Splay Trees

Sleator & Tarjan
1985

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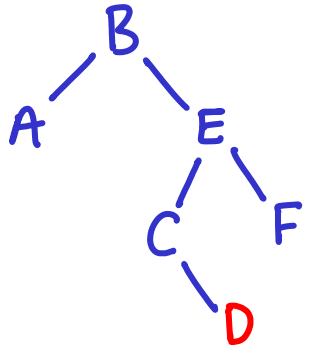
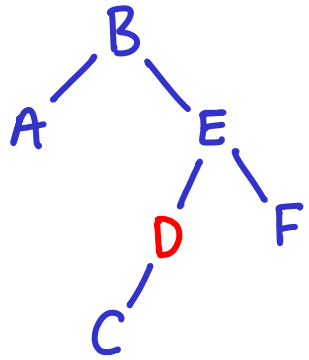
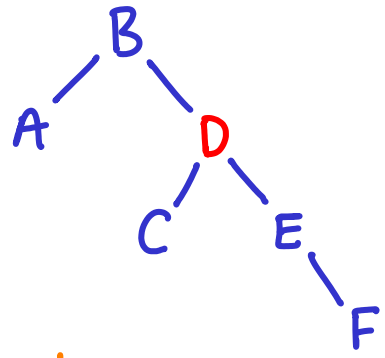
*More specifically, if already taking time t to search,
(even for delete and unsuccessful search)

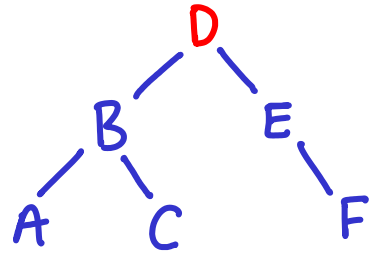
use $O(t)$ time to do rotations

Easy to move to top with rotations









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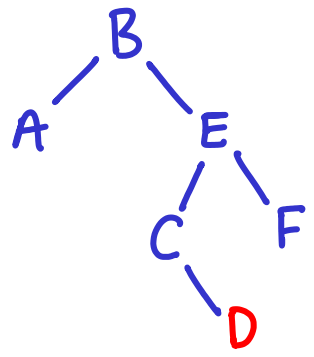

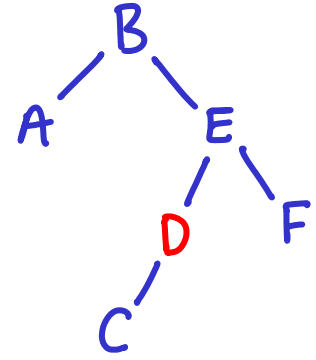

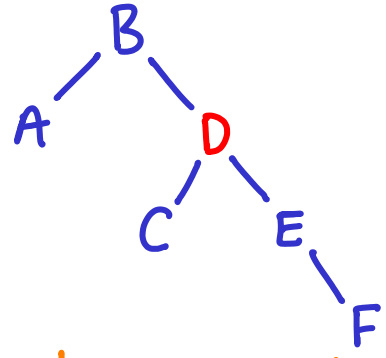

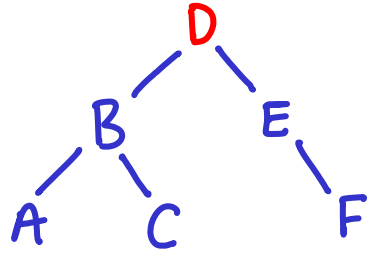

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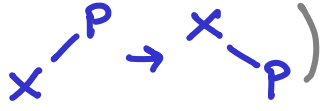
(if no G then 

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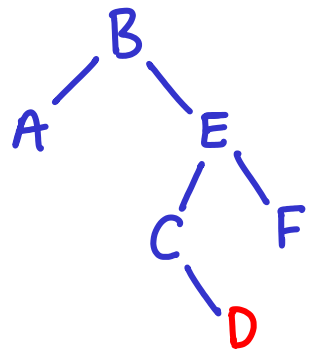

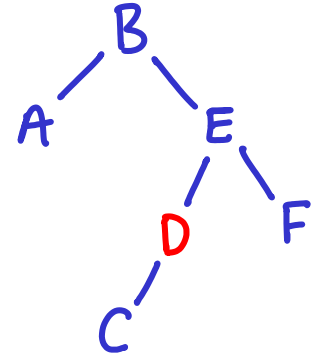

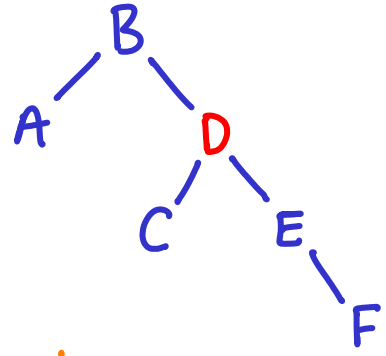

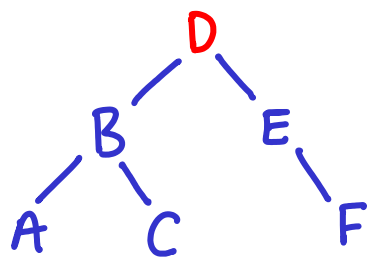

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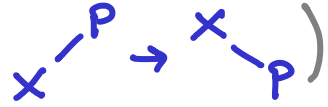
(if no G then 

