidence in one direction
accounting for all edges

except for
1 endpoint on
<u>some</u> lines

We want an upper bound for I (better than $n \cdot m$)

Given n points & m lines, if $I < m + 4n$, we're happy.

Otherwise, use $E \geqslant I - m \geqslant 4n = 4V$

We know $\underbrace{cr(G) \geqslant \frac{1}{64} \cdot \frac{E^3}{V^2}}_{\text{for all graphs with } E \geqslant 4V\,!}$ and $\underbrace{cr(G) \leq \binom{m}{2}}_{\text{trivially}}$ $\Big\}\; E^3 \leq 64 n^2 \binom{m}{2}$

$(I-m)^3 \leq 32 n^2 m^2$

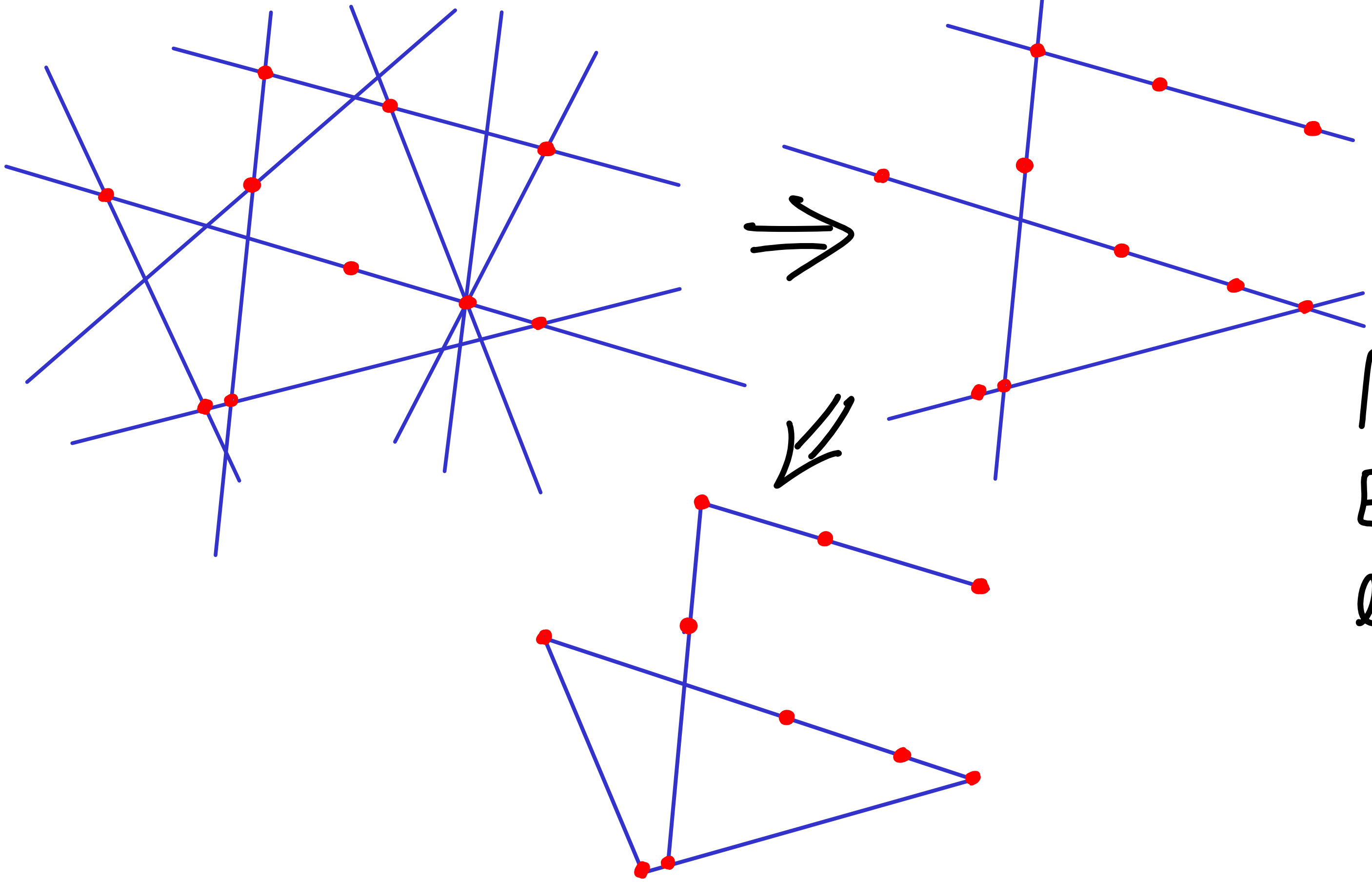
...

Szemerédi-Trotter thm: $I = O\left(n + m + (mn)^{2/3}\right)$

In fact for all m,n, there are examples matching this asymptotically.

Proof on wiki

Keep only lines w/ ≥ 3 points



(All other lines contribute $\leq 2m$ to I.)

