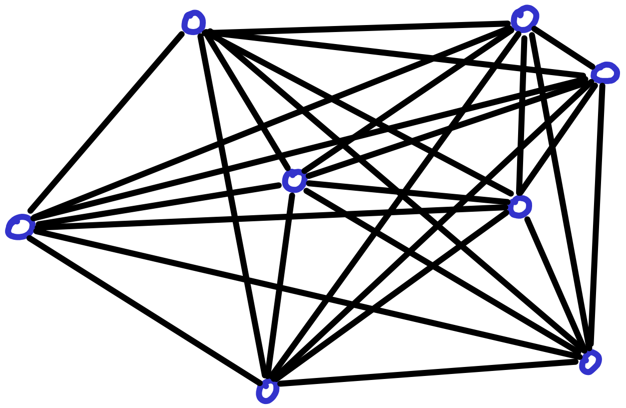
$K_n = (V, E)$: a network allowing efficient communication

... but it is expensive : $\binom{n}{2}$ edges



$K_n = (V, E)$: a network allowing efficient communication

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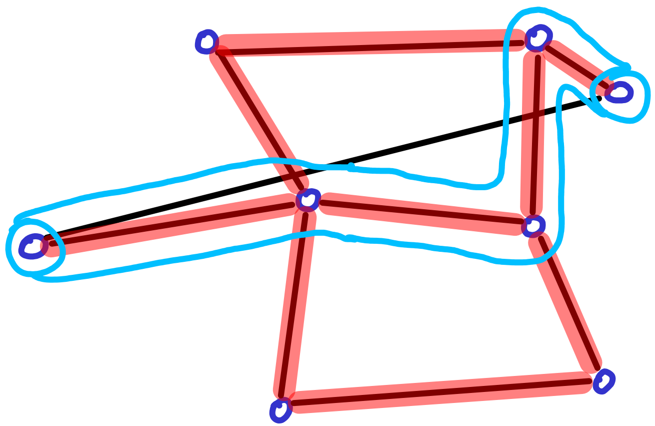


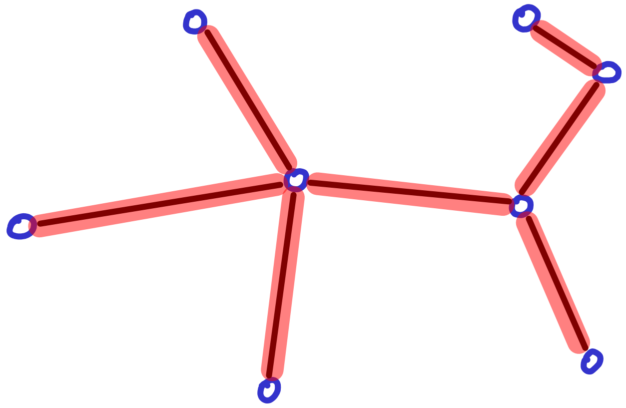
Can we use a (less costly) subset $G = (V, E')$ that still ensures reasonable communication?

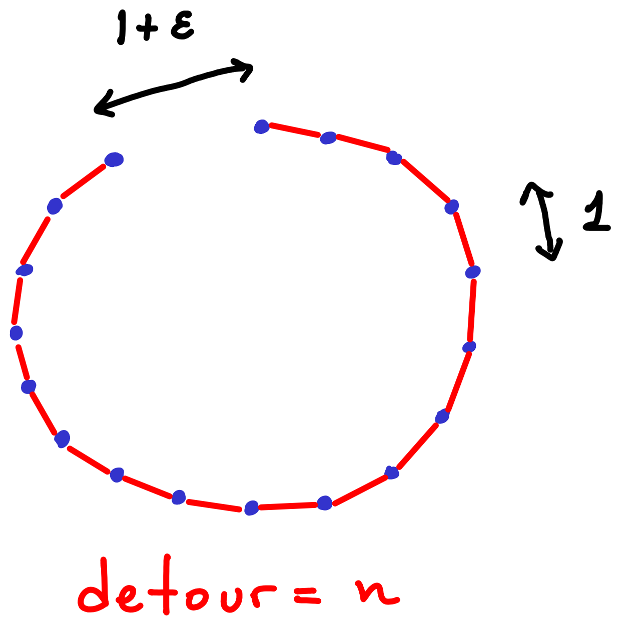
Measuring

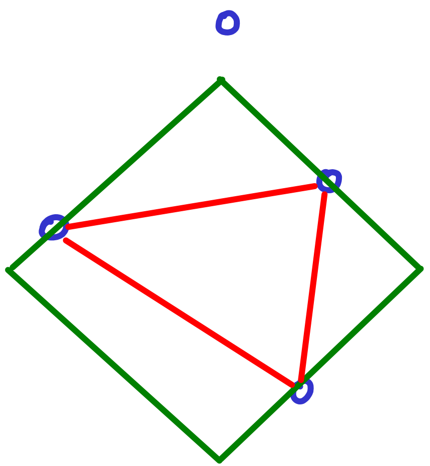
- cost → # edges
- "reasonable" → (max) detour → $\dfrac{d_G(a,b)}{d_{K_n}(a,b)}$ → shortest path vs Euclidean

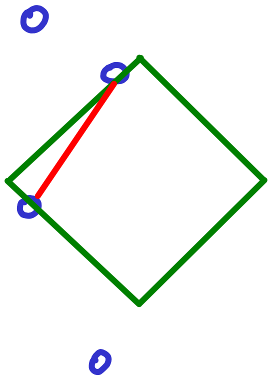
over all pairs of vertices a,b

Does the MST give a good detour ratio? <u>NO</u>

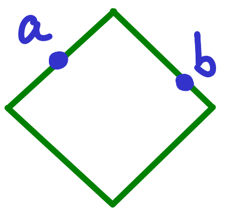




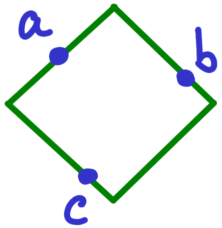


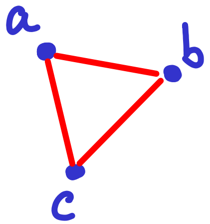


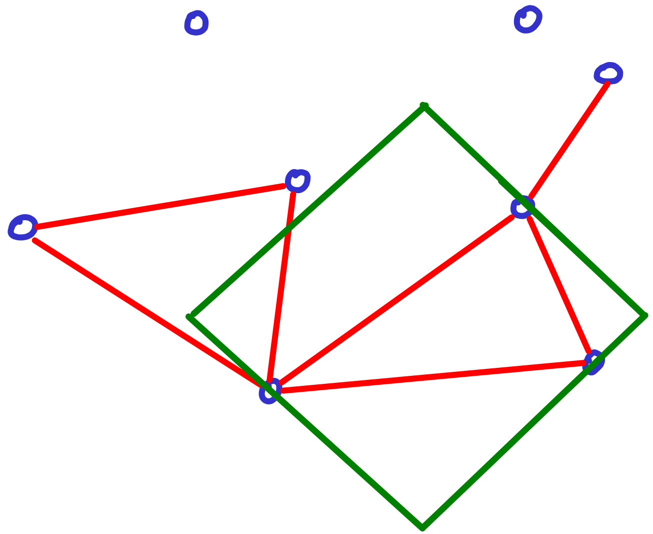
T_{L_1} : keep any edge $\overline{a,b}$ iff a,b are on some empty "diamond"



↳ square tilted 45°

Notice if 

then 



T_{L_1} : keep any edge $\overline{a,b}$ iff

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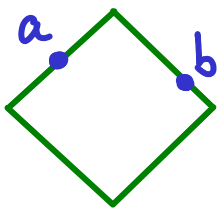

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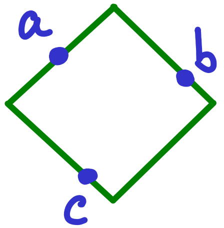

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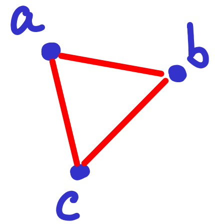

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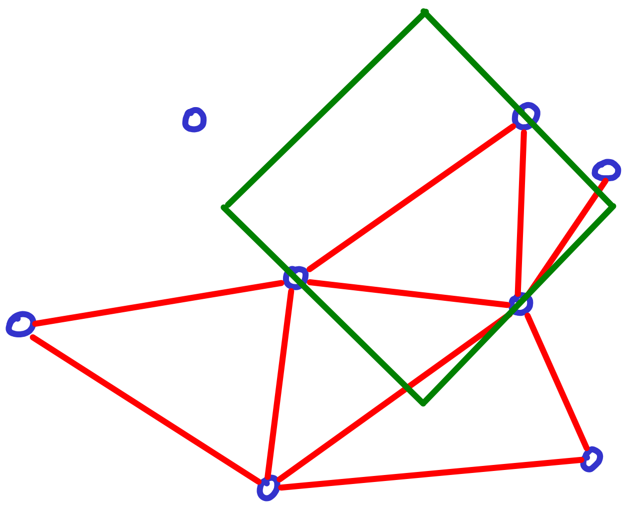
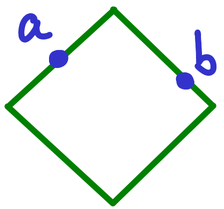


a,b are on some empty "diamond"



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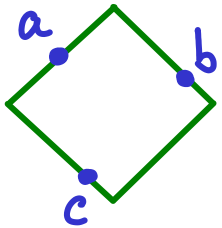

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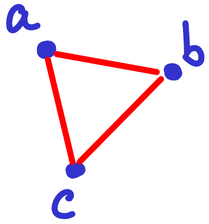

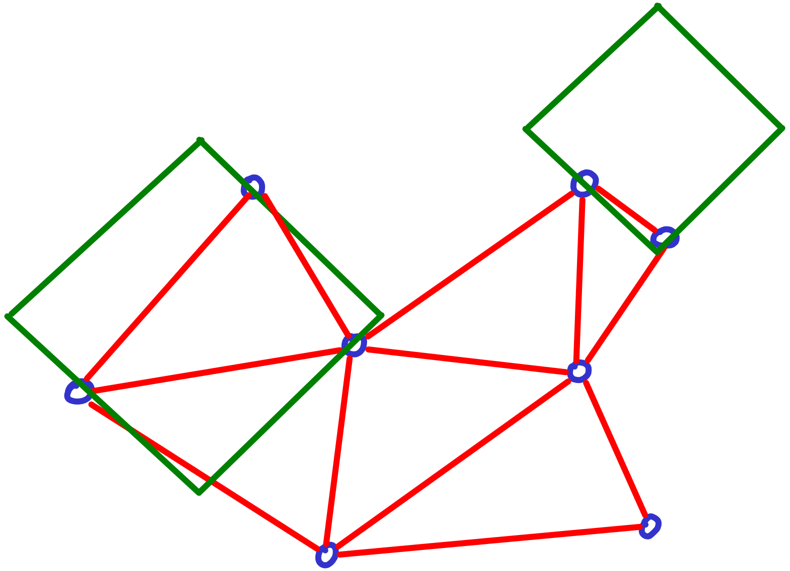

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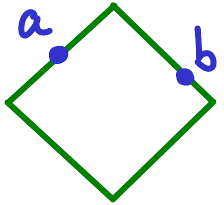

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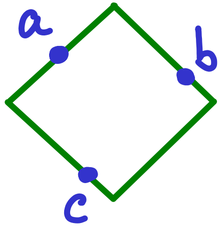
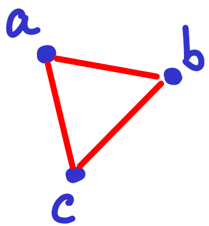
Notice if 

then 

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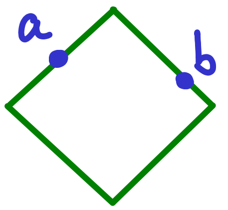

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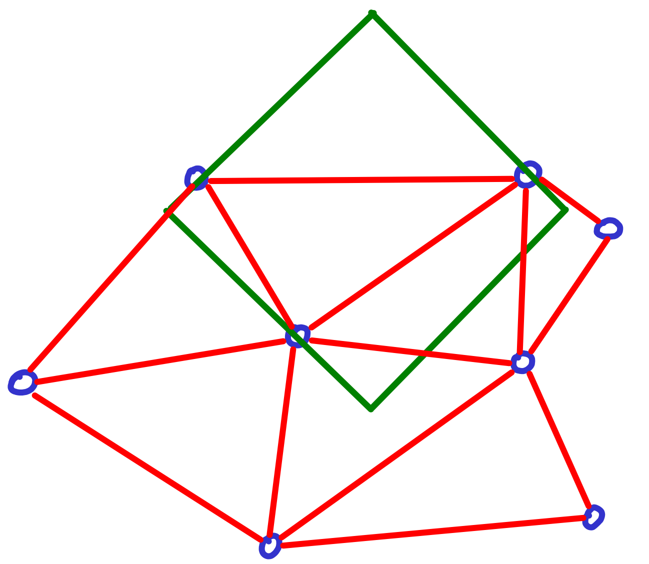

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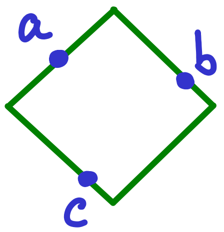

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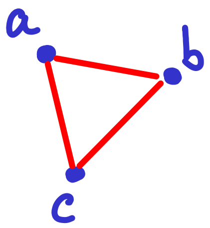

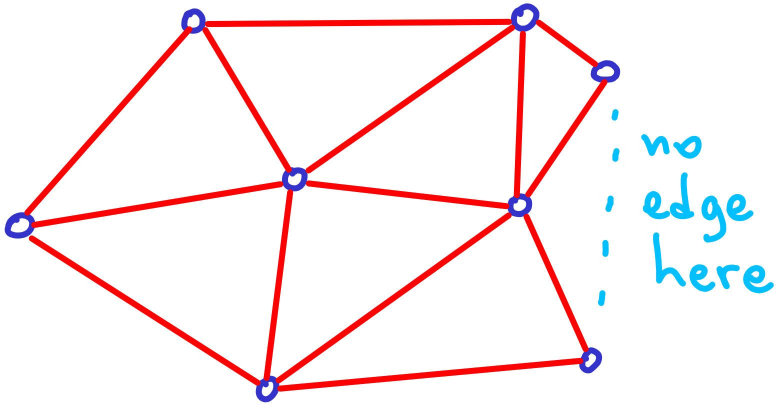

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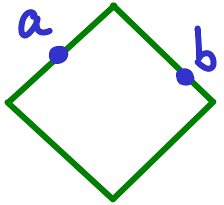