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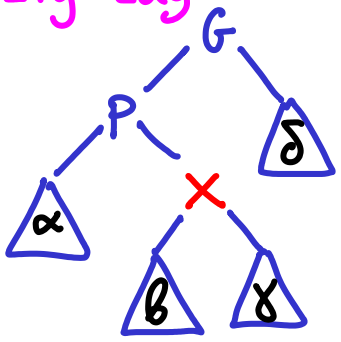

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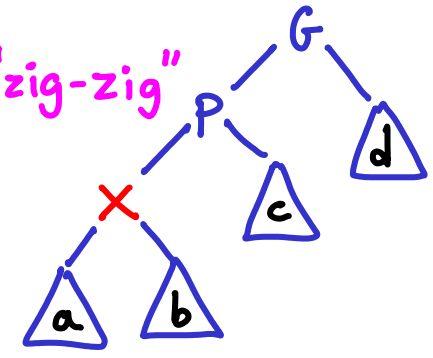

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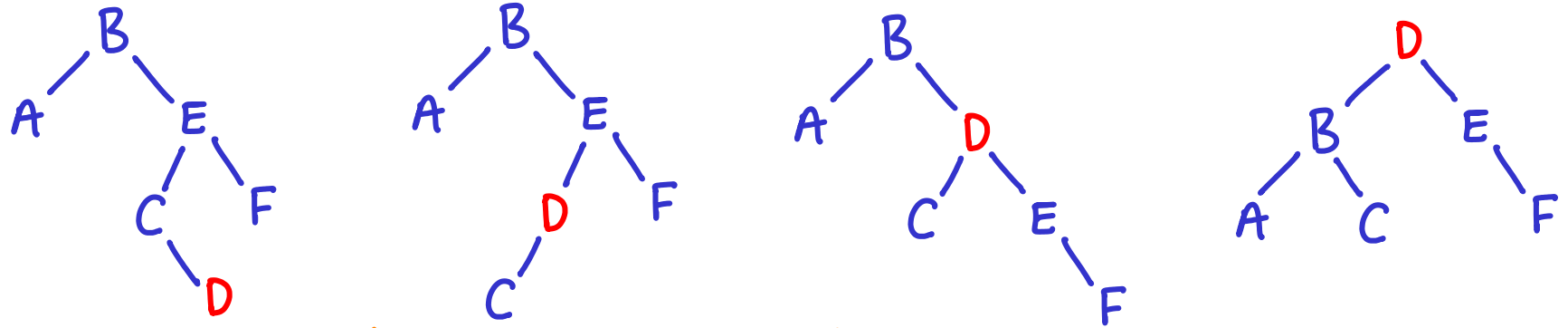

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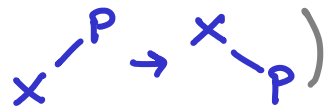
(if no G then 

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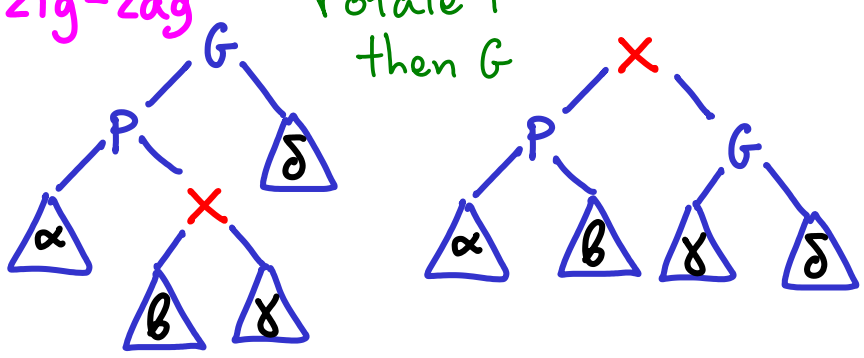
"zig-zag" rotate P then G



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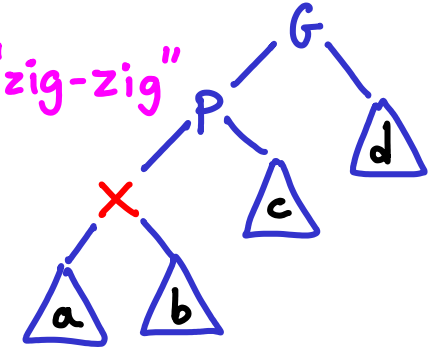

