Analysis of Open Addressing

Assuming **Uniform Hashing**: each key is equally likely to have any of the m! permutations as probe sequence (independent of other keys).

For a random h, every slot is equally likely.
Analysis of open addressing

Assuming uniform hashing: each key is equally likely to have any of the $m!$ permutations as probe sequence (independent of other keys)

even though all we have so far is $O(m^2)$

For a random $h$, every slot is equally likely
Analysis of Open Addressing

Assuming Uniform Hashing: each key is equally likely to have any of the \(m!\) permutations as probe sequence (independent of other keys)

Recall \(n < m\), so \(\alpha < 1\).

Claim: \(\mathbb{E}[\text{# probes}] \leq \frac{1}{1 - \alpha} \left(\frac{m}{m - n}\right)\) (search)
Analysis of open addressing

Assuming uniform hashing: each key is equally likely to have any of the $m!$ permutations as probe sequence (independent of other keys)

Recall $n < m$, so $\alpha < 1$. Claim: $E[\text{#probes}] \leq \frac{1}{1-\alpha} \left( \frac{m}{m-n} \right)$

If true, then for $n \ll m$ $E[\text{#probes}] = O(1)$
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If true, then for $n \ll m$ $E[\# \text{probes}] = O(1)$

$\Rightarrow n = \frac{1}{2} m \rightarrow 2$ probes

$\Rightarrow 90\%$ full table $\rightarrow 10$ probes
Analysis of open addressing. 

Assuming uniform hashing: each key is equally likely to have any of the \( m! \) permutations as probe sequence (independent of other keys).

Recall \( n < m \), so \( \alpha < 1 \).

Claim: \( E[\#\text{probes}] \leq \frac{1}{1-\alpha} \left( \frac{m}{m-n} \right) \)

If true, then for \( n \ll m \quad E[\#\text{probes}] = O(1) \)

\( \Rightarrow \quad n \approx \frac{1}{2} m \rightarrow 2 \text{ probes} \)

\( \Rightarrow \quad 90\% \text{ full table} \rightarrow 10 \text{ probes} \)

Works well if you can afford a table \( \sim \text{data } \times 2 \)
Claim: $E[\# \text{probes}] \leq \frac{1}{1-\alpha}$  

Look at unsuccessful search
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Remember, probe sequence is a permutation.
Never check one slot twice.
Claim: $E[\# \text{probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

$P[1st \text{ probe collides}] = \frac{n}{m}$

Remember, probe sequence is a permutation.
Never check one slot twice.
Claim: $E[\text{#probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

$P[1\text{st probe collides}] = \frac{n}{m} \quad \rightarrow \text{need 2nd probe}$

Remember, probe sequence is a permutation.
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Look at unsuccessful search

\[
P[\text{1st probe collides}] = \frac{n}{m} \quad \rightarrow \quad \text{need 2nd probe}
\]

\[
P[\text{2nd probe collides}] = \frac{n-1}{m-1} \quad \rightarrow \quad \text{need 3rd probe}
\]

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$\vdots$

$\frac{n-i}{m-i} \rightarrow \text{need } i+1 \text{ probes}$

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Claim:  $E[\#\text{probes}] \leq \frac{1}{1-\alpha}$  

Look at unsuccessful search

$P[1\text{st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

$P[2\text{nd probe collides}] = \frac{n-i}{m-1} \rightarrow \text{need 3rd probe}$

$\vdots$

$\frac{n-i}{m-i} < \frac{n}{m} = \alpha$

Remember, probe sequence is a permutation.
Never check one slot twice.
Claim: \( E[\#\text{probes}] \leq \frac{1}{1-\alpha} \)

- \( P[\text{1st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe} \)
- \( P[\text{2nd probe collides}] = \frac{n-i}{m-1} \rightarrow \text{need 3rd probe} \)
- \( \vdots \)
- \( \frac{n-i}{m-i} < \frac{n}{m} = \alpha \)

\[
E[\#\text{probes}] = 1 + \frac{n}{m} \left( \text{need at least a 2nd probe} \right)
\]

Look at unsuccessful search

Remember, probe sequence is a permutation.
Never check one slot twice.
Claim: \( E[\# \text{probes}] \leq \frac{1}{1 - \alpha} \)

Look at unsuccessful search

\[
P[\text{1st probe collides}] = \frac{n}{m} \quad \rightarrow \text{need 2nd probe}
\]

\[
* \quad P[\text{2nd probe collides}] = \frac{n-1}{m-1} \quad \rightarrow \text{need 3rd probe}
\]

\[
\vdots
\]

\[
\frac{n-i}{m-i} < \frac{n}{m} = \alpha
\]

\[
E[\# \text{probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( \text{need a 3rd probe} \right) \right)
\]
Claim: $E[\text{#probes}] \leq \frac{1}{1-\alpha}$  

Look at unsuccessful search

$P[1\text{st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

$P[2\text{nd probe collides}] = \frac{n-1}{m-1} \rightarrow \text{need 3rd probe}$

$\vdots$

$\frac{n-i}{m-i} \leq \frac{n}{m} = \alpha$

$E[\text{#probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \ldots \right) \right) \right)$

Remember, probe sequence is a permutation.

