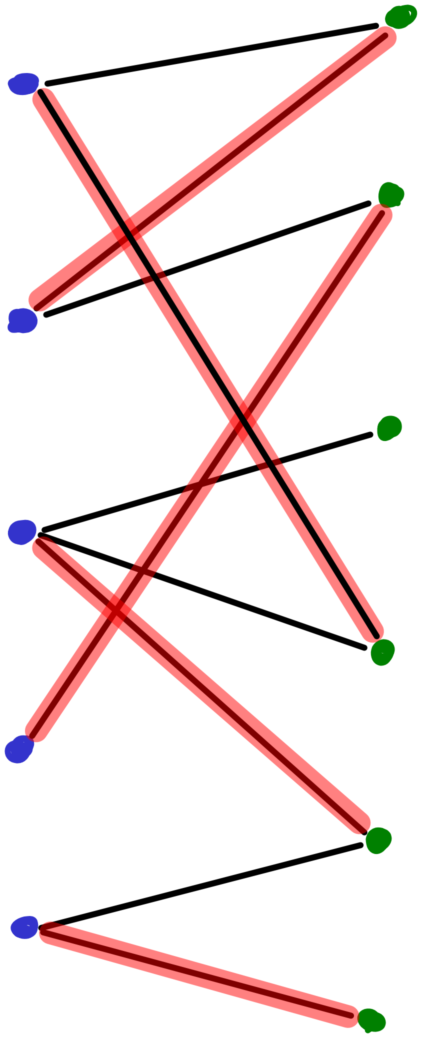
MATCHING in a BIPARTITE GRAPH



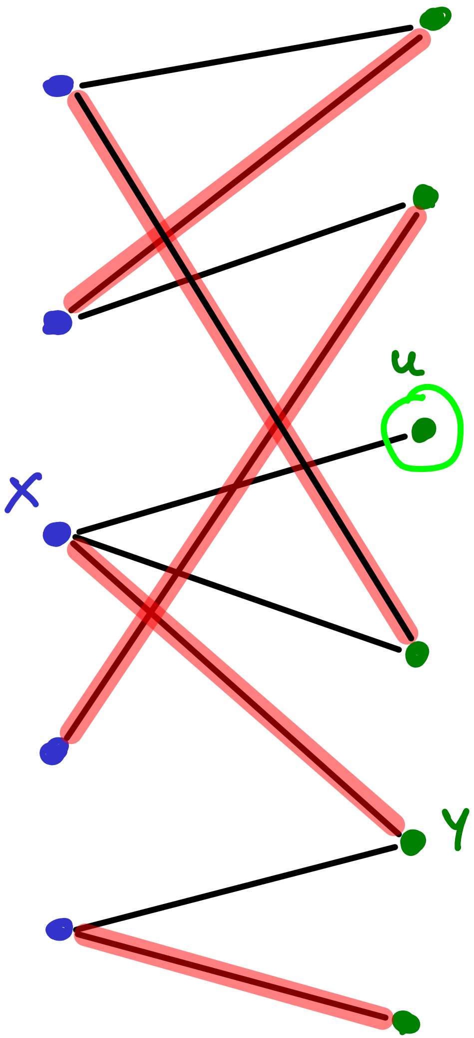
ex: edges represent mutual consent

[wiki: E represent men approved by women,
& all men will take any woman
who wants them!]

Goal: maximize # independent edges

(like 1 round of greedy edge-coloring)