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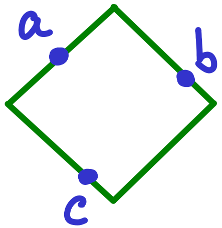

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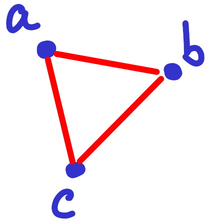

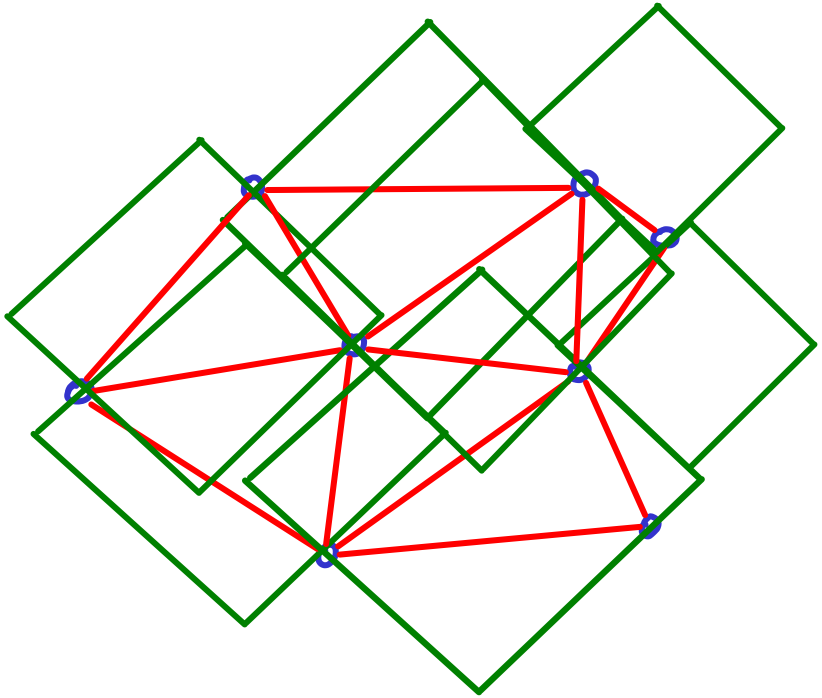

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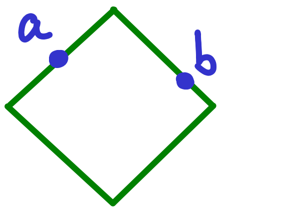
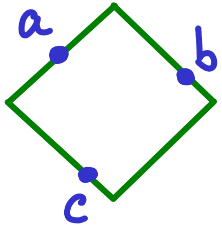
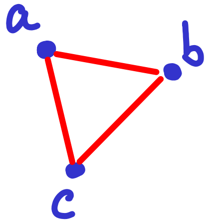


→ square tilted 45°

Notice if

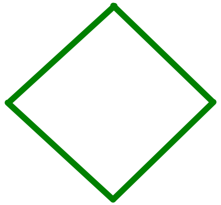


then



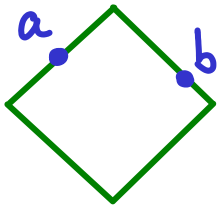
Why T_{L_1} ?

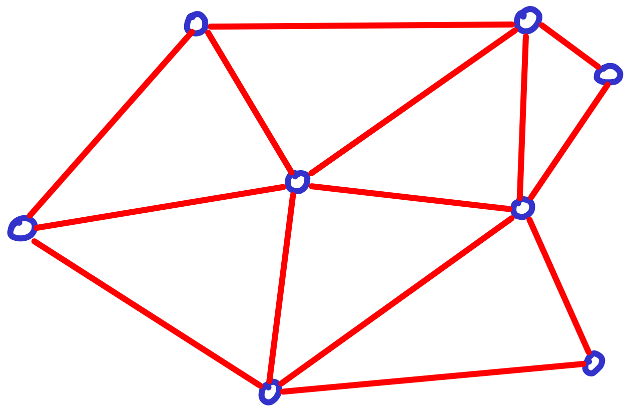
It's a "triangulation"
sort of



is a unit circle in the L_1 metric

T_{L_1} : keep any edge $\overline{a,b}$ iff a,b are on some empty "diamond"





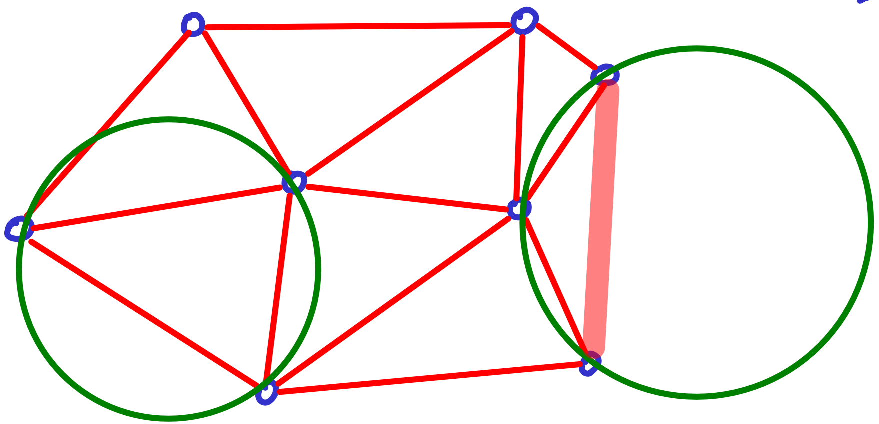
Assume "general position"

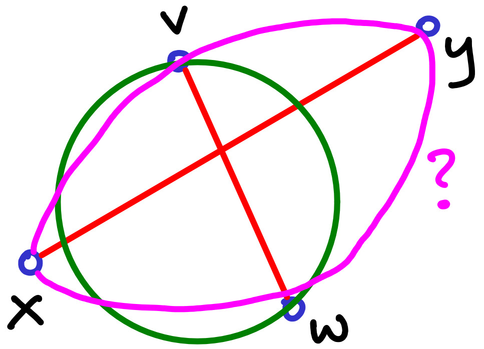
↳ no 4 points on an empty diamond

(it is just a technicality)

T_{L_2} : Delaunay Triangulation

based on regular empty circles





Can 2 edges cross in T_{L_2}? No

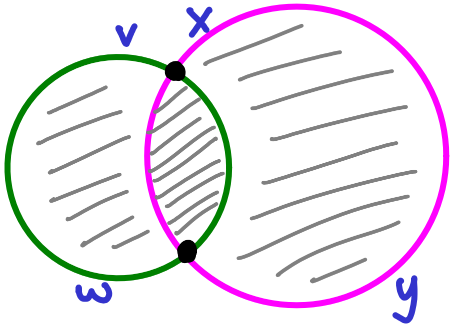
Suppose $\overline{xy}$ crosses $\overline{vw}$

$\exists$ empty circle ○ through v,w

$\exists$ " " ○ " x,y.

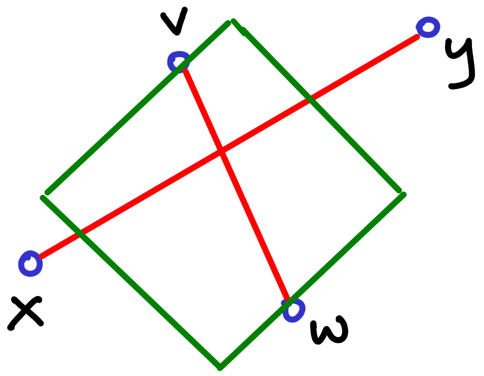
Assuming general position (no 4 on a circle)

○ ○ are distinct & must intersect exacty twice.



No way to place x,y,v,w.

$\square$



Can 2 edges cross in T_{L_1} ? No

Suppose $\overline{xy}$ crosses $\overline{vw}$

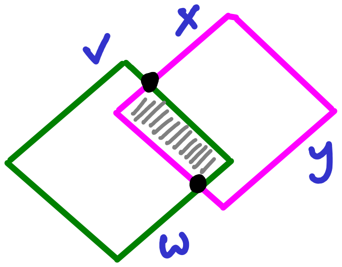
$\exists$ empty "circle" $\Diamond$ through v,w

$\exists$ " " $\Diamond$ " x,y.

Assuming general position (no 4 on a $\Diamond$)

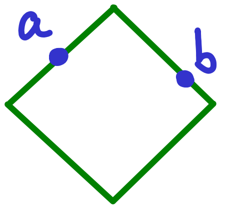
$\Diamond$ $\Diamond$ are distinct & must intersect exacty twice.

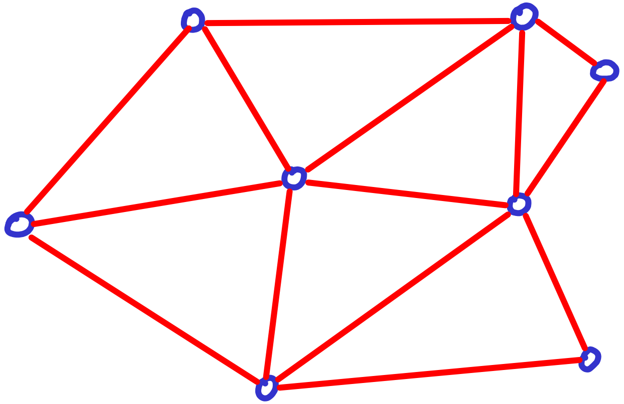
[Same for all convex pseudocircles]



No way to place x,y,v,w.

T_{L_1} : keep any edge $\overline{a,b}$ iff a,b are on some empty "diamond"



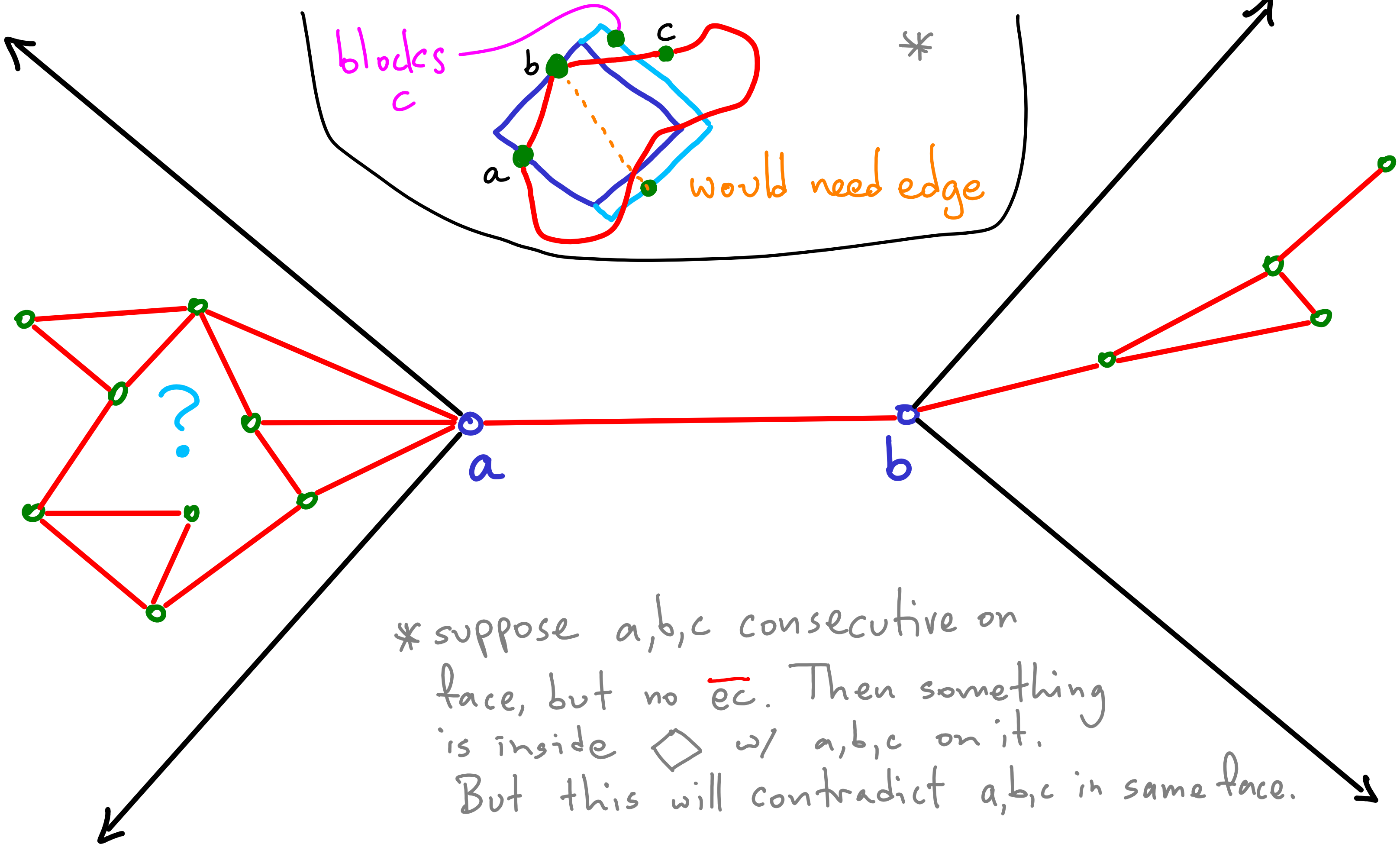


No edges cross : planar graph

↳ $O(V)$ edges.

