ANALYSIS OF QUICKSORT

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X_{ij} = \begin{cases} 
1 & \text{if } Z_i \text{ is ever compared to } Z_j \\
0 & \text{otherwise} 
\end{cases}
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Every \( x_i \) & \( x_j \) is compared once or never.
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Every \( x_i \& x_j \) is compared once or never.

We are interested in \( E[X] \).
$E[X] = E[\text{total # comparisons}] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right]$
$E[X] = E[\text{Total # comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} E[X_{ij}]$

linearity of expectation
\[ E[X] = E[\text{Total # comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} E[X_{ij}] \]

define \( Z_{ij} = \{z_i \ldots z_j\} \) (subsequence of \( \{z_1 \ldots z_n\} \))
\[ E[X] = E[\text{total # comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \]

define \( Z_{ij} = \{z_i \ldots z_j\} \) (subsequence of \( \{z_1 \ldots z_n\}\))

\( z_i \) will be compared to \( z_j \) unless any \( z_k \) (\( i < k < j \)) is a pivot before them.

\( z_1, \ldots z_{i-1} \)

\&

\( z_{j+1} \ldots z_n \)

are irrelevant
\[ E[X] = E[\text{Total \# comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} E[X_{ij}] \]

Define \( Z_{ij} = \{z_i, \ldots, z_j\} \) (subsequence of \( \{z_1, \ldots, z_n\} \))

\( z_i \) will be compared to \( z_j \) unless any \( z_k \) \((i < k < j)\) is a pivot before them.

\[ \begin{align*}
S_o \quad E[X_{ij}] &= P_r\{z_i \text{ is chosen first among } Z_{ij}\} \\
&+ P_r\{z_j \text{ is chosen first among } Z_{ij}\}
\end{align*} \]

\[ E[Y] = \sum_t t \cdot P(Y=t) \]

For I.R.V.: \( E[Y] = P[Y=1] \)
\[ E[X] = E[\text{Total # comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} E[X_{ij}] \]

Define \( Z_{ij} = \{ z_i, \ldots, z_j \} \) (subsequence of \( \{ z_1, \ldots, z_n \} \))

\( z_i \) will be compared to \( z_j \) unless any \( z_k \) (\( i < k < j \)) is a pivot before them.

So \( E[X_{ij}] = \Pr \{ z_i \text{ is chosen first among } Z_{ij} \} \) + \( \Pr \{ z_j \text{ is chosen first among } Z_{ij} \} \) = \( \frac{2}{j-i+1} \)
\[ E[X] = E[\text{Total \# comparisons}] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{\tilde{n}} E[X_{ij}] \]

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define \( Z_{ij} = \{ z_i \ldots z_j \} \) (subsequence of \( \{ z_1 \ldots z_n \} \))

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\( z_{j+1} \ldots z_n \) are irrelevant

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\[ E[X] = \sum_{i=1}^{n-1} \frac{2}{j-i+1} \]
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\[ E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{t=1}^{n-i} \frac{2}{t+1} < \sum_{i=1}^{n-1} \sum_{t=1}^{n} \frac{2}{t} \]
\[ E[X] = E[\text{Total \# comparisons}] = E\left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \]

define \( Z_{ij} = \{ z_i, \ldots, z_j \} \) (subsequence of \( \{ z_1, \ldots, z_n \} \))

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So \( E[X_{ij}] = \Pr \{ Z_i \text{ is chosen first among } Z_{ij} \} \]
\[ + \Pr \{ Z_j \text{ is chosen first among } Z_{ij} \} = \frac{2}{j-i+1} \]

\[ E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{t=1}^{n-i} \frac{2}{t+1} < \sum_{i=1}^{n-1} \sum_{t=1}^{n} \frac{2}{t} = \sum_{i=1}^{n-1} O(\log n) \]
\[ E[X] = \mathbb{E}[\text{Total # comparisons}] = \mathbb{E}\left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[X_{ij}] \]

Define \( Z_{ij} = \{z_i, \ldots, z_j\} \) (subsequence of \( \{z_1, \ldots, z_n\} \))

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So \[ E[X_{ij}] = \Pr\{Z_i \text{ is chosen first among } Z_{ij}\} + \Pr\{Z_j \text{ is chosen first among } Z_{ij}\} = \frac{2}{j-i+1} \]

\[ E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{t=1}^{n-i} \frac{2}{t+1} < \sum_{i=1}^{n-1} \sum_{t=1}^{n} \frac{2}{t} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n) \]