MATCHING in a BIPARTITE GRAPH



unmatched ☹

- if no incident edges, no hope
- if $\exists$ edge at u & it is not marked

then $\exists$ x that is matched elsewhere

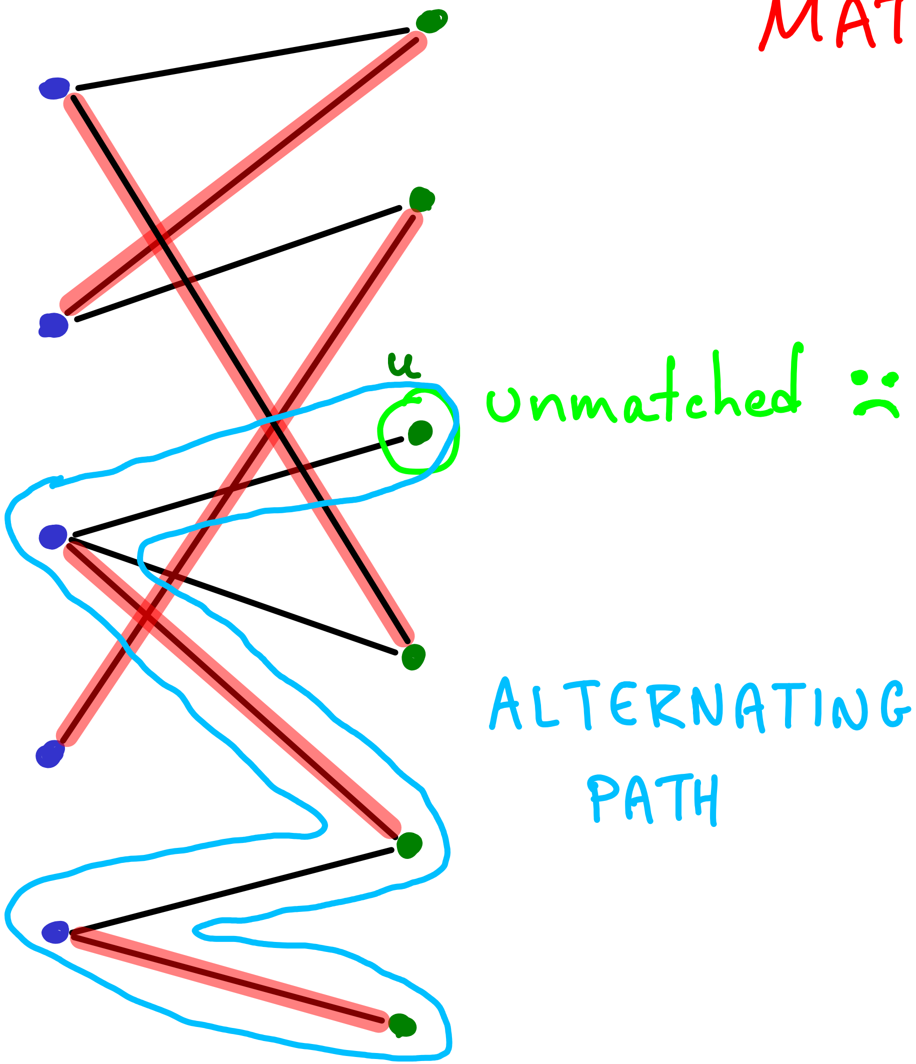
(otherwise match $x \leftrightarrow u$)