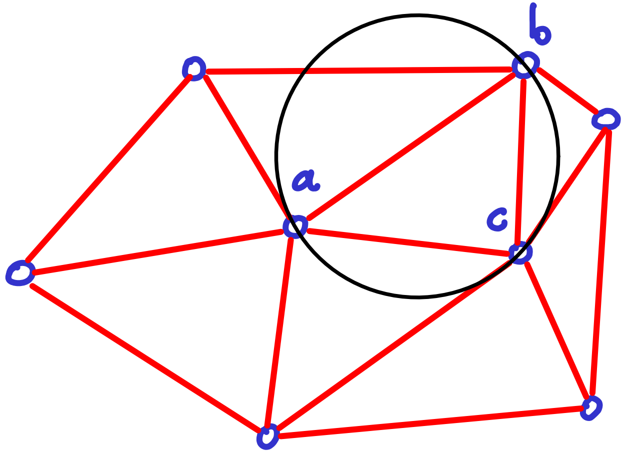
T_{L_1}: Not really a triangulation in the sense that all faces are triangles

Can a bounded face not be a triangle? ... No



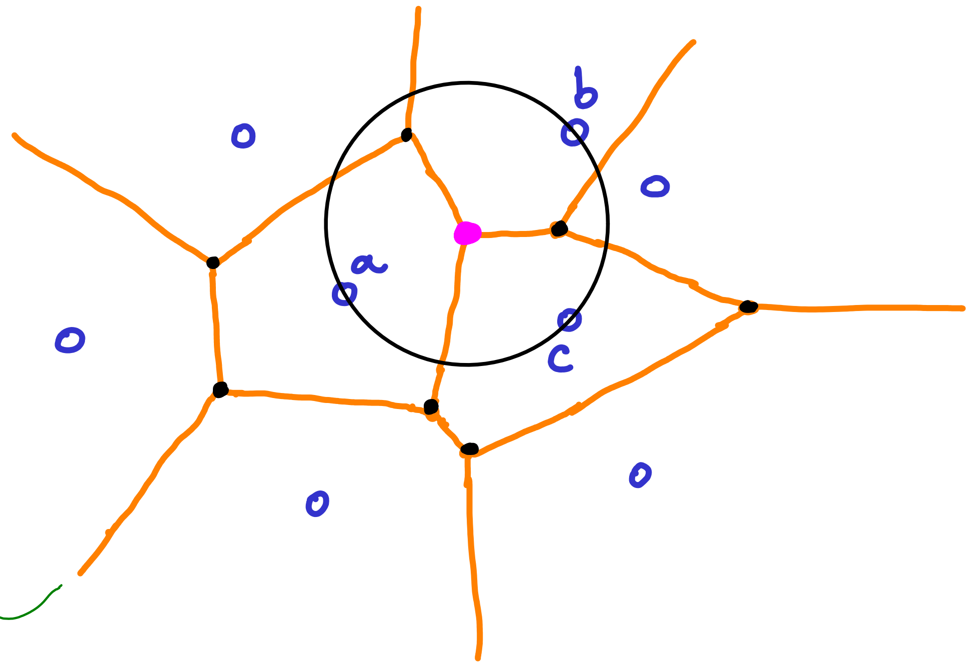
* suppose a,b,c consecutive on face, but no $\overline{ac}$. Then something is inside ◇ w/ a,b,c on it.
But this will contradict a,b,c in same face.

defines regions closest to o sites



Dual of VORONOI DIAGRAM

L_2

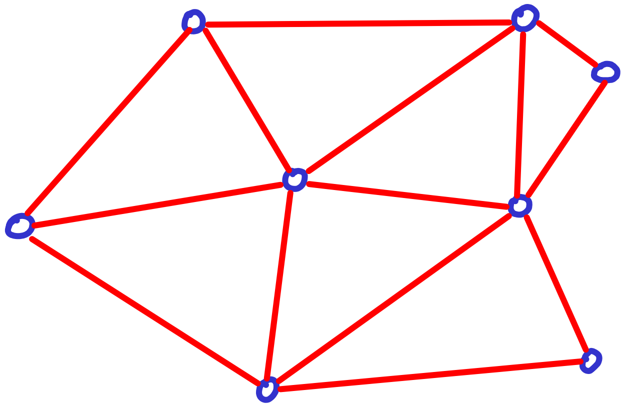


a, b, c form (empty) triangle
because ∃ empty circle on a, b, c

therefore center of circle is equidistant to a, b, c & further from other o

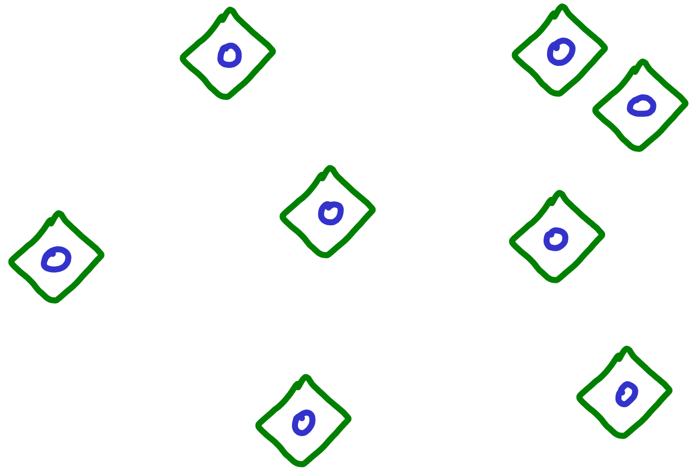
so it is a vertex of

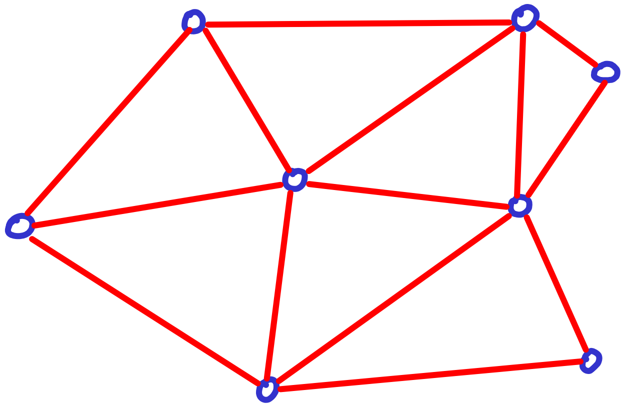
can be constructed by expanding empty "circles"



Dual of VORONOI DIAGRAM

L_1





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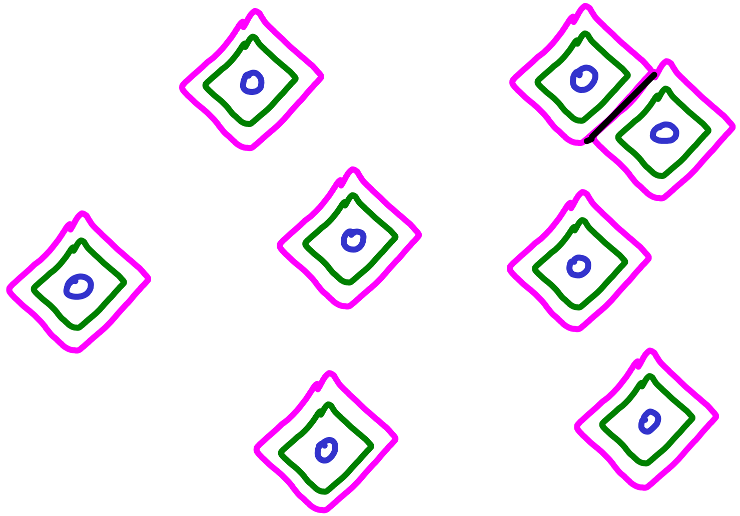

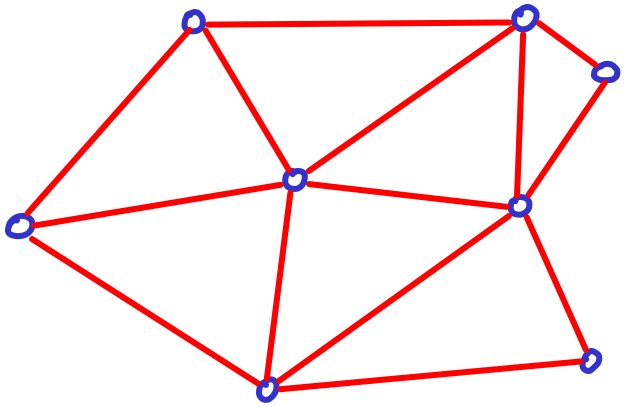

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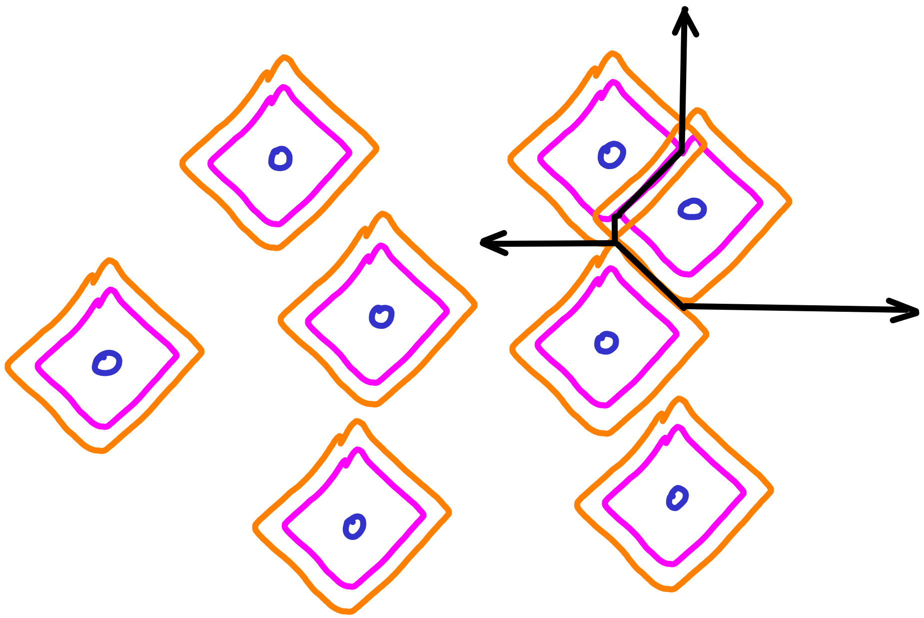

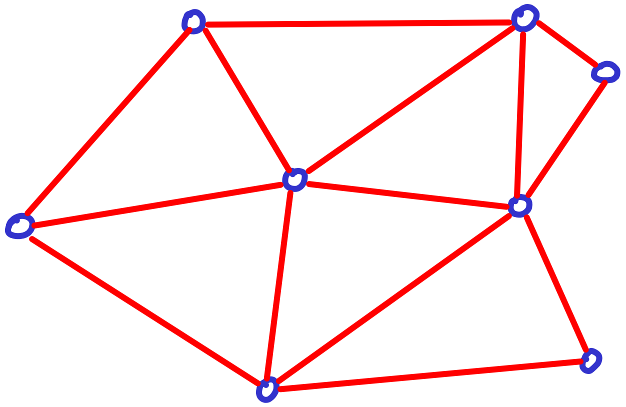

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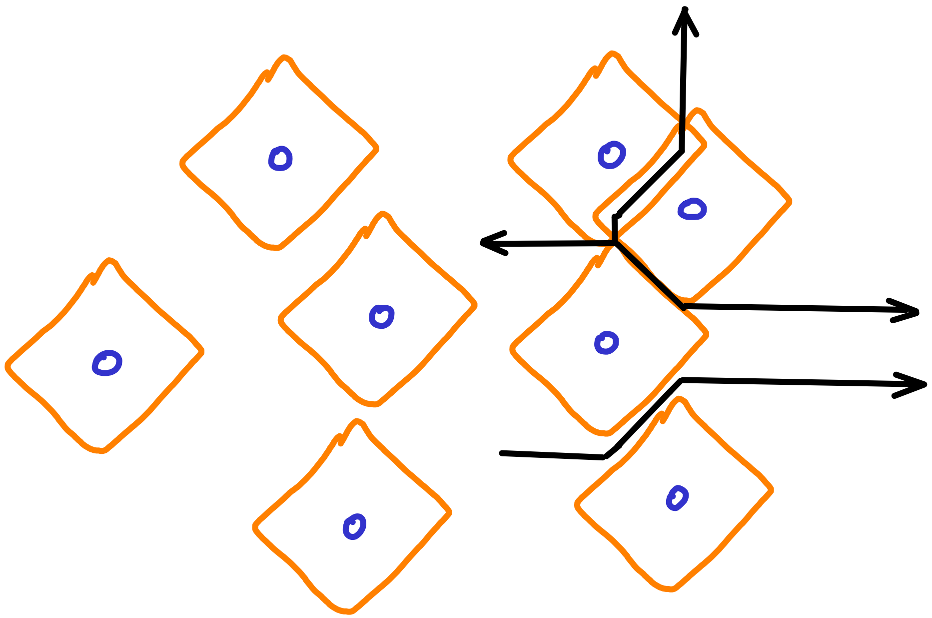
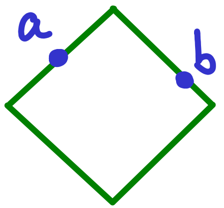
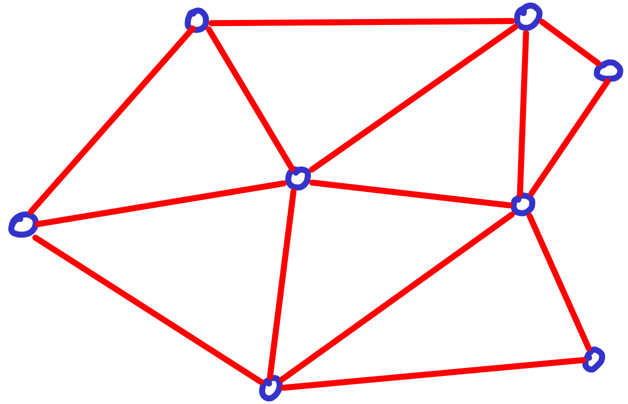


T_{L_1} : keep any edge $\overline{a,b}$ iff a,b are on some empty "diamond"





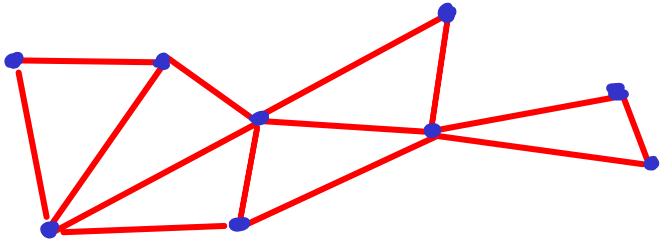
No edges cross : planar graph

↳ $O(V)$ edges.

