Analysis of open addressing

Assuming uniform hashing: each key is equally likely to have any of the $m!$ permutations as probe sequence (independent of other keys)

simple uniform hashing

For a random $h$, every slot is equally likely
Analysis of Open Addressing

Assuming Uniform Hashing: each key is equally likely to have any of the m! permutations as probe sequence (independent of other keys)

Recall n < m, so \( \alpha < 1 \). Claim: \( E[\#\text{probes}] \leq \frac{1}{1-\alpha} \left( \frac{m}{m-n} \right) \)

If true, then for \( n \ll m \) \( E[\#\text{probes}] = O(1) \)

\( \Rightarrow n = \frac{1}{2} m \rightarrow 2 \text{ probes} \)

\( \Rightarrow 90\% \text{ full table} \rightarrow 10 \text{ probes} \)

Works well if you can afford a table ~ data x 2
Claim: $E[\#\text{probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

$* \quad P[1\text{st probe collides}] = \frac{n}{m} \quad \to \quad \text{need 2nd probe}$

$P[2\text{nd probe collides}] = \frac{n-1}{m-1} \quad \to \quad \text{need 3rd probe}$

\[ \vdots \]

\[ \frac{n-i}{m-i} < \frac{n}{m} = \alpha \]

$E[\#\text{probes}] = 1 + \frac{n}{m} \left( \text{need at least a 2nd probe} \right)$

Remember, probe sequence is a permutation.
Never check one slot twice.
Claim: $E[\#\text{probes}] \leq \frac{1}{1-\alpha}$  

Look at unsuccessful search

$P[\text{1st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

$* \quad P[\text{2nd probe collides}] = \frac{n-1}{m-1} \rightarrow \text{need 3rd probe}$

$\vdots$

$\frac{n-i}{m-i} < \frac{n}{m} = \alpha$

$E[\#\text{probes}] = 1 + \frac{n}{m} \left(1 + \frac{n-1}{m-1} \left(\text{need a 3rd probe}\right)\right)$
Claim: $E[\#\text{probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

$P[\text{1st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

$P[\text{2nd probe collides}] = \frac{n-1}{m-1} \rightarrow \text{need 3rd probe}$

$\vdots$

$\frac{n-i}{m-i} \leq \frac{n}{m} = \alpha$

$E[\#\text{probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \cdots \left( 1 + \frac{1}{m-n} \right) \right) \right) \right) \leq 1 + \alpha \left( 1 + \alpha \left( 1 + \alpha \left( \cdots \left( 1 + \alpha \right) \right) \right) \right) \cdots \text{n terms}$

$\leq 1 + \alpha + \alpha^2 + \alpha^3 + \cdots \cdots \text{\infty terms}$

$= \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$

see CLRS for alternate analysis incl. successful search