so $\exists$ y s.t. $x \leftrightarrow y$

which means all other edges at y

are not selected, etc

MATCHING in a BIPARTITE GRAPH

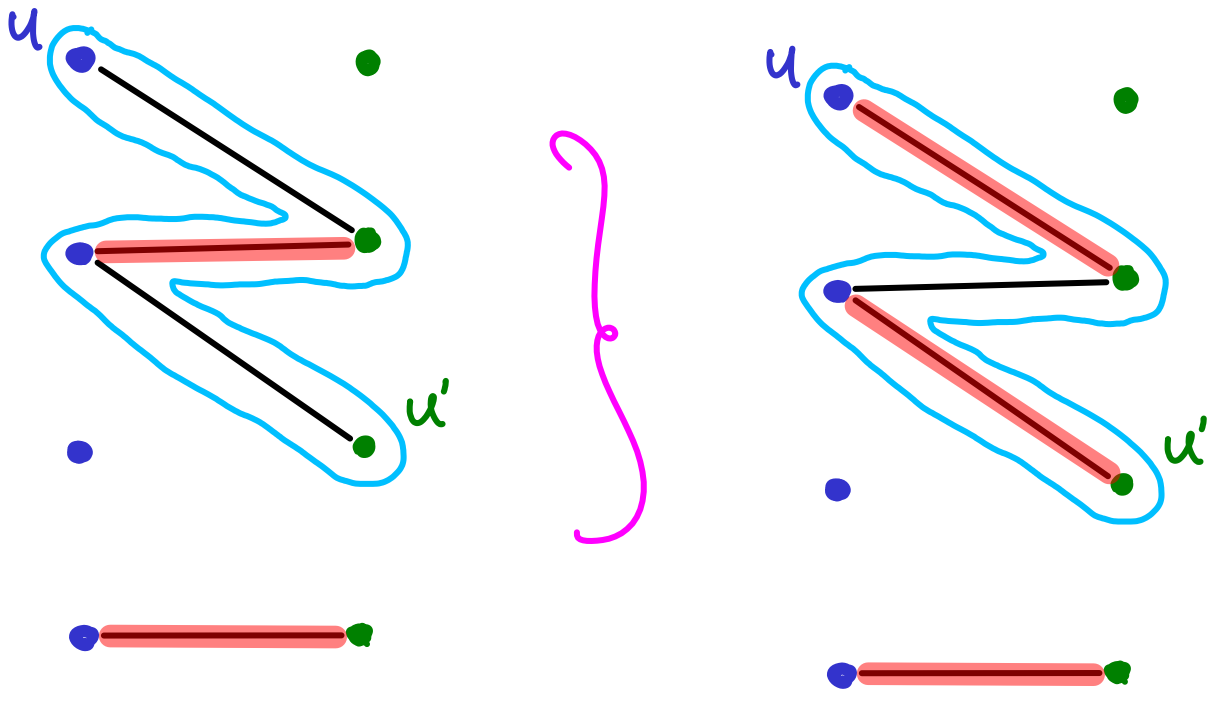


- if no incident edges, no hope
- if $\exists$ edge at u & it is not marked

then $\exists$ x that is matched elsewhere (otherwise match $x \leftrightarrow u$)

So $\exists$ y s.t. $x \leftrightarrow y$

which means all other edges at y are not selected, etc



AUGMENTING PATH

Is a matching optimal if no augmenting path exists?

"homework"

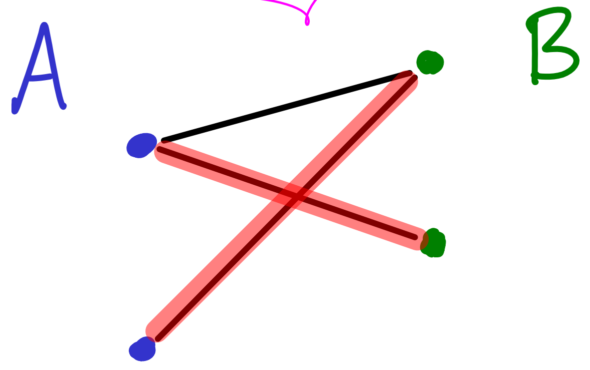
Algorithm & time complexity to find an avg. path ?

... or an optimal matching?

All vertices of A matched → <u>necessary</u> & sufficient conditions?

↳ every vertex has a neighbor

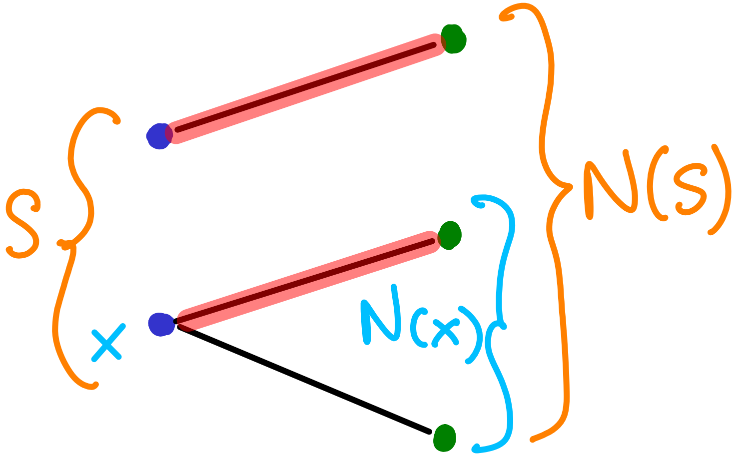
↳ every group of S vertices in A has $\geq |S|$ neighbors.