Claim this gives a worst-case detour of $\sqrt{10}$.

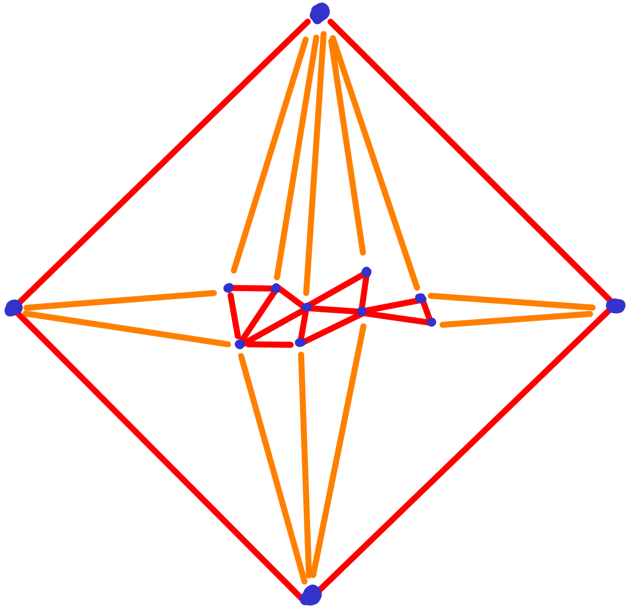
i.e it is a "$\sqrt{10}$-spanner" (t-spanner w/ $t=\sqrt{10}$)

(result by Paul Chew ~1986)



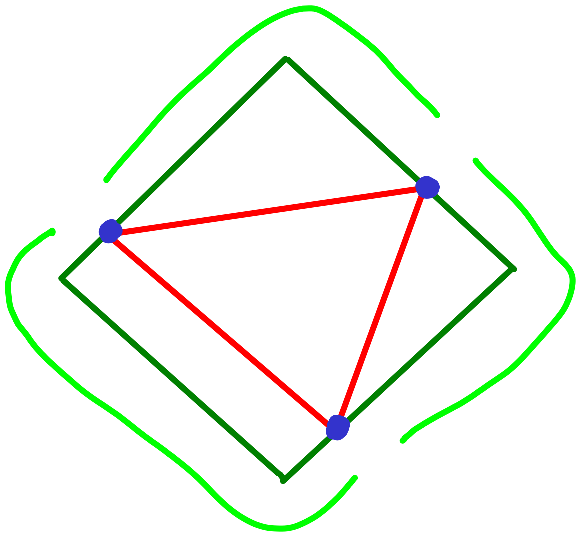
to help w/ proof:

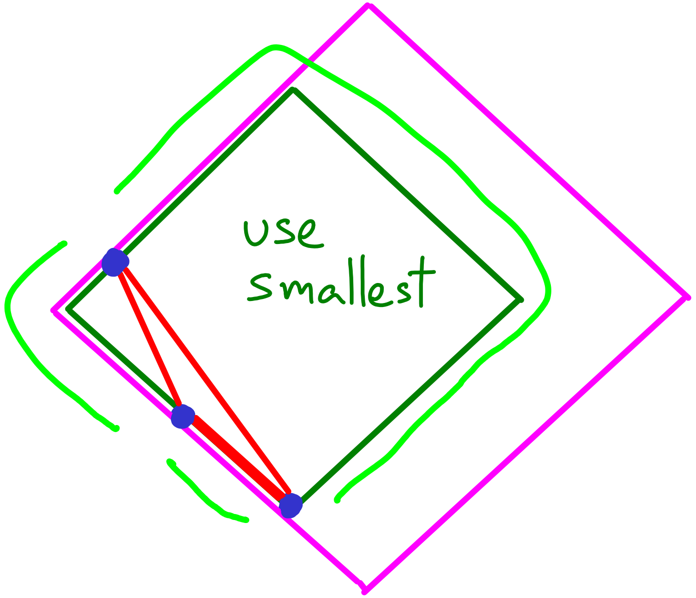
$\xrightarrow[\text{augment}]{}$

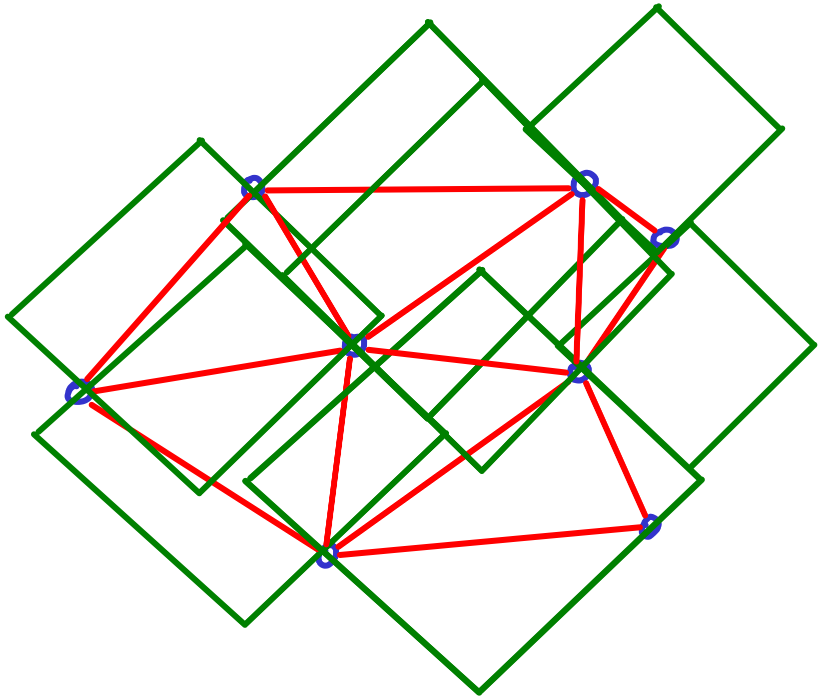


Suppose 2 points u, w have same y-coord:

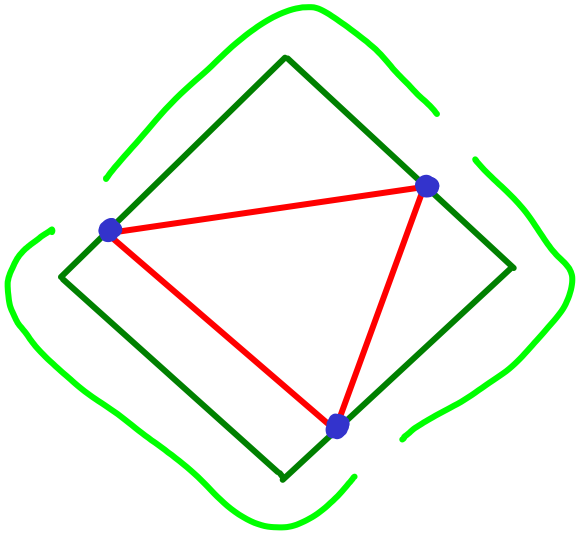
We will show that T_{L_1} is a $\sqrt{8}$-spanner for $\overline{uw}$.

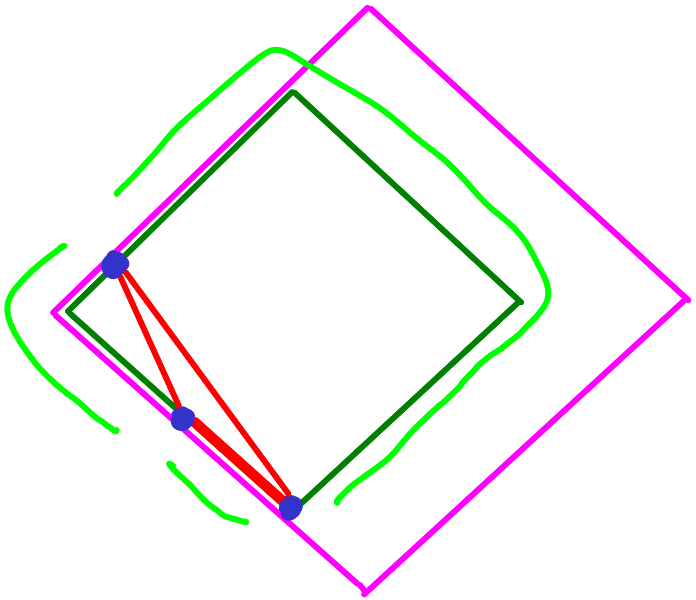




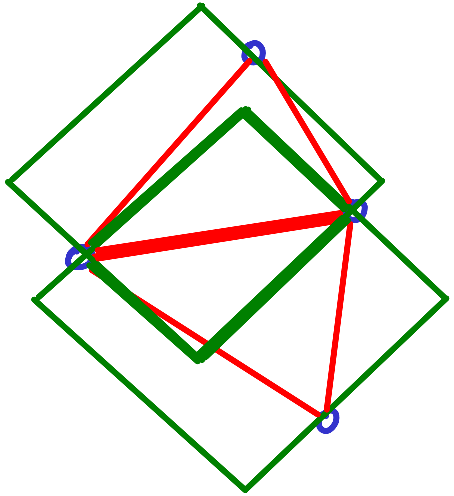


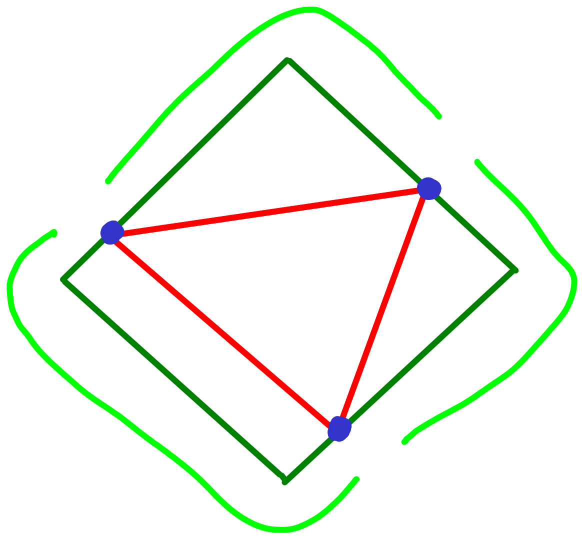
Circle graph

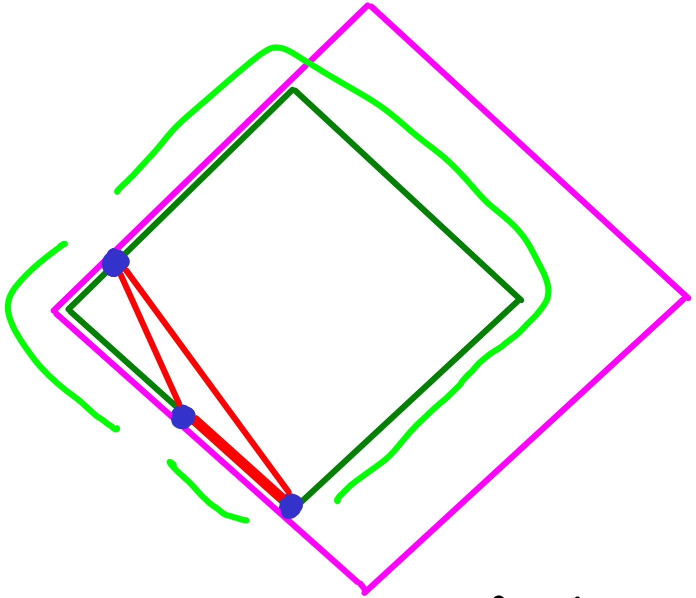




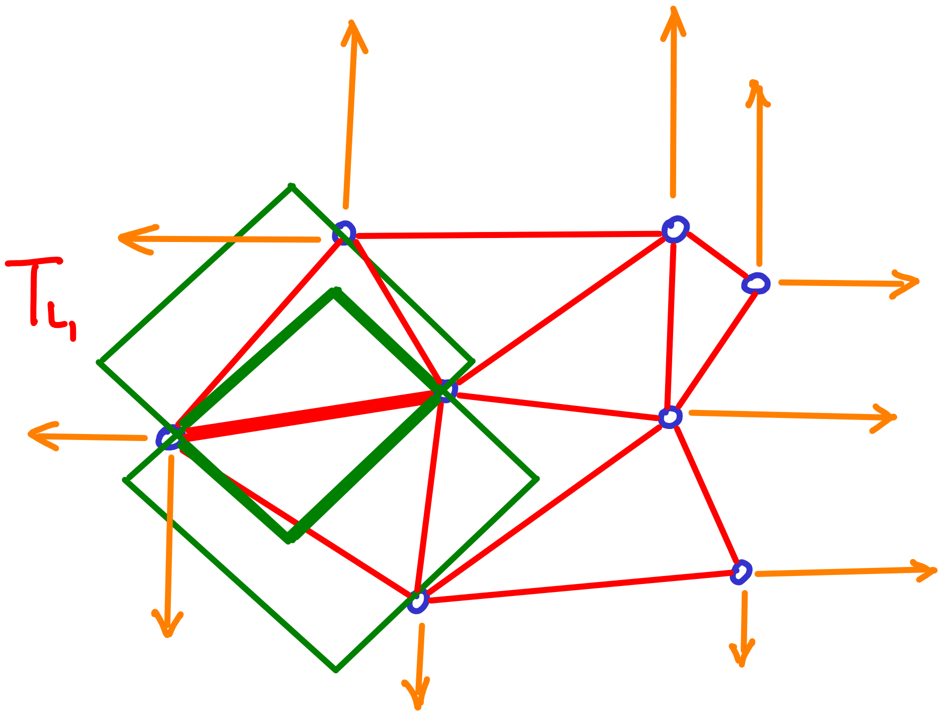
Each edge in T_{L_1} contributes 2 circle graph edges