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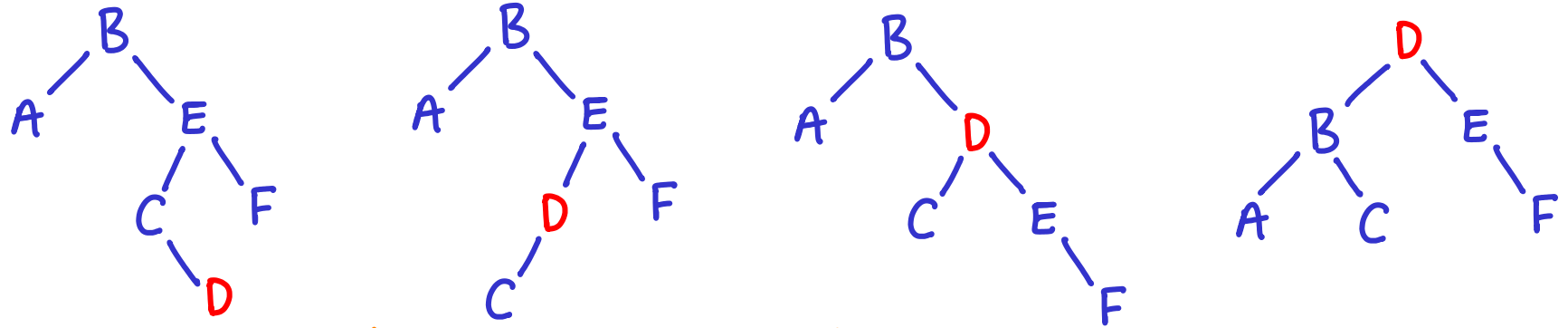

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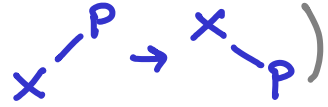
(if no G then 

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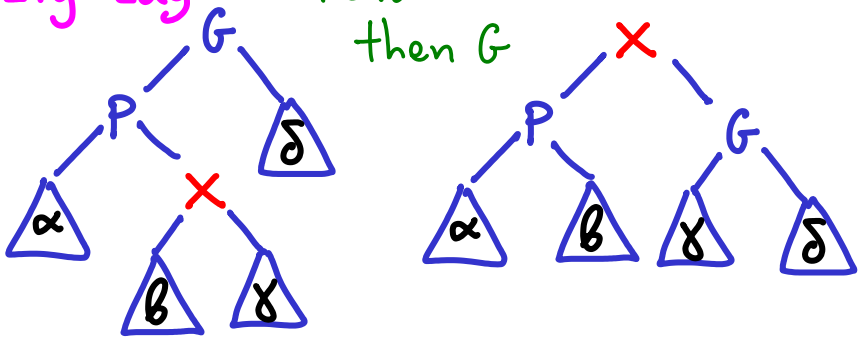

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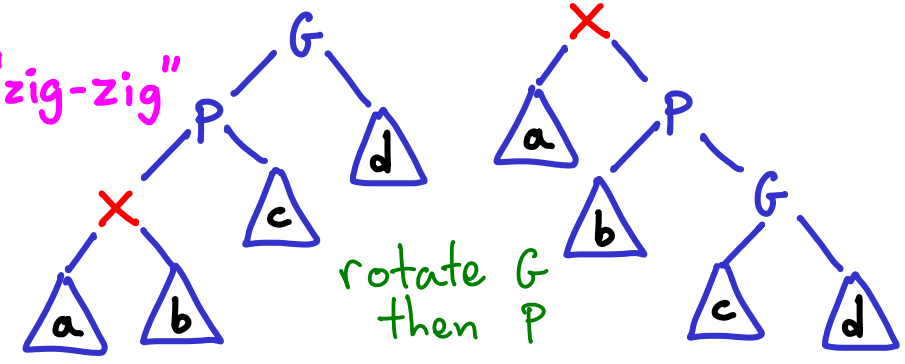
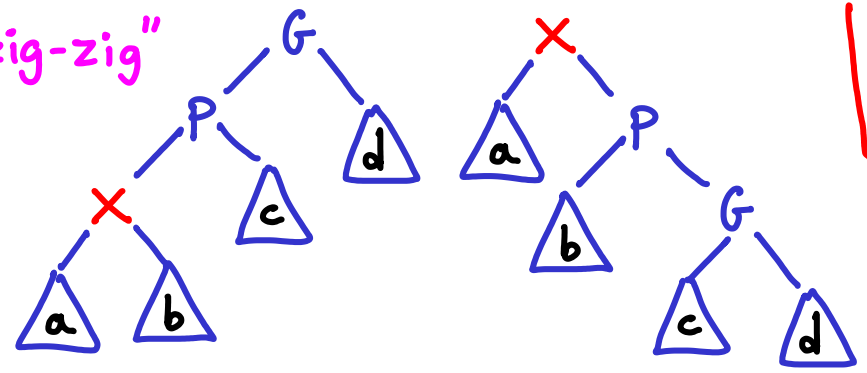