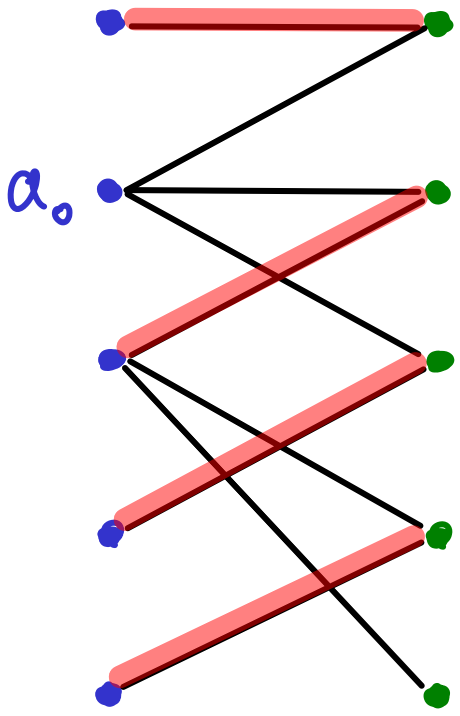
Also sufficient: (Hall's theorem)

All vertices in A will be matched if for every $S \subseteq A$

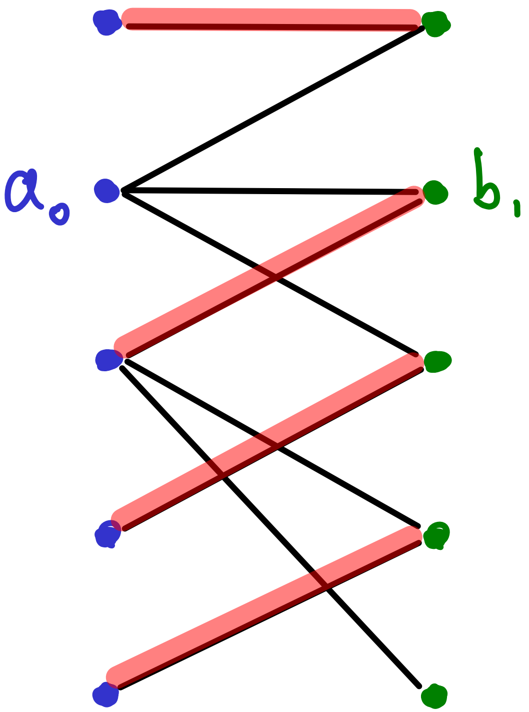
$$|N(S)| \geq |S|$$



Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched

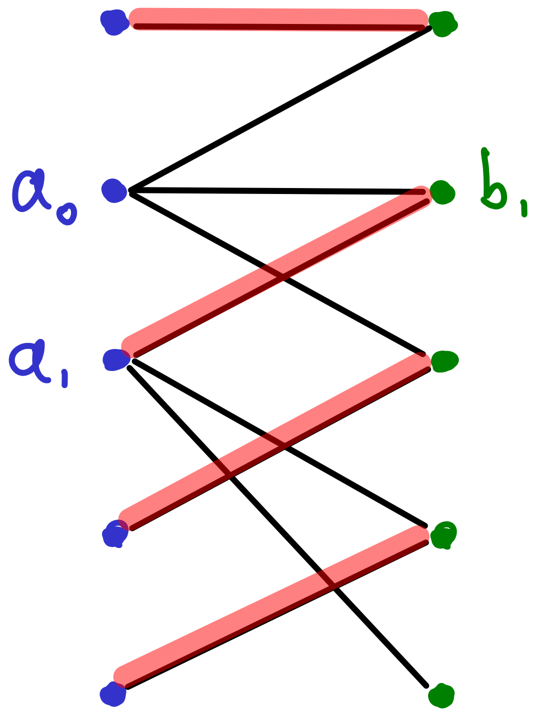


Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$\exists\ b_1$ adjacent to a_0 $\left(|N(a_0)| \geq 1 \right)$

Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched

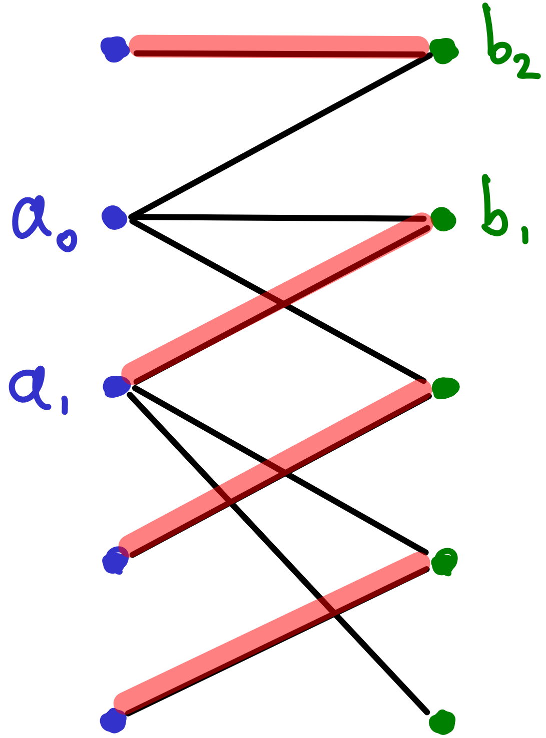


$\exists$ b_1 adjacent to a_0 $(|N(a_0)| \geq 1)$

b_1 matches to some a_1

(otherwise match a_0 to b_1)

Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$\exists$ b_1 adjacent to a_0 $(|N(a_0)| \geq 1)$

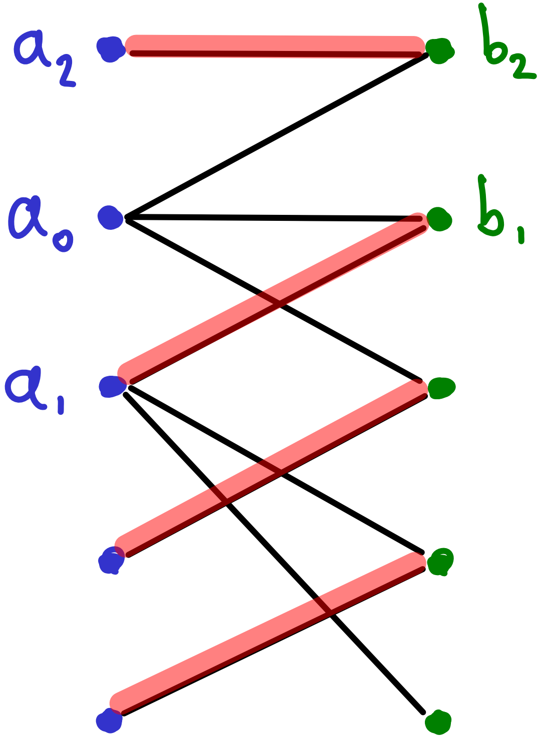
b_1 matches to some a_1

(otherwise match a_0 to b_1)

Next, if $\exists$ other vertex adjacent to a_0 or a_1 label it b_2

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$\exists\ b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