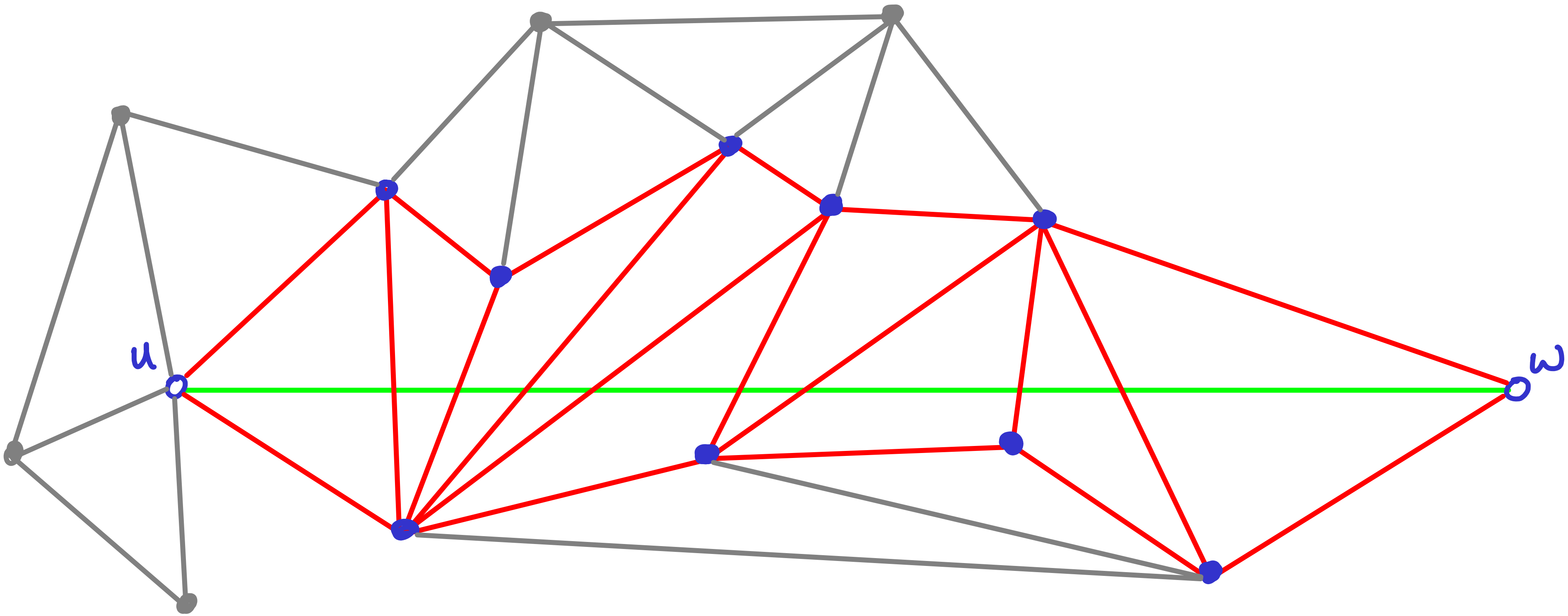
Each edge in T_{L_1} contributes 2 circle graph edges



In fact we will find an $\sqrt{8}$-approximation path in the Circle graph

(using original edges clearly gives shortcuts)

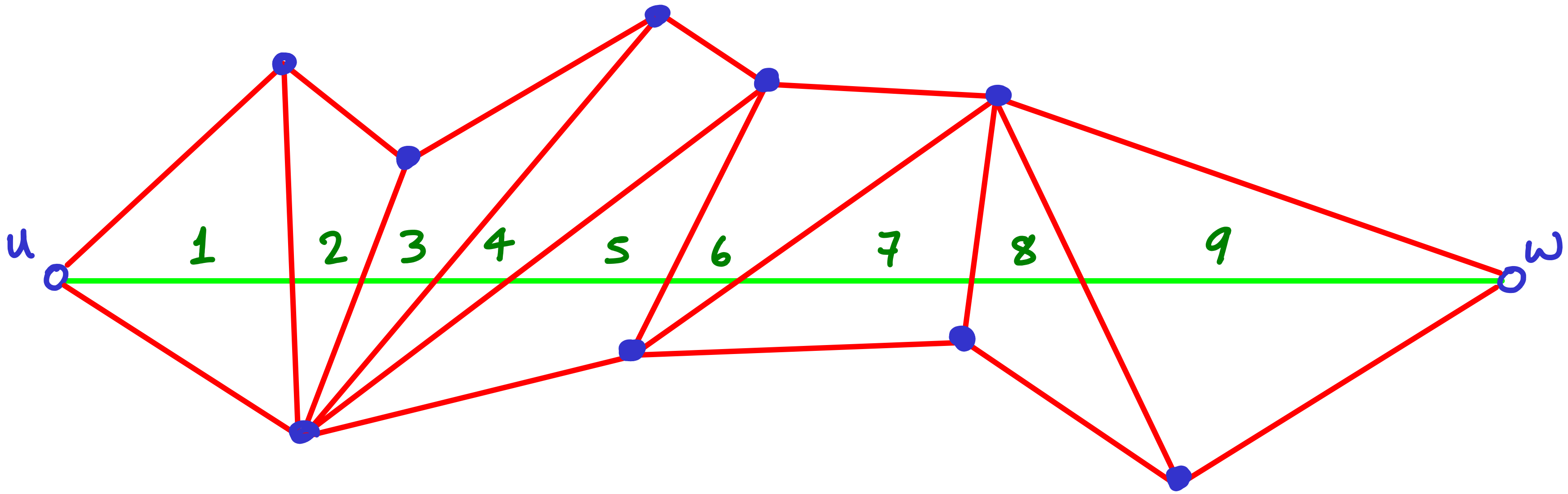
We only need the subgraph of triangles that contain part of $\overline{uw}$



We only need the subgraph of triangles that contain part of $\overline{uw}$

The path that approximates $\overline{uw}$ will only use edges of this subgraph.

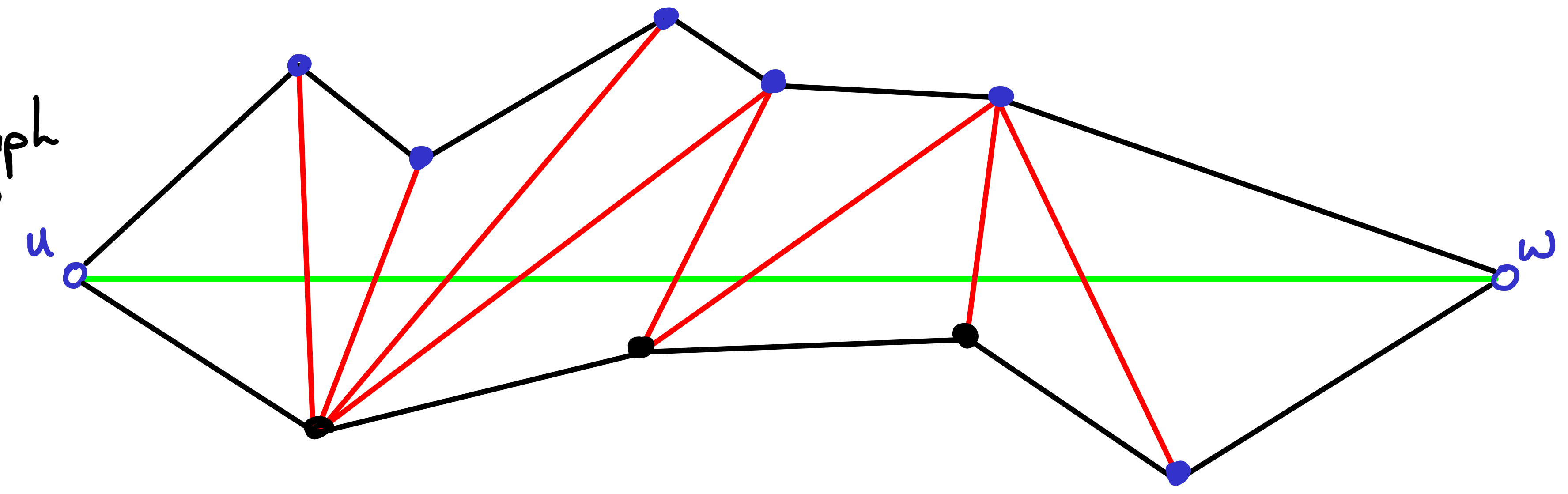
↳ in fact it will make steady progress, visiting triangles in order.



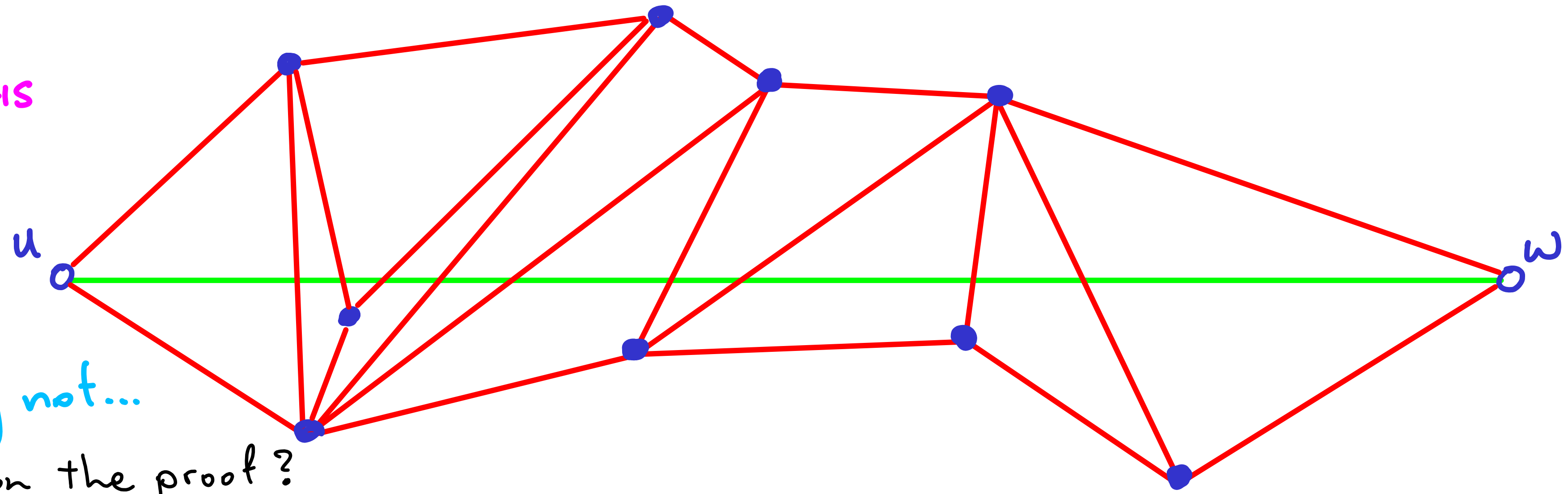
Assume no vertex on $\overline{uw}$ otherwise recurse

Aside:
will our subgraph look like this?

↳ top chain
(links between)
bottom chain



can we get this ?

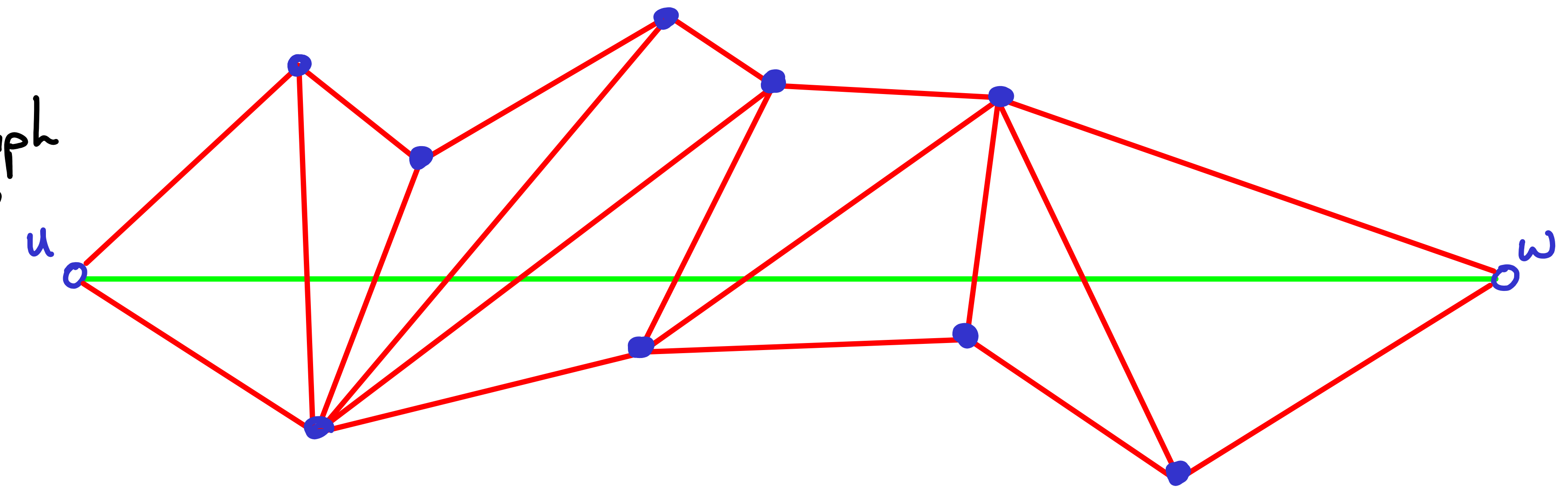


Actually yes, why not...