Never check one slot twice.
Claim: $E[\#\text{probes}] \leq \frac{1}{1-\alpha}$

Look at unsuccessful search

$P[\text{1st probe collides}] = \frac{n}{m} \rightarrow \text{need 2nd probe}$

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$\vdots$

$\frac{n-i}{m-i} < \frac{n}{m} = \alpha$

$E[\#\text{probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \cdots \left( 1 + \frac{1}{m-n} \right) \right) \right) \right)$

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P[\text{1st probe collides}] = \frac{n}{m} \quad \rightarrow \text{need 2nd probe}
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\[
\vdots
\]

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\frac{n-i}{m-i} \quad < \quad \frac{n}{m} = \alpha
\]

\[
E[\#\text{probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \cdots \left( 1 + \frac{1}{m-n} \right) \right) \right) \right)
\]

\[
\leq 1 + \alpha \left( 1 + \alpha \left( 1 + \alpha \left( \cdots \left( 1 + \alpha \right) \right) \right) \right) \quad \cdots \text{n terms}
\]

Remember, probe sequence is a permutation.

Never check one slot twice.
Claim: $E[\# \text{probes}] \leq \frac{1}{1-\alpha}$  

Look at unsuccessful search

$P[1\text{st probe collides}] = \frac{n}{m} \quad \rightarrow \text{need 2nd probe}$

$P[2\text{nd probe collides}] = \frac{n-1}{m-1} \quad \rightarrow \text{need 3rd probe}$

\[ \vdots \]

\[ \frac{n-i}{m-i} \quad \leq \quad \frac{n}{m} = \alpha \]

$E[\# \text{probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \ldots \left( 1 + \frac{1}{m-n} \right) \right) \right) \right)$

$\leq 1 + \alpha \left( 1 + \alpha \left( 1 + \alpha \left( \ldots \left( 1 + \alpha \right) \right) \right) \right) \ldots \text{n terms}$

$\leq 1 + \alpha + \alpha^2 + \alpha^3 + \ldots \quad \ldots \infty \text{ terms}$

Remember, probe sequence is a permutation.

Never check one slot twice.
Claim: $E[\text{#probes}] \leq \frac{1}{1-\alpha}$  

Look at unsuccessful search

$$P[1\text{st probe collides}] = \frac{n}{m} \quad \rightarrow \text{need 2nd probe}$$

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$$\vdots$$

$$\frac{n-i}{m-i} \leq \frac{n}{m} = \alpha$$

$$E[\text{#probes}] = 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \cdots (1 + \frac{1}{m-n}) \right) \right) \right)$$

$$\leq 1 + \alpha \left( 1 + \alpha \left( 1 + \alpha \left( \cdots (1 + \alpha) \right) \right) \right) \quad \ldots \ n \text{ terms}$$

$$\leq 1 + \alpha + \alpha^2 + \alpha^3 + \cdots \quad \ldots \infty \text{ terms}$$

$$= \sum_{i=0}^{\infty} \alpha^i$$
Claim: \( E[\# \text{probes}] \leq \frac{1}{1-\alpha} \)

Look at unsuccessful search

\[
P[1\text{st probe collides}] = \frac{n}{m} \quad \rightarrow \text{need 2nd probe}
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P[2\text{nd probe collides}] = \frac{n-1}{m-1} \quad \rightarrow \text{need 3rd probe}
\]

\[
\vdots
\]

\[
\frac{n-i}{m-i} \leq \frac{n}{m} = \alpha
\]

\[
E[\# \text{probes}] = 1 + \frac{n}{m} \left(1 + \frac{n-1}{m-1} \left(1 + \frac{n-2}{m-2} \left(\cdots \left(1 + \frac{1}{m-n}\right)\right)\right)\right)
\leq 1 + \alpha \left(1 + \alpha \left(1 + \alpha \left(\cdots \left(1 + \alpha\right)\right)\right)\right) \quad \cdots \quad n \text{ terms}
\leq 1 + \alpha + \alpha^2 + \alpha^3 + \cdots \quad \cdots \quad \infty \text{ terms}
\]

\[
= \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}
\]

\[\text{see CLRS for alternate analysis}\]
\[\text{incl. successful search}\]