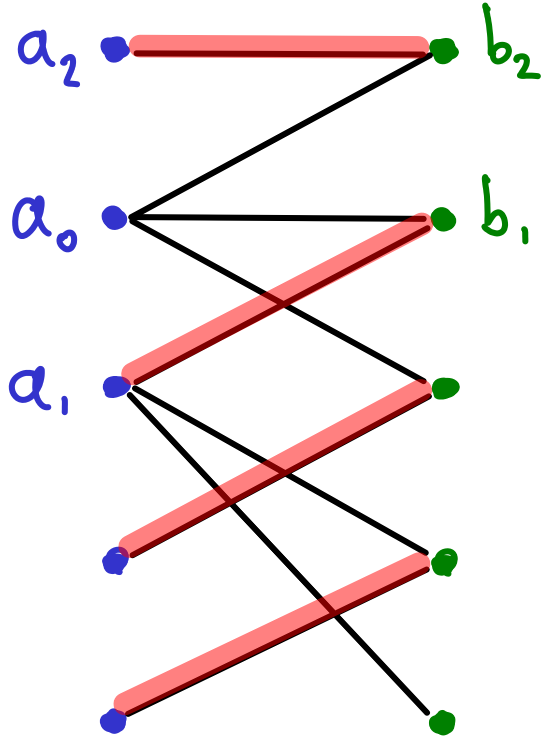
b_1 matches to some a_1

(otherwise match a_0 to b_1)

Next, if $\exists$ other vertex adjacent to a_0 or a_1 label it b_2

and if b_2 matches to something, label it a_2

Start w/ best matching.



$a_0 b_1 a_1 b_2 a_2 \ldots$

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

$\exists$ b_1 adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

b_1 matches to some a_1

(otherwise match a_0 to b_1)

Next, if $\exists$ other vertex adjacent to a_0 or a_1
label it b_2

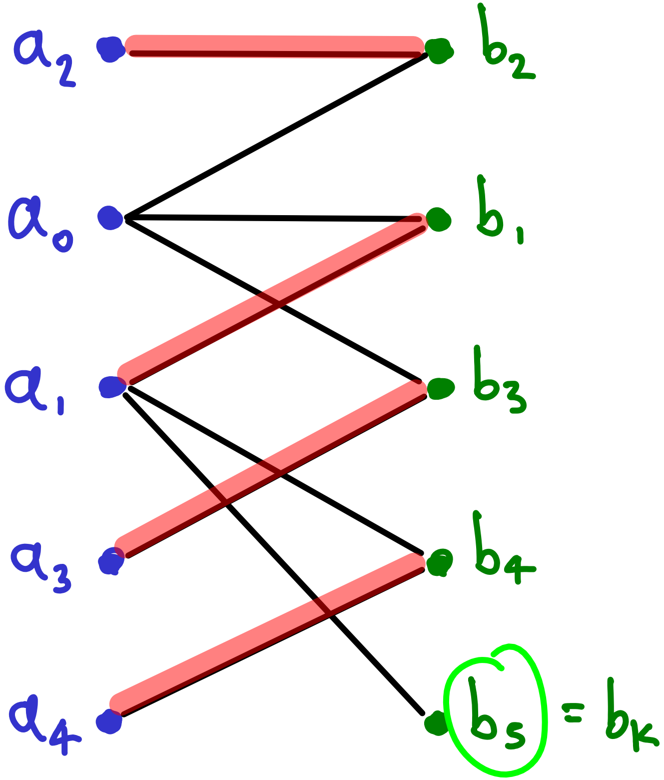
and if b_2 matches to something, label it a_2

While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \ldots a_{i-1})$

Start w/ best matching.



$a_0 b_1 a_1 b_2 a_2 \dots b_k$

or

$a_0 b_1 a_1 b_2 a_2 \dots a_k$?