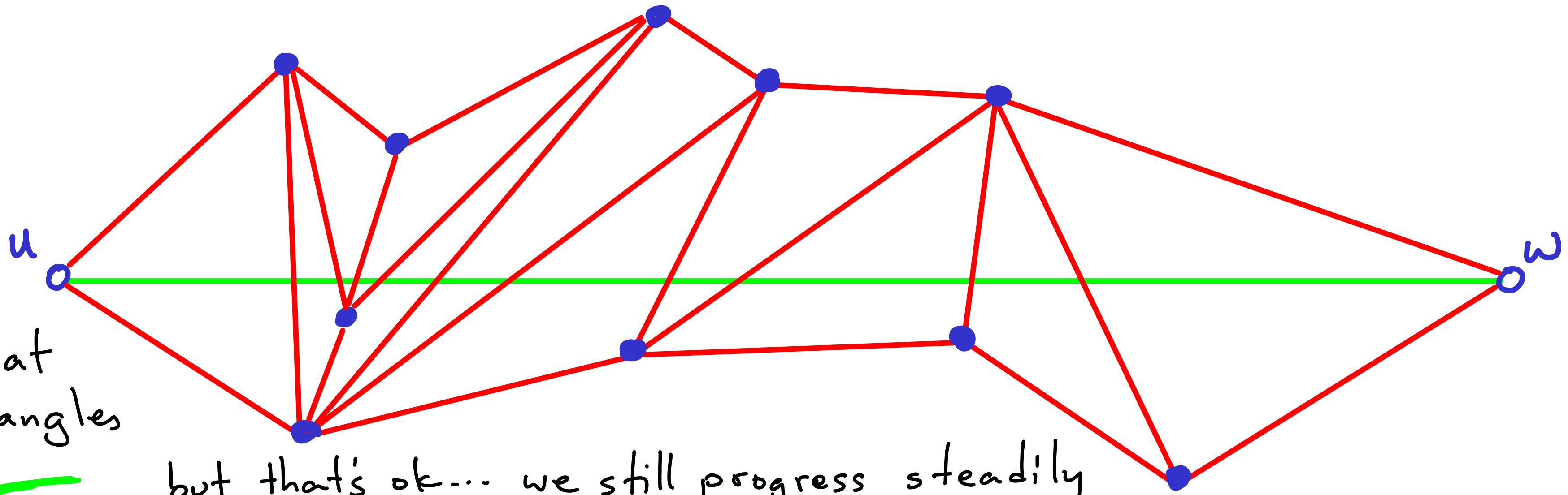
is there any effect on the proof?

Aside:
will our subgraph
look like this?

↳ top chain
(links between)
bottom chain

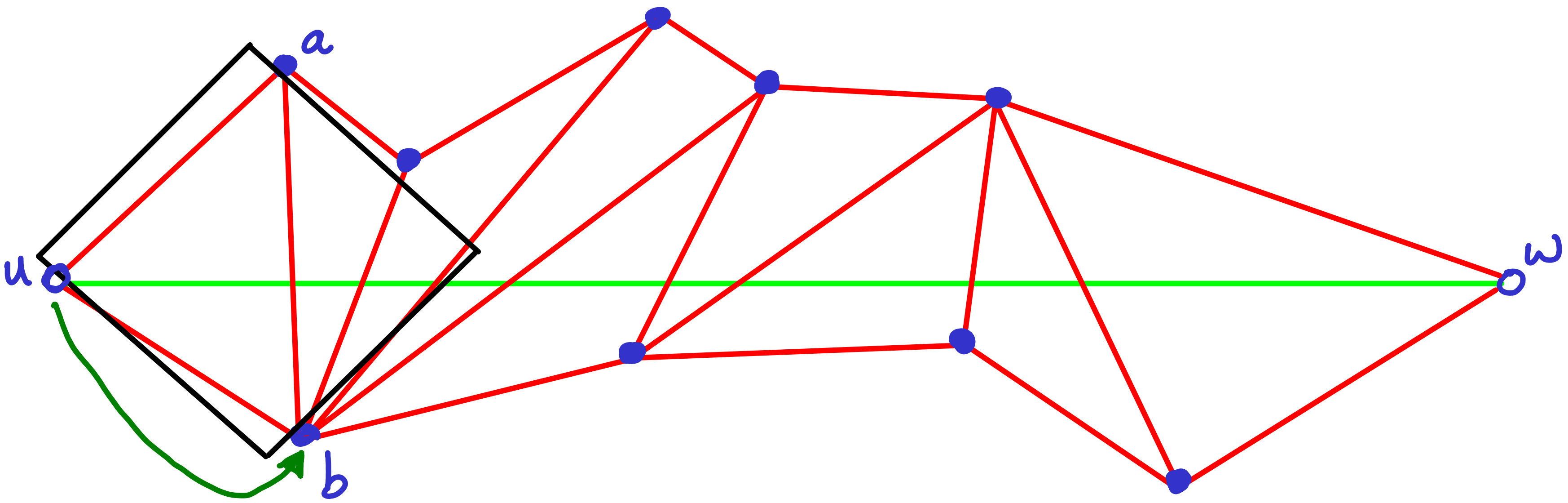


??



this would imply that
we don't visit all triangles
that touch ... but that's ok... we still progress steadily

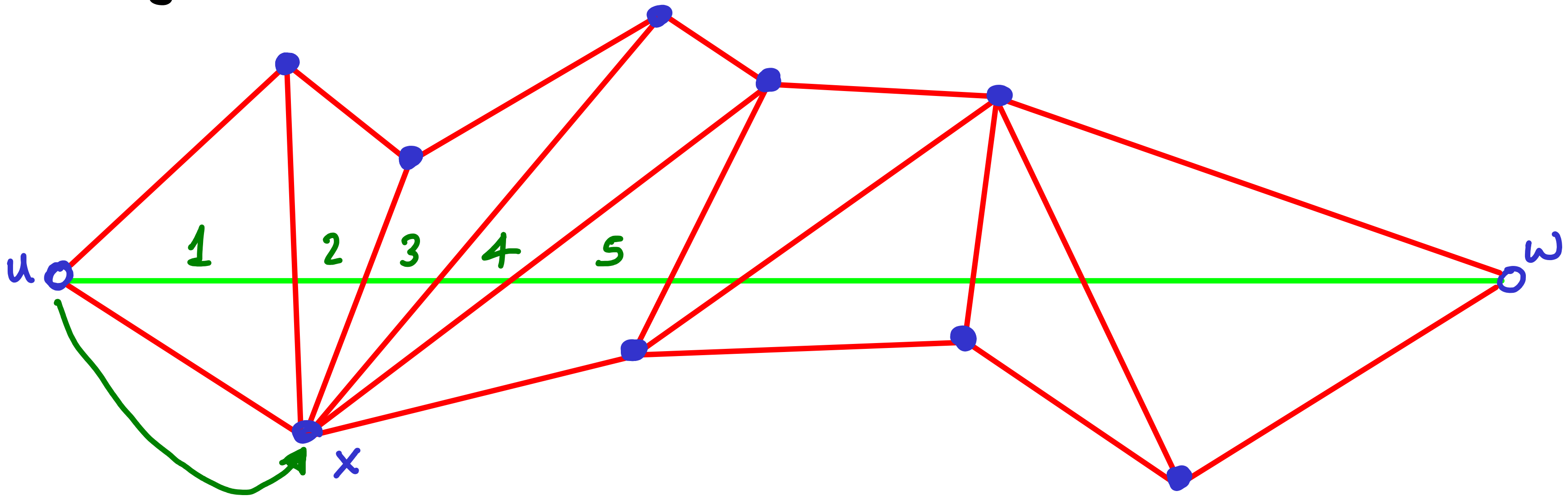
Notice u belongs only to 1 triangle uab, with a above & b below $\overline{uw}$



In fact u is on the ◇ or ◇ side of the empty diamond on uab.

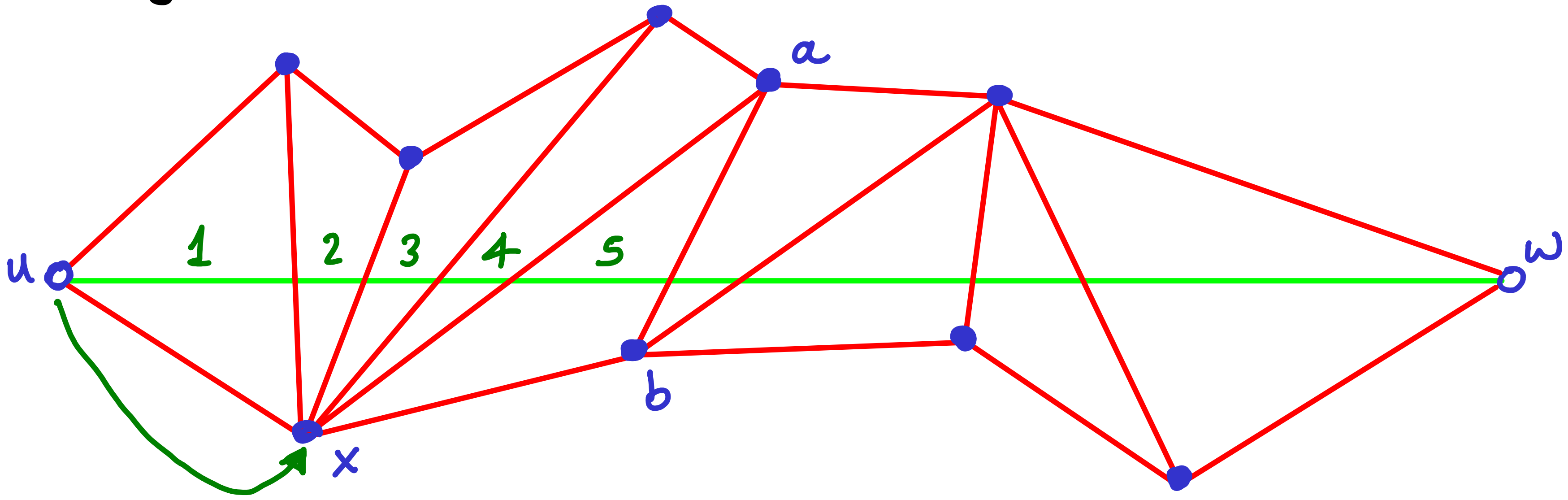
Start our path with u ↪ b [along ◇] because u is on ◇

x belongs to many triangles. We care about the rightmost one.

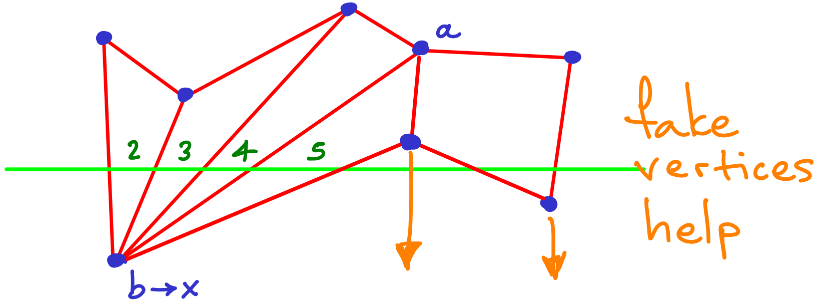


current vertex x

x belongs to many triangles. We care about the rightmost one.

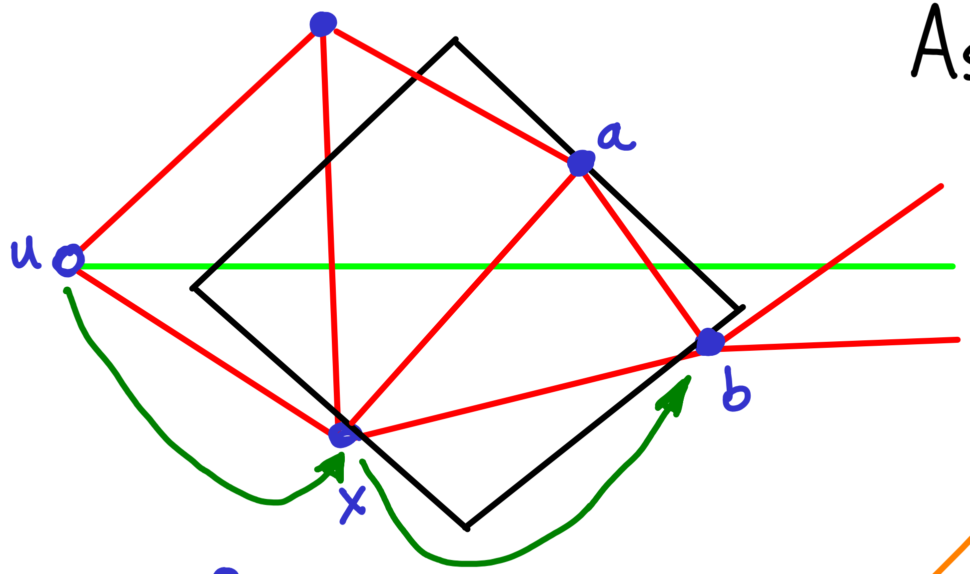


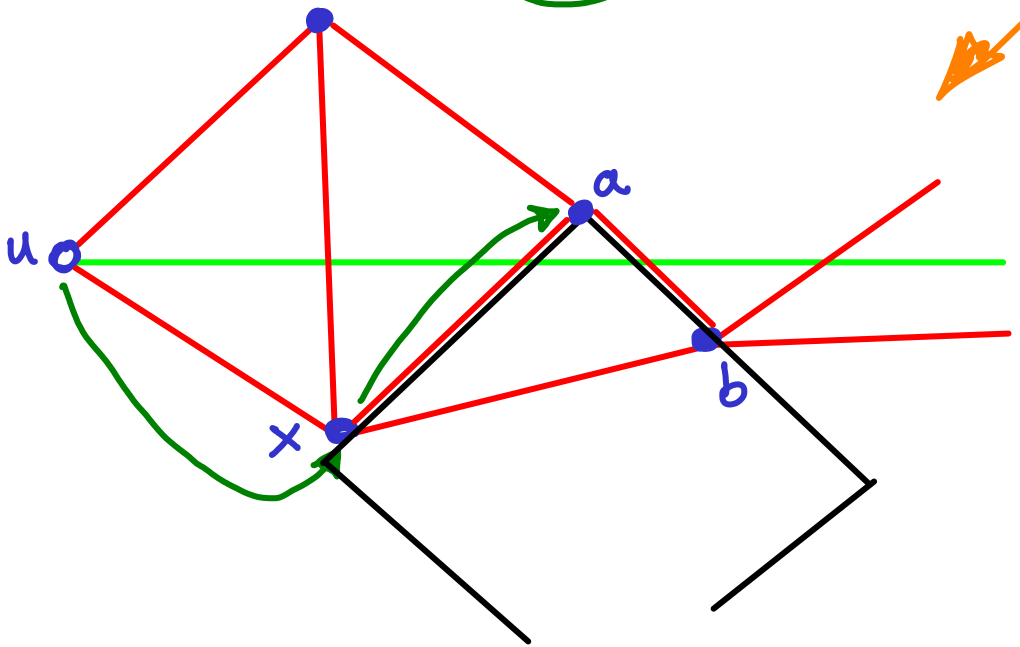
Again we have the property that the current vertex x on our path is in a triangle xab, w/ a above & b below uw