- Splay node x if we access it (search or insert)
- Splay parent of any deleted node (possibly parent of successor)
- Splay last visited node if a search is unsuccessful

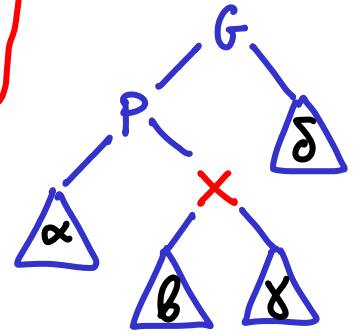
Bottom line: splay up (to root) from deepest position of search
(the search buys us splaying time)

There are other variants, e.g. splay-before-delete or top-down splaying

"zig-zig"



What does one step accomplish?



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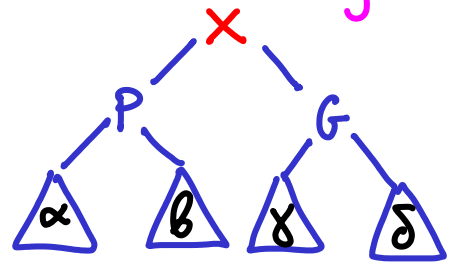

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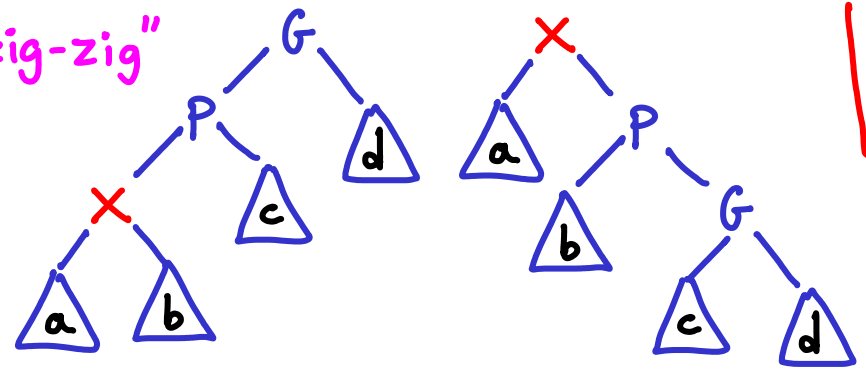

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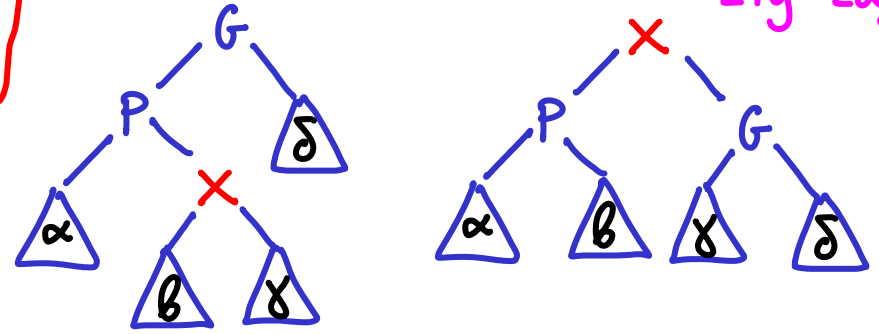

sort of has a tendency to decrease depth

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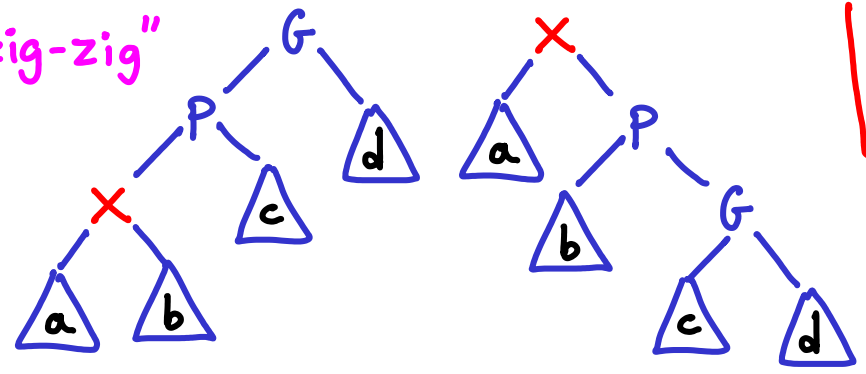

