Dynamic Programming - Longest Increasing Subsequence

S: 23, 3, 5, 18, 10, 101, 12, 14, 4, 105

\[ L(S) = 3, 5, 10, 12, 14, 105 \]

\[ |L(S)| = 6 \]

Brute-force: try all subsequences, see if they are increasing: O(2^n)

For dynamic programming we would like

- a recursive expression w/ repeated subproblems
- an easy, fast way to use solved subproblems
23, 3, 5, 18, 10, 101, 12, 14

\[ L(S) \sim L_{i..n}(s) \sim L_n \]

|Ln| using |Ln-1| ?

if \( S[n] > \) \underline{last element in Ln-1} then \( |L_n| = |L_{n-1}| + 1 \)

\( \Rightarrow \) could be at any position in S

else \( S[n] \leq \) \underline{last (Ln-1)}

\( \Rightarrow \) keep |Ln-1| ?

\( \Rightarrow \) add \( S[n] \) to suboptimal solution from \( S[1..n-1] \)?

\( \text{enegy} \)
Redefine: $L_n =$ longest increasing subsequence that actually uses $S[n]$

$$|L_n| = 1 + \max_{\{\text{all } j \text{ s.t. } S[j] < S[n]\}} |L_j|$$

look at all $L_j$ ($j < n$)
\[ |L_n| = 1 + \max_{\text{all } j \text{ s.t. } S[j] < S[n]} |L_j| \]

Recursion: \( L_n \)

BAD

\[ L_{n-1} \quad L_{n-2} \quad \ldots \quad L_1 \]

\[ L_{n-2} \quad L_{n-3} \quad \ldots \quad \text{etc} \]

Dyn. Prog: Build solutions, "bottom up"

When it's time to solve \( |L_k| \) we have stored all \( |L_j| \) (\( j < k \)) in an array.

\[ T(k) = \Theta(k) \]

\[ T(n) = \sum_{i=1}^{n} T(k) = \Theta(n^2) \]

Space = \( \Theta(n) \)
$23, 3, 5, 18, 10, 101, 12, 14, 4$

$1, 1, 2, 3, 3, 4, 4, 5, 2$

$T(n) = \Theta(n^2)$

$\text{space} = \Theta(n)$

$$|L_n| = 1 + \max_{\text{all } j \text{ s.t. } S[j] < S[n]} |L_j|$$

What about $|L.I.S.|$?

$$|L.I.S.| = \max_{j=1..n} |L_j|$$

What about L.I.S.?

Keep the pointers: for each $S[j]$ store any $S[i]$ pointer that generated $|L_j|$. 

$\blacksquare$
A quick solution for L.I.S. ... but still $O(n^2)$ & dyn-prog.

$S = \{23, 3, 5, 18, 10, 101, 12, 14\}$

sort $S$

$S_2 = \{3, 5, 10, 12, 14, 18, 23, 101\}$

Find longest common subsequence!

- any common subsequence is increasing
  so $\text{LCS}(S, S_2)$ qualifies as a solution

- L.I.S. must exist in $S_2$, so it is a candidate for LCS.