As before, if x is on move

else if x is on move

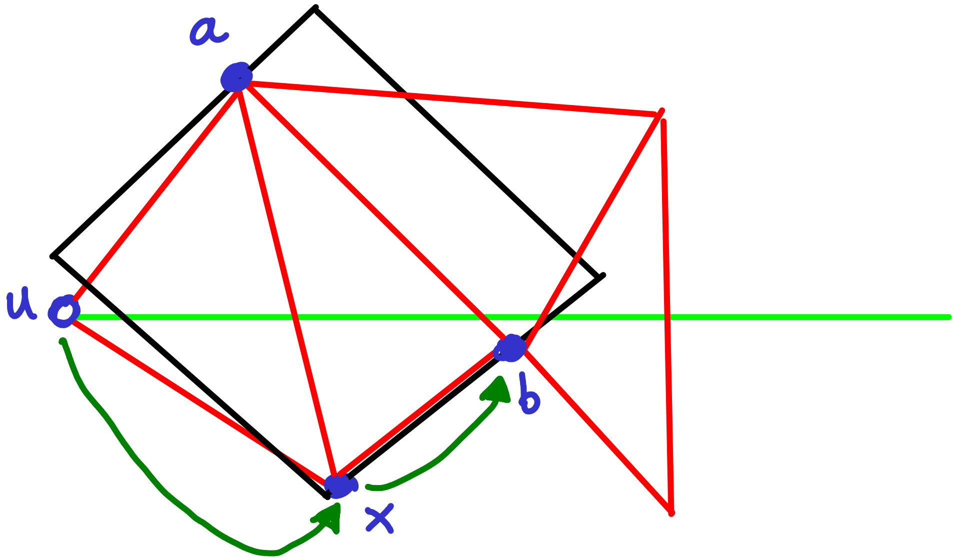




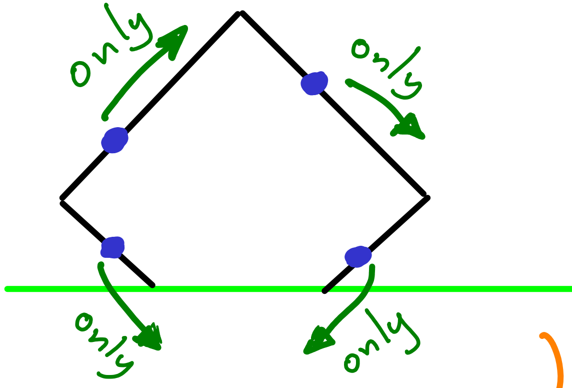
<u>else</u> // x is on

if x is above (below) $\overline{uw}$

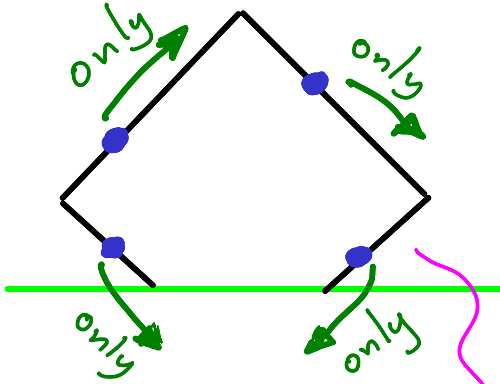
move ( )



when moving above $\overline{uw}$

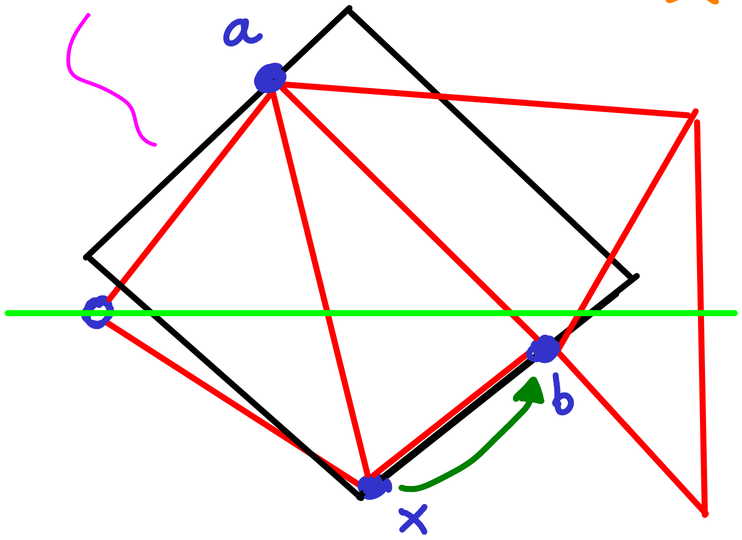


if you start at a • point, we have established direction when moving above $\overline{uw}$

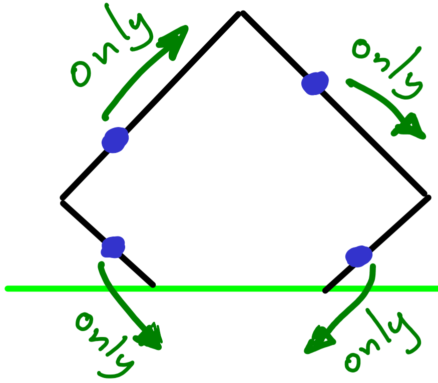


So we can't continue above

Starting from below & ending up above:

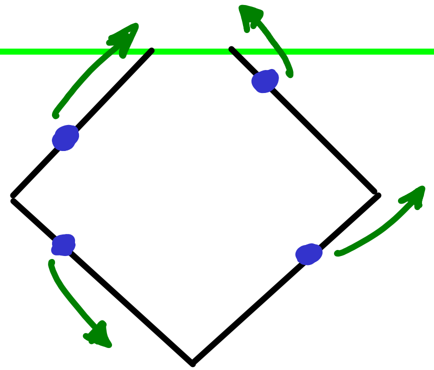


Notice that b must be below $\overline{uw}$ if x is on or

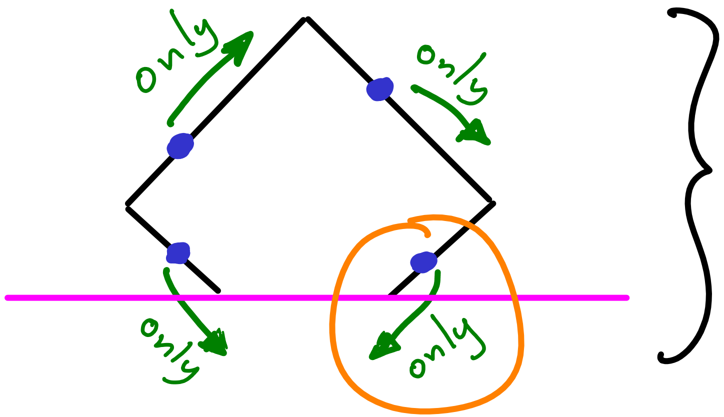


when moving above $\overline{uw}$ $\Bigg\}$ 1. never go $\nwarrow$

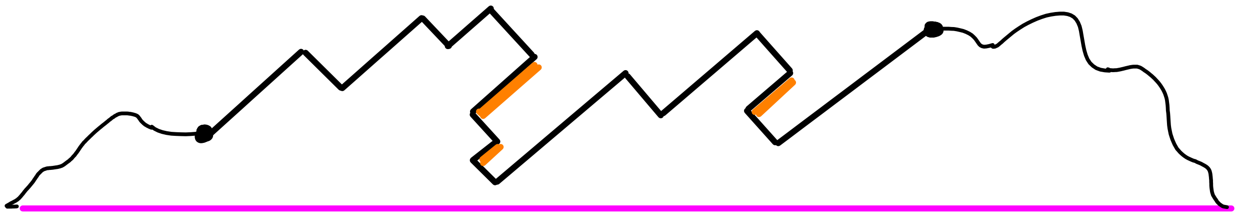
2. go $\nearrow$ iff on



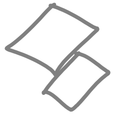
when moving below $\overline{uw}$ //symmetric



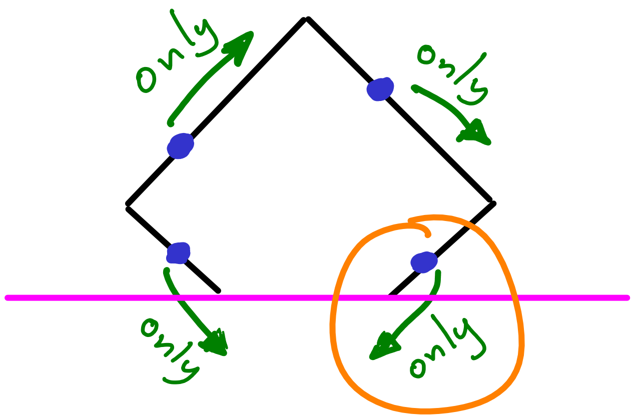
while above...



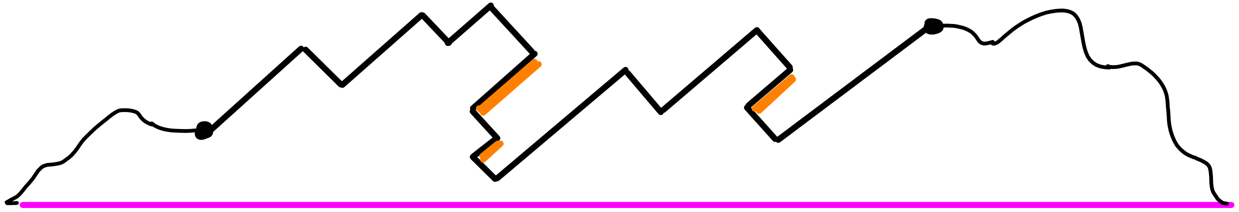
(why can't we switch from ↙ to ↗ ?)

would involve placement of 

s.t. the corresponding triangles inside would not overlap — in proper order.

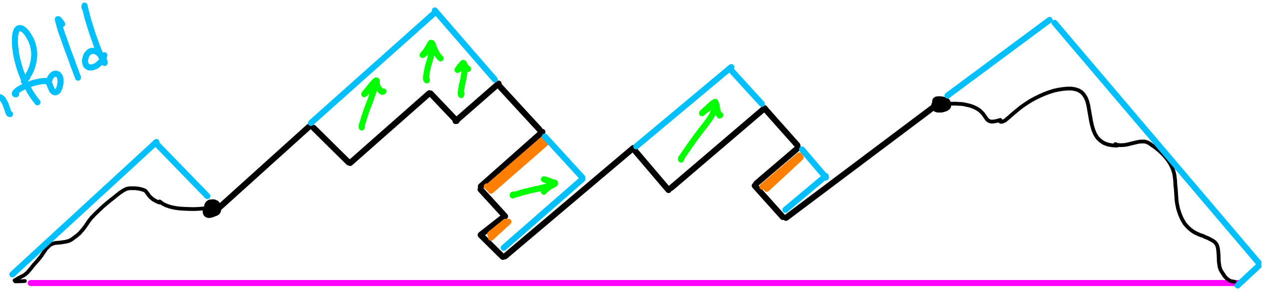


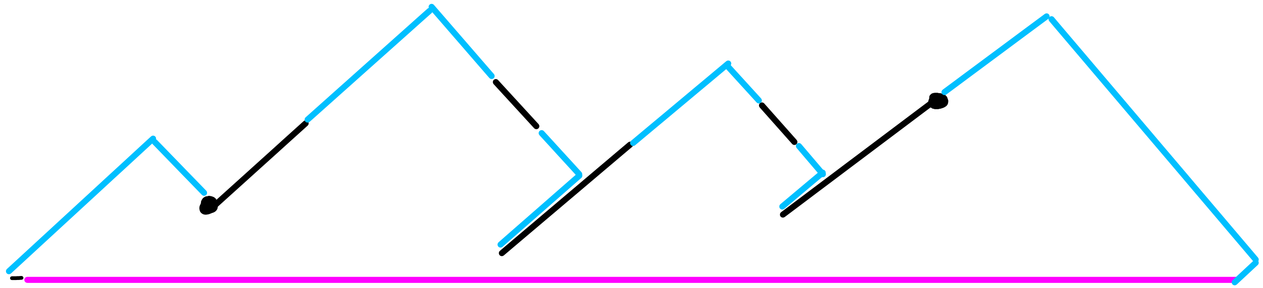
while above...