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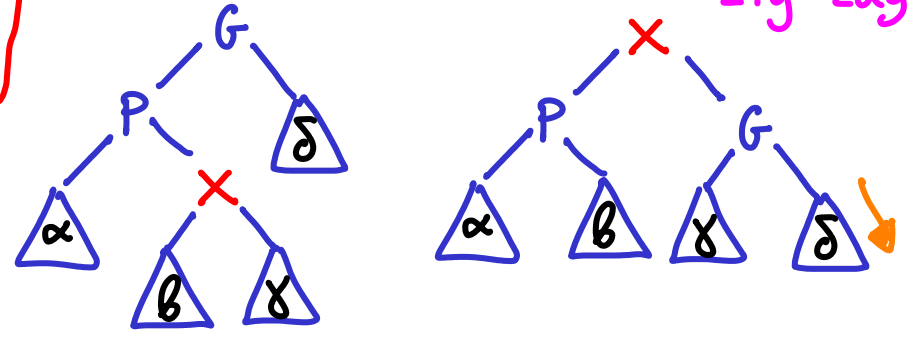

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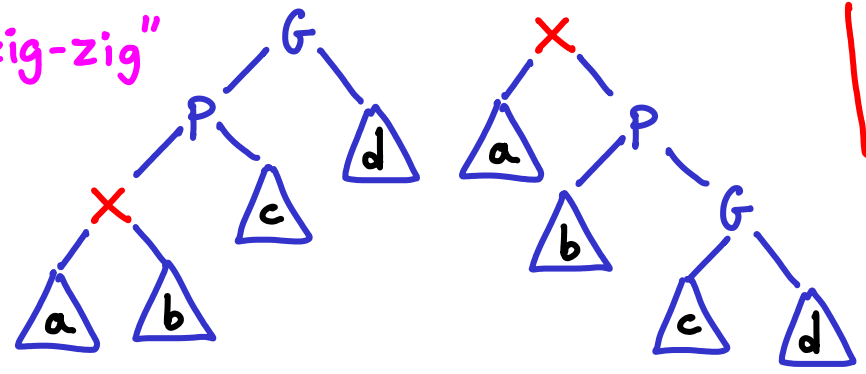

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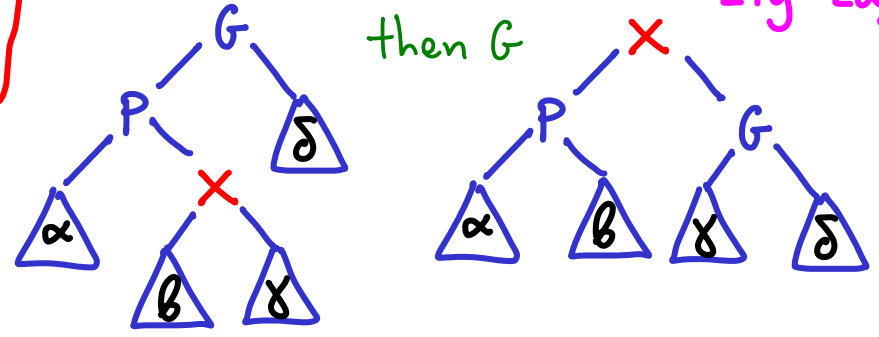

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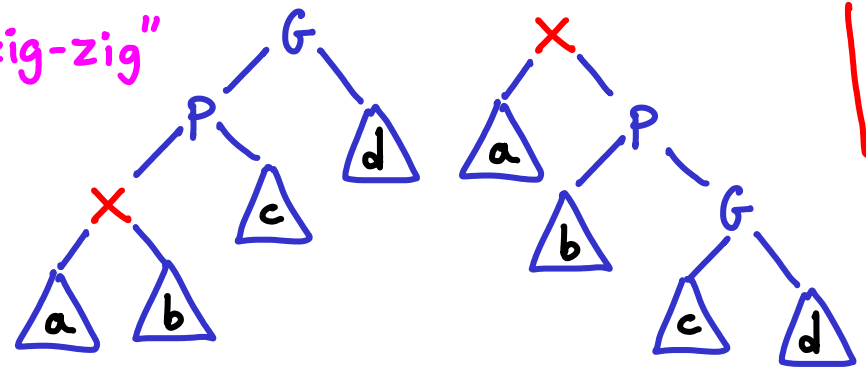

