Finding strongly connected components

G
DFS from arbitrary vertex

DFS 1→2→8→4
DFS \[4 \rightarrow 3 \rightarrow 5\] done
DFS: 5 → 3 → 8 → 10 → 9 → 12 → 9 → 10

done  done  done  done
DFS: \[10 \rightarrow 8 \rightarrow 2 \rightarrow 1 \rightarrow 6 \rightarrow 11 \rightarrow 6\]

done done done
Obtain finishing times

DFS: 1 — restart at 7, trivially done
Finishing times don't provide SCC.
\[ G \]

\[ G^\top \]

DFS from 7 discovers only 7.
\( G \)

\[ 7 \ 1 \ 6 \ 11 \ 2 \ 8 \ 10 \ 9 \ 12 \ 3 \ 5 \ 4 \]

\( G^T \)

DFS from 1

... discovers 1, 11, 6
G

7 1 6 11

2 8 10 9 12 3 5 4

G^T

DFS from 2

... discovers 2, 3, 8, 5
Correctness

When processing a vertex \( v \) (e.g. 10),
(1) all vertices to the left (e.g. 7, 1, 6, 11, 2, 8)
    are already assigned to their SCC. (by induction)
Correctness

When processing a vertex $v$ (e.g., 10),

1. all vertices to the left (e.g., 7, 1, 6, 11, 2, 8) are already assigned to their SCC. (by induction)

2. $v$ will find its SCC because there is a path to every member vertex. (by definition)
Correctness

When processing a vertex $v$ (e.g. 10),
1. all vertices to the left (e.g. 7, 1, 6, 11, 2, 8) are already assigned to their SCC. (by induction)
2. $v$ will find its SCC because there is a path to every member vertex. (by definition)
3. $v$ won't find any other vertex (not in $\text{SCC}(v)$). To do so, there would have to be an edge from $\text{SCC}(v)$ to an unmarked vertex $x$, in $G^T$. Then, $\exists$ edge from $x$ to $\text{SCC}(v)$ in $G$.
Assuming $x$ is not in $\text{SCC}(v)$, $\exists$ edge from $\text{SCC}(v)$ to $x$ in $G$.
Then $x$ must finish after $v$ in DFS on $G$, which contradicts (i).
Correctness

When processing a vertex $v$ (e.g. 10),
(1) all vertices to the left (e.g. 7, 1, 6, 11, 2, 8) are already assigned to their SCC. (by induction)
(2) $v$ will find its SCC because there is a path to every member vertex.
(3) $v$ won't find any other vertex (not in $\text{SCC}(v)$).

(2) & (3) establish (1) for next processed vertex.