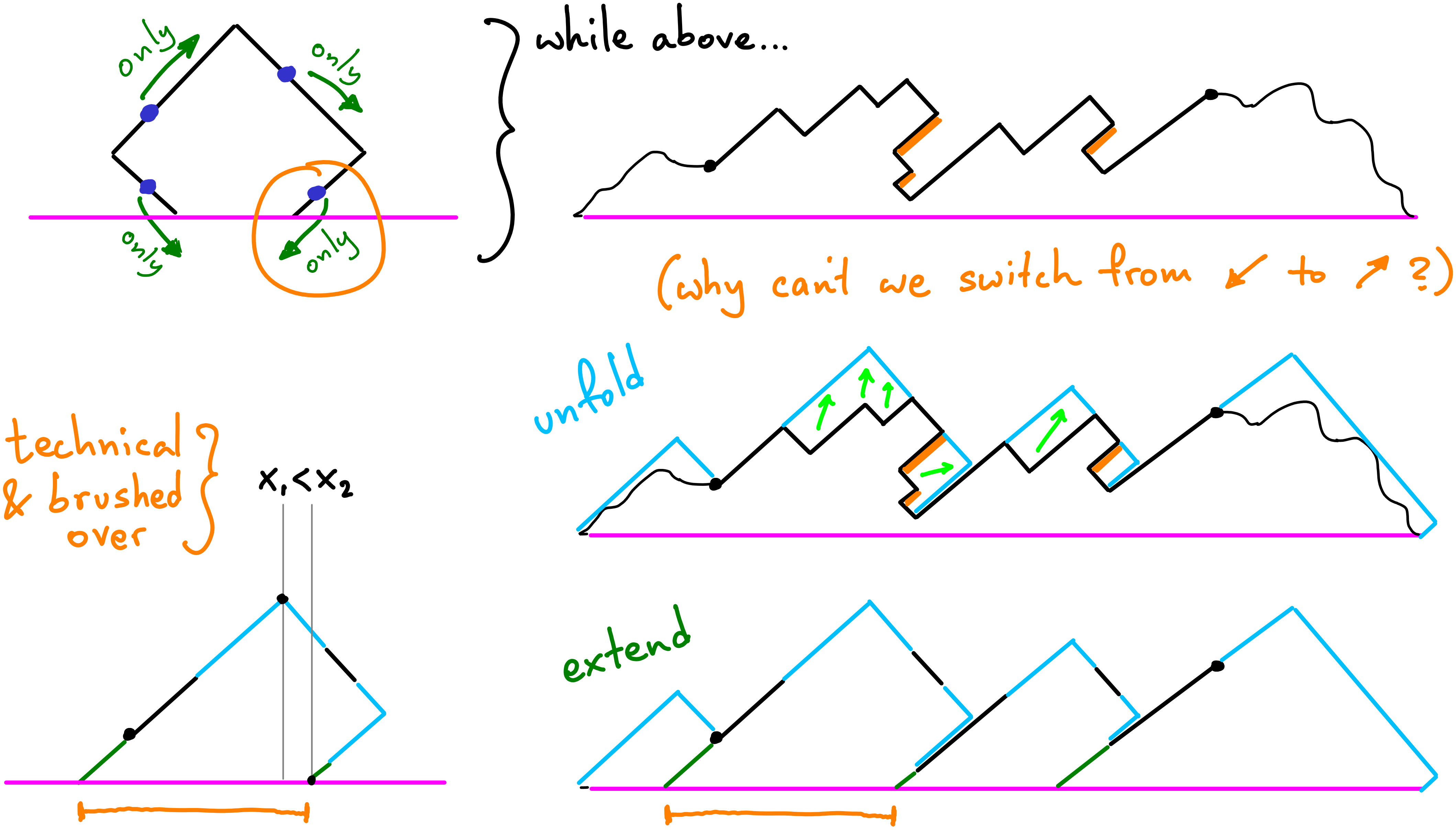


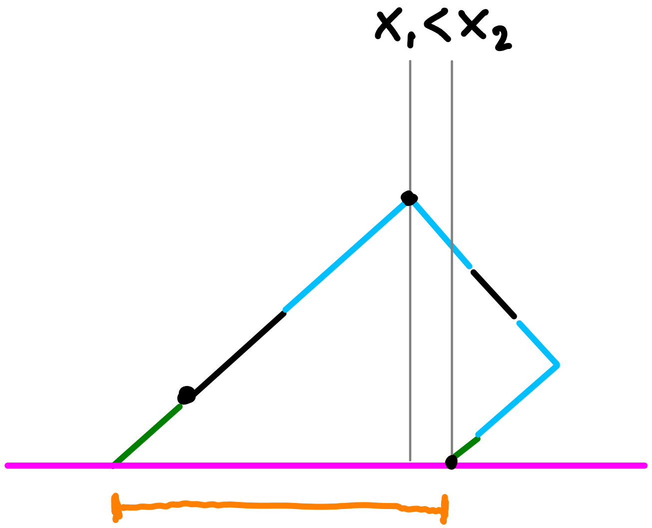
(why can't we switch from ↙ to ↗ ?)

unfold

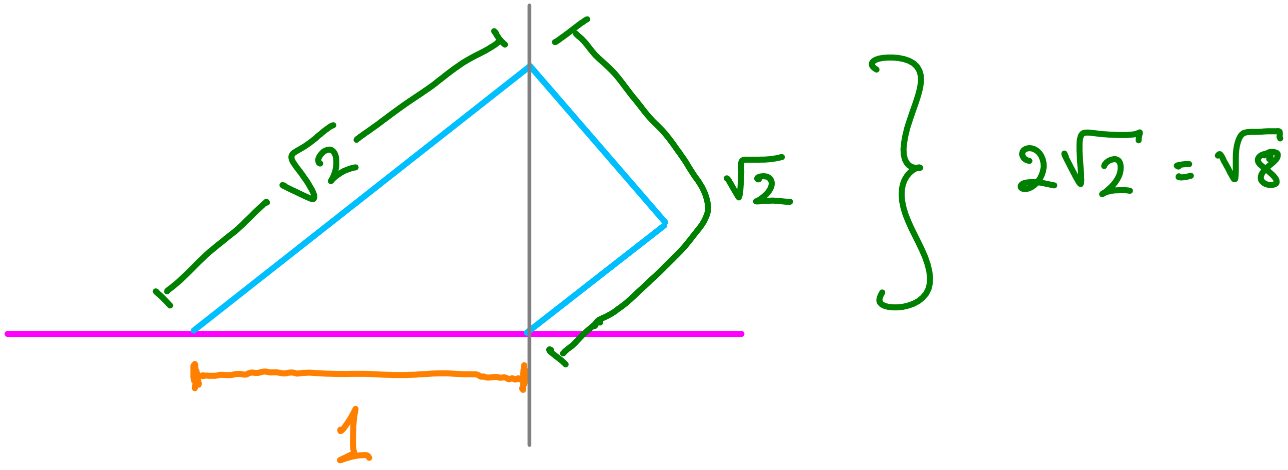


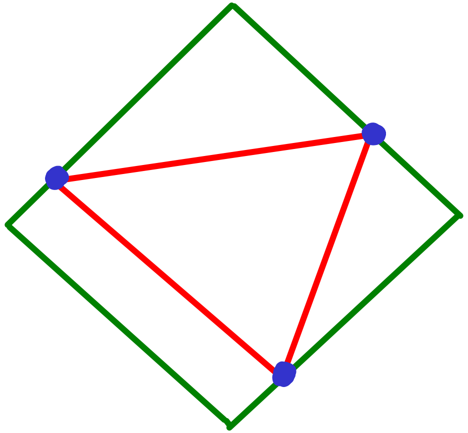


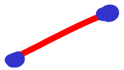

only
only
only
only
while above...
(why can't we switch from ↙ to ↗?)
unfold
technical & brushed over
x₁ < x₂
extend

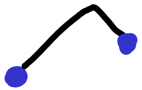


worst case

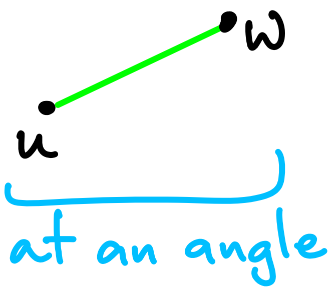




In fact we travel on 

not on 

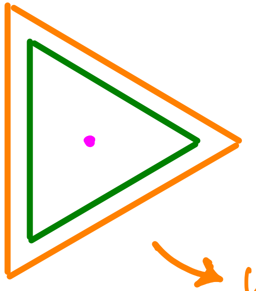
so the bound is better.

Dealing w/ 

is mostly skipped but claimed to give $\sqrt{10}$

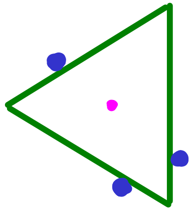
Computation : Delaunay triangulation (L_1 or L_2) : $\Theta(n \log n)$

Journal version contains improvement : 2-spanner (from $\sqrt{10}$)

↳ use  as distance

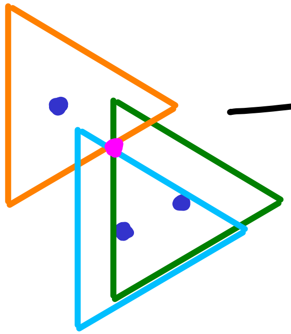
would use this to "grow" Voronoi diagram

// notice it's not a metric but we don't care

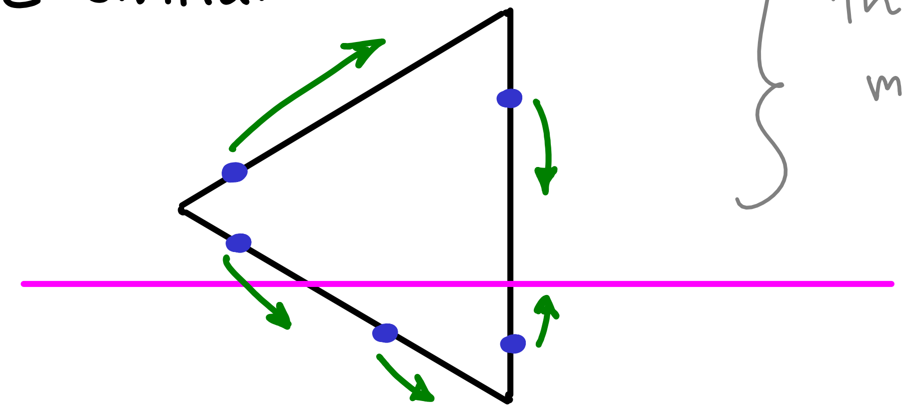
 ← however an empty circle is inverted

// 3 points at same distance from center

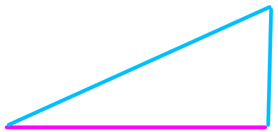
All that matters is that we form a graph using ◁

 → center of circle reached simultaneously from Voronoi "seeds"

Rules are similar



this might make many of the proofs easier

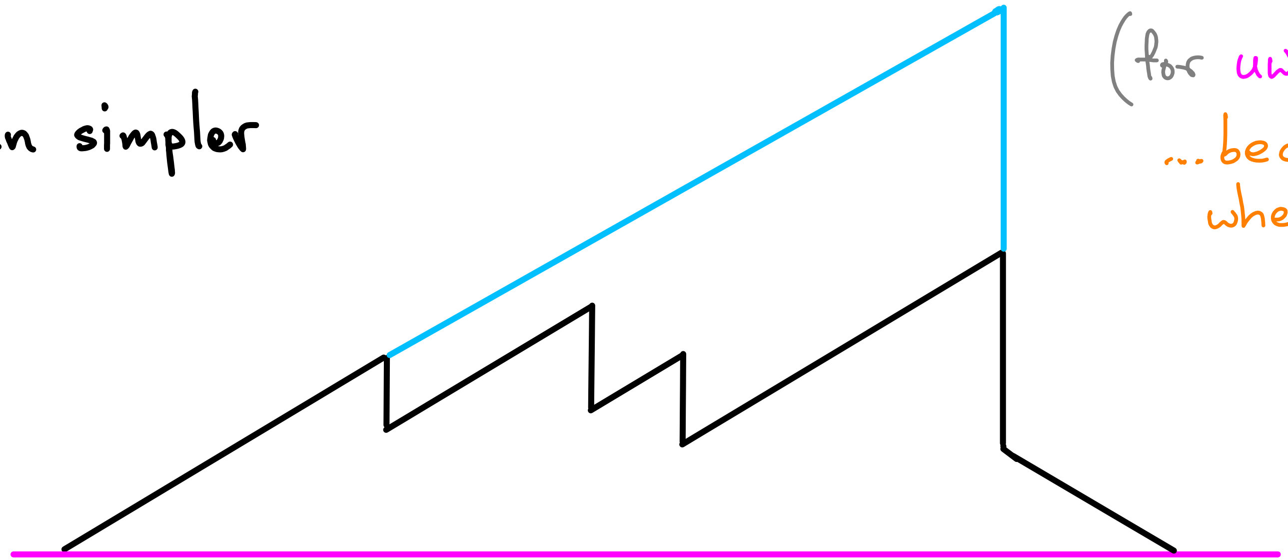


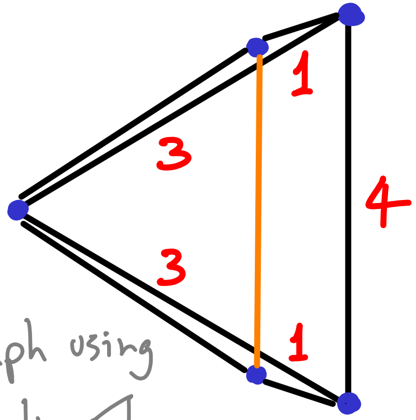
worst case ratio: $\sqrt{3}$

(for uw horizontal)

...becomes 2 when tilted

Resulting shape is even simpler



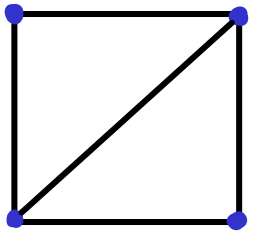


graph using empty ◁

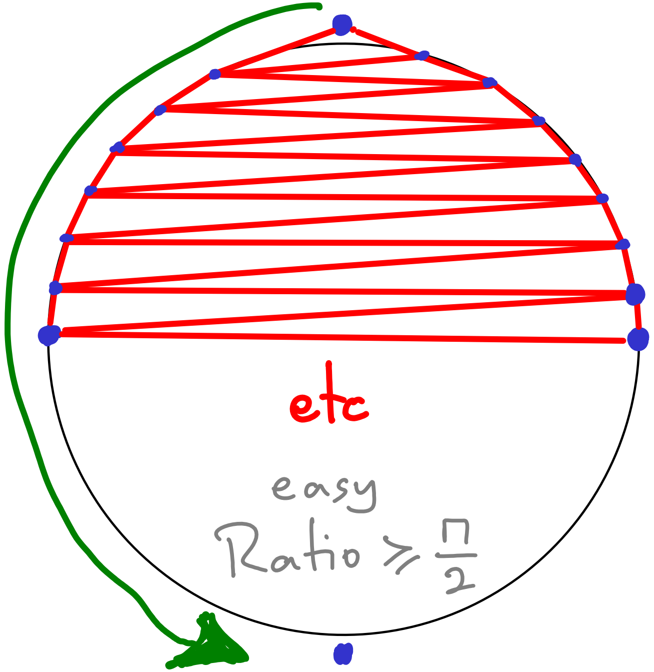
Euclidean $= 3+\varepsilon$

Detour $\simeq 6$

upper bound is tight



Simple lower bound of $\sqrt{2}$ for any planar spanner



perturbed co-circular points: can choose any Delaunay triangulation

Ratio $> \frac{\pi}{2}$

(recent update)

Bose et al.

& known $\leq \dfrac{2\pi}{3\cos\frac{\pi}{6}} \sim 2.42$