ANALYSIS OF QUICKSORT

Let \( \{ Z_1, Z_2, Z_3, \ldots, Z_n \} \) be the given data arranged in sorted order.

\[
X_{ij} = \begin{cases} 
1 & \text{if } Z_i \text{ is ever compared to } Z_j \\
0 & \text{otherwise}
\end{cases}
\]

\[
X = \text{total # comparisons} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}
\]

Every \( x_i \) & \( x_j \) is compared once or never.

We are interested in \( E[X] \)
$E[X] = E[\text{total \# comparisons}] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$

linearity of expectation
\[ E[X] = E[\text{Total \# comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} E[X_{ij}] \]

define \( Z_{ij} = \{ z_i, \ldots, z_j \} \) (subsequence of \( \{ z_1, \ldots, z_n \} \))

\( z_i \) will be compared to \( z_j \) unless any \( z_k \) (\( i < k < j \)) is a pivot before them

So \( E[X_{ij}] = \Pr\{ z_i \text{ is chosen first among } Z_{ij} \} + \Pr\{ z_j \text{ is chosen first among } Z_{ij} \} \)

\[ E[Y] = \sum_t t \cdot P(y=t) \quad \text{For I.R.V.: } E[Y] = P[Y=1] \]
\[
E[X] = E[\text{Total \# comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]
\]

Define \( Z_{ij} = \{ Z_i, ..., Z_j \} \) (subsequence of \( \{ Z_1, ..., Z_n \} \))

\( z_i \) will be compared to \( z_j \) unless any \( z_k \) \( (i < k < j) \) is a pivot before them

So \( E[X_{ij}] = Pr\{ Z_i \text{ is chosen first among } Z_{ij} \} + Pr\{ Z_j \text{ is chosen first among } Z_{ij} \} = \frac{2}{j-i+1} \)

\[
E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{t=1}^{n-i} \frac{2}{t+1} < \sum_{i=1}^{n-1} \sum_{t=1}^{n} \frac{2}{t} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n)
\]