Minimum Spanning Trees

Assume connected undirected graph w/ distinct real edge weights.
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MST = a connected tree spanning all vertices,

\[ W(T) = \sum_{e \in E} w(e) = \sum_{(u,v) \in E} w(u,v) \]
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\[ w(\text{min weight}) \quad w(T) = \sum_{e \in E} w(e) = \sum_{(u,v) \in E} w(u,v) \]

\[ o - 9 & 15 \text{ must be in MST trivially} \]
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- 3 must be in otherwise 14 leads to a leaf worse
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- 6 must be in otherwise 12
**MINIMUM SPANNING TREES**

assume connected undirected graph w/ distinct real edge weights

\[ \text{MST} = \text{a connected tree spanning all vertices,} \]

\[ w(\text{MST}) = \sum_{e \in \text{E}} w(e) = \sum_{(u,v) \in \text{E}} w(u,v) \]

- 9 & 15 must be in MST trivially
- 3 must be in otherwise 14 leads to a leaf; worse
- 6 must be in otherwise 12

Need 3 more edges : 7 total = V-1

Every edge added reduces # connected components by 1
**Minimum Spanning Trees**

Assume connected undirected graph with distinct real edge weights.

\[ \text{MST} = \text{a connected tree spanning all vertices,} \]

With min weight, \( W(T) = \sum_{e \in E} w(e) = \sum_{(u,v) \in E} w(u,v) \)

- 9 & 15 must be in MST trivially
- 3 must be in otherwise 14 leads to a leaf: worse
- 6 must be in otherwise 12

Need 3 more edges: 7 total = \( V - 1 \)

Every edge added reduces the number of connected components by 1.

Remaining: 5, 7, 8, 10, 12... 5, 7, 8 happen to work: optimal
Every edge splits a tree into 2 subtrees

\[ W(T) = e + W(T_1) + W(T_2) \]
Every edge splits a tree into 2 subtrees
\[ W(T) = e + W(T_1) + W(T_2) \]

If there was a \( T_1' \) s.t. \( W(T_1') < W(T_1) \)
Every edge splits a tree into 2 subtrees
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If there was a \( T'_1 \) s.t. \( W(T'_1) < W(T_1) \)
then there would be a \( T' \) s.t. \( W(T') < W(T) \)
Every edge splits a tree into 2 subtrees
\[ W(T) = e + W(T_1) + W(T_2) \]

If there was a \( T_2' \) s.t. \( W(T_2') < W(T_1) \)
then there would be a \( T' \) s.t. \( W(T') < W(T) \)

So every subtree of a MST is also a MST on the vertex set spanned by the subtree.
**Greedy Algorithm for MST**

Greedy choice property: locally optimal choices are also globally optimal

Choose 3 or 6 over 14 or 12
Greedy Algorithm for MST

Greedy choice property: locally optimal choices are also globally optimal for any subset $A \subseteq V$, let $u, v \in E$ be the lightest edge connecting $A$ to $V - A$.

Choose 3 over 14 or 6 over 12
**Greedy Algorithm for MST**

**Greedy choice property:** locally optimal choices are also globally optimal.

**Claim:** for any subset $A \subseteq V$, let $u,v \in E$ be the lightest edge connecting $A$ to $V-A$. Then $u,v$ is in $MST(V)$.

Choose 3 over 14 or 6 over 12.
**Greedy Algorithm for MST**

Greedy choice property: locally optimal choices are also globally optimal

Claim: for any subset \( A \subseteq V \), let \( u,v \in E \) be the lightest edge connecting \( A \) to \( V - A \).

Then \( u,v \) is in \( \text{MST}(V) \).

Proof: suppose \( u,v \) not in \( \text{MST} \).

Choose 3 over 14 or 6 over 12
**Greedy Algorithm for MST**

**Greedy choice property:** locally optimal choices are also globally optimal.

**Claim:** for any subset \( A \subseteq V \), let \( u, v \in E \) be the lightest edge connecting \( A \) to \( V - A \).

Then \( u, v \) is in \( \text{MST}(V) \).

**Proof:** suppose \( u, v \) not in \( \text{MST} \).

Choose 3 over 14, or 6 over 12.

This is the MST. Other edges in \( G \) not shown. (including \( u, v \))
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Claim: for any subset \( A \subseteq V \), let \( u, v \in E \) be the lightest edge connecting \( A \) to \( V - A \). Then \( u, v \) is in MST(\( V \)).

Proof: suppose \( u, v \) not in MST.

Look at unique path from \( u \) to \( v \). It must cross the cut.
**Greedy Algorithm for MST**

Greedy choice property: locally optimal choices are also globally optimal.

Claim: for any subset \( A \subseteq V \), let \( u, v \in E \) be the lightest edge connecting \( A \) to \( V - A \).

Then \( u, v \) is in \( \text{MST}(V) \).

Proof: suppose \( u, v \) not in \( \text{MST} \).

Choose 3 over 14 or 6 over 12.

Look at unique path from \( u \) to \( v \).

It must cross the cut.

Swap the two edges & improve the MST. **Contradiction**