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

$\exists\, b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

b_1 matches to some a_1

(otherwise match a_0 to b_1)

Next, if $\exists$ other vertex adjacent to a_0 or a_1 label it b_2

and if b_2 matches to something, label it a_2

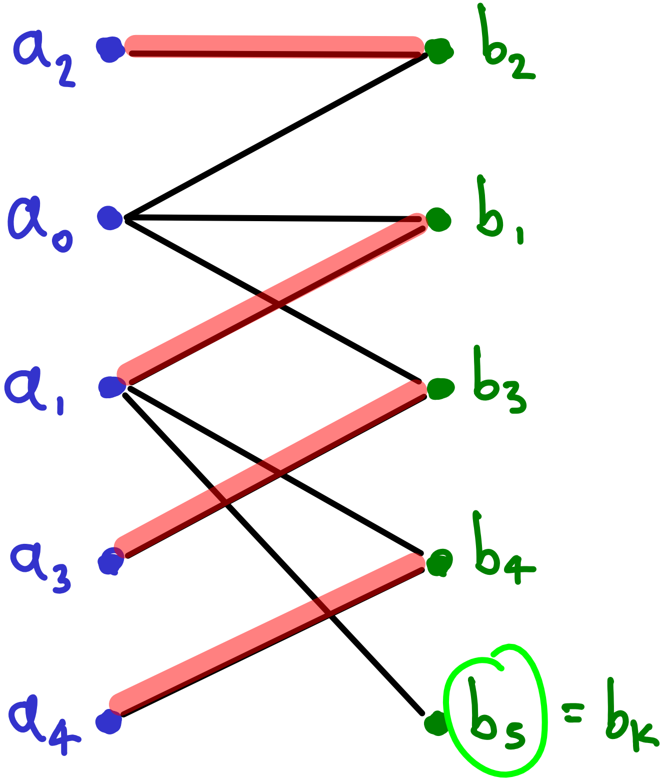
While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \dots a_{i-1})$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \ldots a_{i-1})$

Can this end in A at some a_k ?

No because $|N(a_0 \ldots a_k)| \geq k+1$

& we're only used $b_1 \ldots b_k$

$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_0 \ldots a_k)$

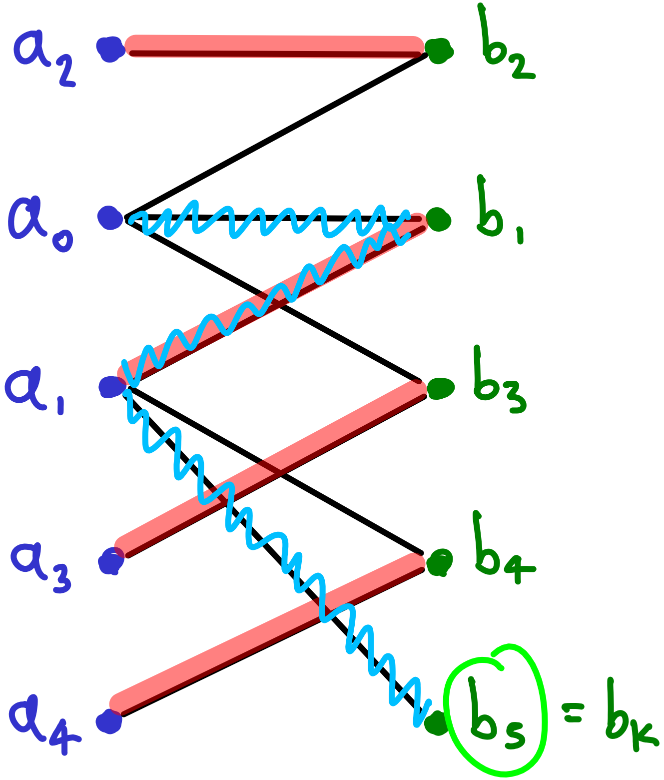
$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

or

$a_0 b_1 a_1 b_2 a_2 \ldots a_k$?

Start w/ best matching. *

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



b_K doesn't match to any $a_0 \ldots a_{K-1}$

by definition

& doesn't match to any $a \neq a_0 \ldots a_{K-1}$

because we could extend the sequence

$b_K \to$ some a_i $(i<k) \to b_i \to$ some a_j $(j<i) \to b_j \to$ etc $\to a_0$

$a_0 b_1 a_1 b_2 a_2 \ldots b_K$ $\sim\neq\sim$ AUGMENTING PATH :

contradict *