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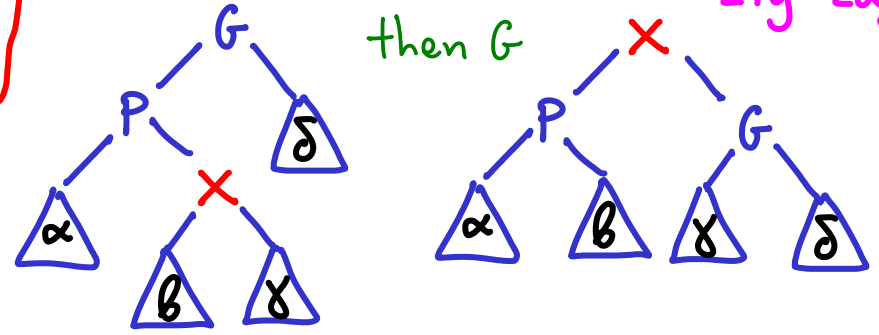

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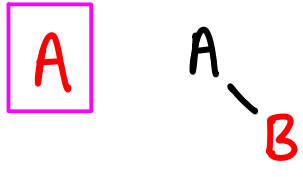

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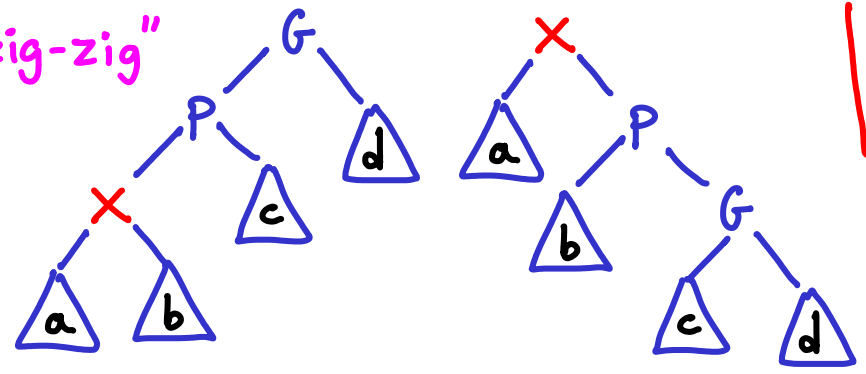

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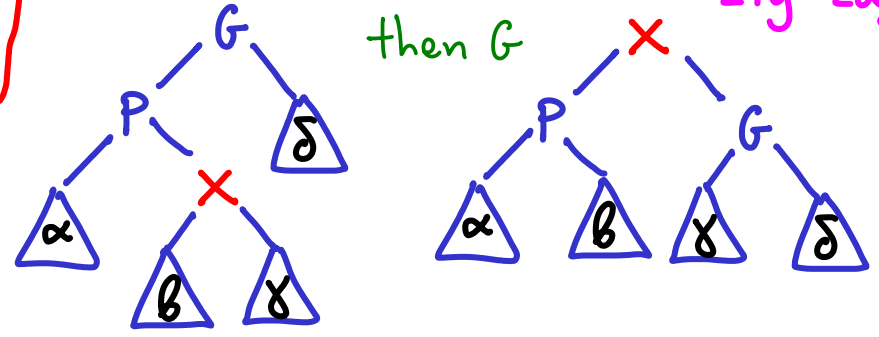

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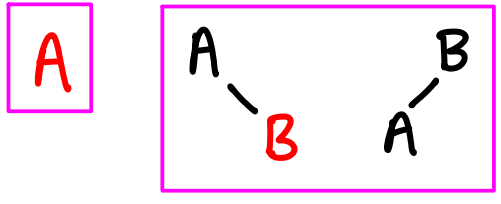

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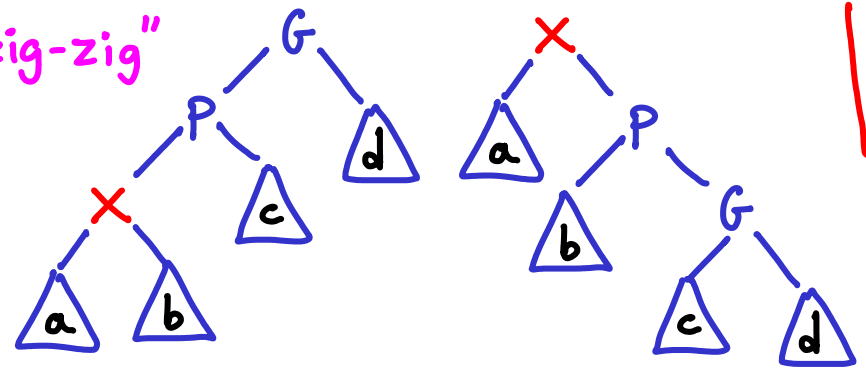

