Analysis of open addressing

Assuming uniform hashing: each key is equally likely to have any of the m! permutations as probe sequence (independent of other keys)

Simple uniform hashing

For a random h, every slot is equally likely
Analysis of Open Addressing

Assuming Uniform Hashing: each key is equally likely to have any of the $m!$ permutations as probe sequence (independent of other keys)

Recall $n < m$, so $\alpha < 1$. Claim: $E[\#\text{probes}] \leq \frac{1}{1-\alpha} \left( \frac{m}{m-n} \right)$

If true, then for $n \ll m$ \( E[\#\text{probes}] = O(1) \)

\[ \Rightarrow n = \frac{1}{2} m \quad \Rightarrow \quad 2 \text{ probes} \]

\[ \Rightarrow 90\% \text{ full table} \quad \Rightarrow \quad 10 \text{ probes} \]

Works well if you can afford a table ~ data $\times 2$
Claim: $E[\#\text{probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

* $P[\text{1st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

$P[\text{2nd probe collides}] = \frac{n-i}{m-i} \rightarrow \text{need 3rd probe}$

\[\vdots\]

$\frac{n-i}{m-i} < \frac{n}{m} = \alpha$

$E[\#\text{probes}] = 1 + \frac{n}{m} \left( \text{need at least a 2nd probe} \right)$

Remember, probe sequence is a permutation.

Never check one slot twice.
Claim:  \( E[\text{#probes}] \leq \frac{1}{1-\alpha} \)  

Look at unsuccessful search

\[
\begin{align*}
P[1\text{st probe collides}] &= \frac{n}{m} \quad \Rightarrow \text{need 2nd probe} \\
\ast \quad P[2\text{nd probe collides}] &= \frac{n-1}{m-1} \quad \Rightarrow \text{need 3rd probe} \\
& \quad \vdots \\
\frac{n-i}{m-i} &< \frac{n}{m} = \alpha
\end{align*}
\]

\[
E[\text{#probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \right) \quad \text{(need a 3rd probe)}
\]
Claim: $E[\# \text{probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

$P[1\text{st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

$P[2\text{nd probe collides}] = \frac{n-1}{m-1} \rightarrow \text{need 3rd probe}$

$\vdots$

$\frac{n-i}{m-i} < \frac{n}{m} = \alpha$

$E[\# \text{probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \ldots (1 + \frac{1}{m-n}) \right) \right) \right) \leq 1 + \alpha \left( 1 + \alpha \left( 1 + \alpha \left( \ldots (1 + \alpha) \right) \right) \right) \ldots \ n \ \text{terms}$

$\leq 1 + \alpha + \alpha^2 + \alpha^3 + \ldots \ldots \infty \ \text{terms}$

$= \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$

See CLRS for alternate analysis incl. successful search