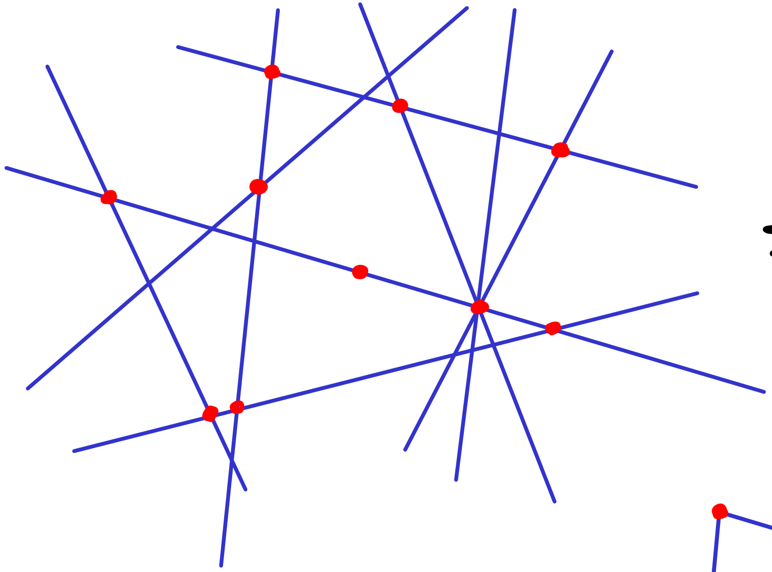
Make plane graph $G = (V, E)$

$E \geq I'/2$ because every line contains ≥ 2 edges in E.

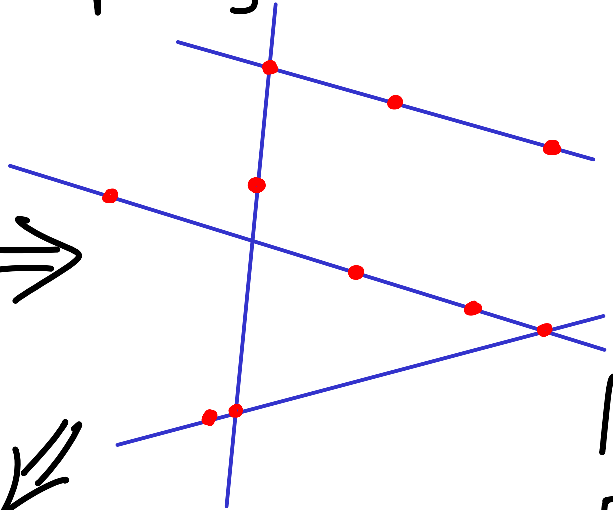
I': incidences that were kept

Proof on wiki

Keep only lines w/ ≥ 3 points



$\Rightarrow$



(All other lines contribute $\leq 2m$ to I.)

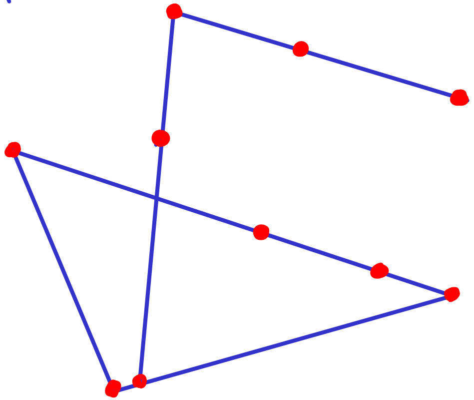
$\downarrow$

Make plane graph $G=(V,E)$

$E \geq I'/2$ because every line contains ≥ 2 edges in E.

So it suffices to upperbound E.

$\Swarrow$



- Trivial : $cr(G) \leq m^2 \rightarrow m^2 \geq \frac{1}{64} \cdot \frac{E^3}{V^2} \rightarrow E = O(m^{2/3} n^{2/3})$

should have added the +m+n part in the big-O

An equivalent statement is:

Given n points, you can only draw $O\left(\frac{n^2}{k^3}+\frac{n}{k}\right)$ lines s.t. each line contains $\geq k$ points. $(k \geq 2)$

Trivially true for any constant k: statement becomes $O(n^2)$ & we can only draw $\binom{n}{2}$ lines even if $k=2$.

If you can draw $\underline{m}$ lines, each containing k points, then you generate $m \cdot k$ incidences, i.e. $I = mk$.

m doesn't dominate in $O(\ldots)$

We know $I = m \cdot k = O(n^{2/3}m^{2/3} + m + n)$ and $mk = \omega(m)$.

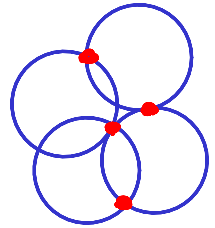
If $n > n^{2/3}m^{2/3}$, $m = O\left(\frac{n}{k}\right)$ /else/ $m^{1/3} = O\left(\frac{n^{2/3}}{k}\right) \rightarrow m = O\left(\frac{n^2}{k^3}\right)$ $\square$

Given n points, you can only draw $O\left(\frac{n^2}{k^3}+\frac{n}{k}\right)$ lines s.t.
each line contains $\geq k$ points. $(k \geq 2)$

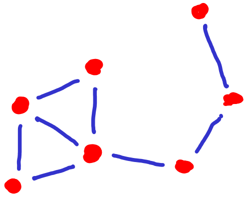
try w/ $k = \frac{n}{c}$ $\quad$ (c = const.)

project: $\quad$ similar point-line theorems
& general position

The same upper bound holds for points & unit circles.

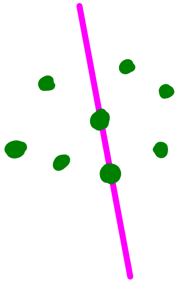


The crossing number result led to other results



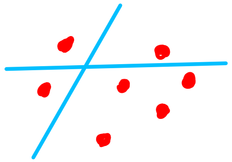
e.g. (1) # unit distances among a set of points $= O(n^{4/3})$

(2) # halving lines in any set of $2n$ points $= O(n^{4/3})$



↳ line through 2 points, splitting others in half.

General result: $O(nk^{1/3})$ "k-sets" → separating k points



$k=2$

Dual result: Complexity of k-level in arrangement of lines.