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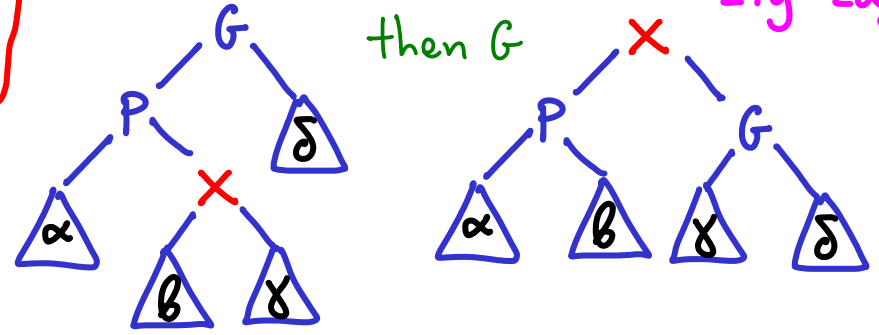

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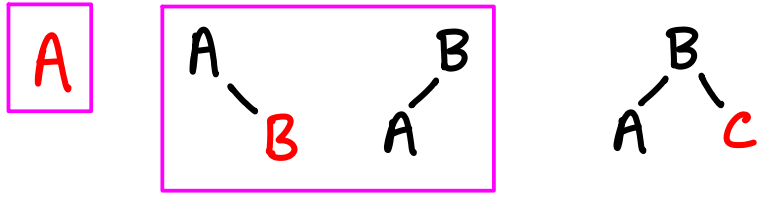

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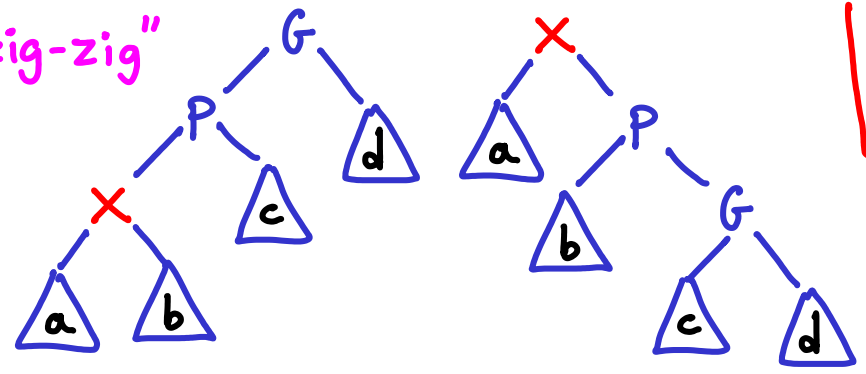

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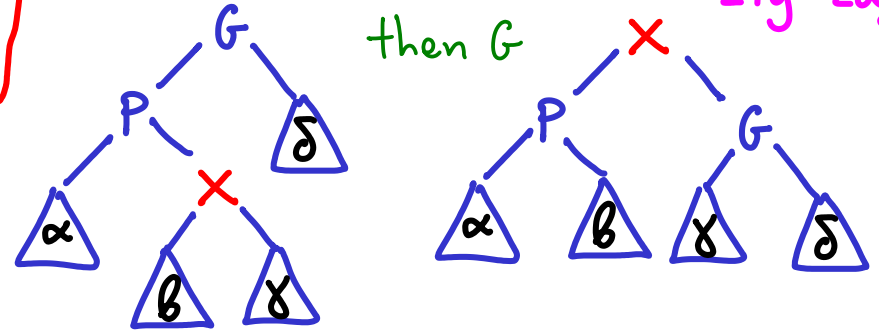

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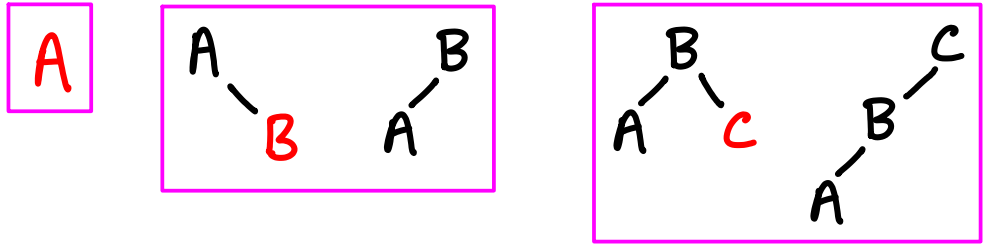

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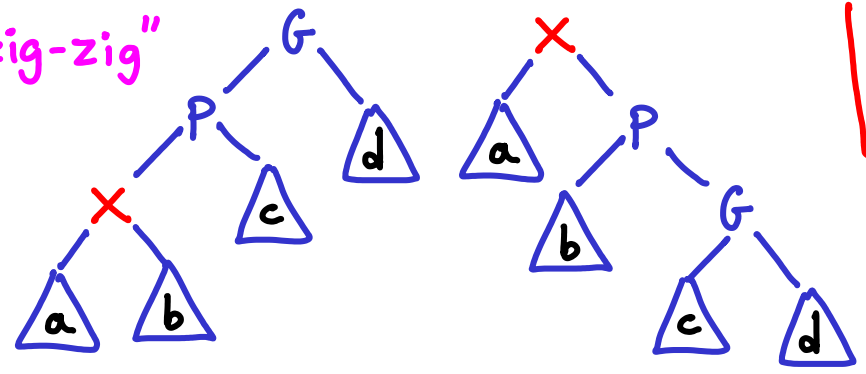

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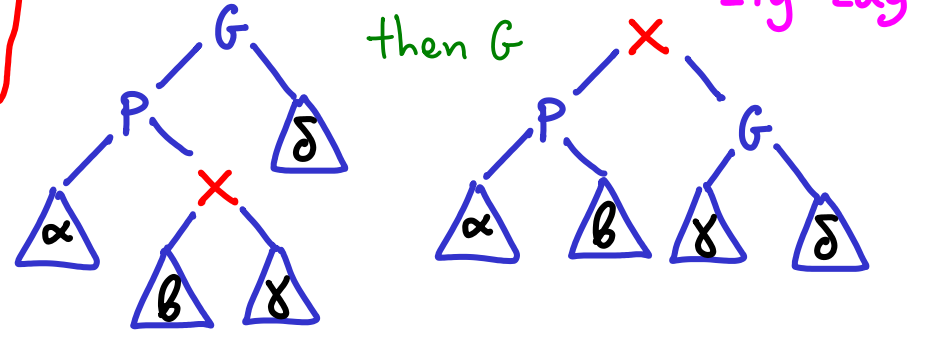

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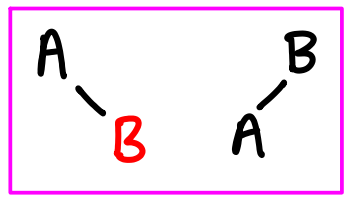

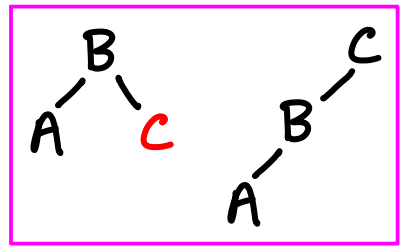

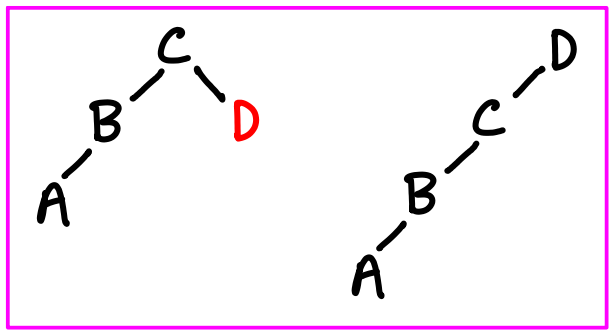

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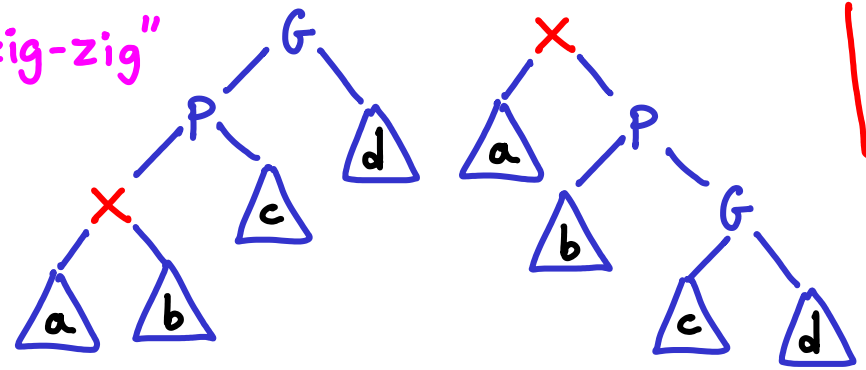

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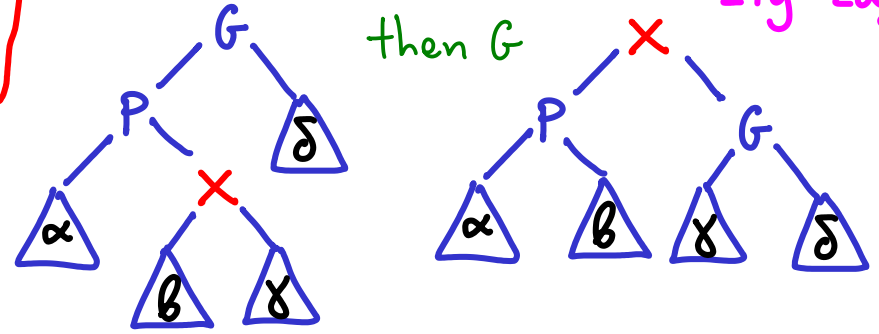

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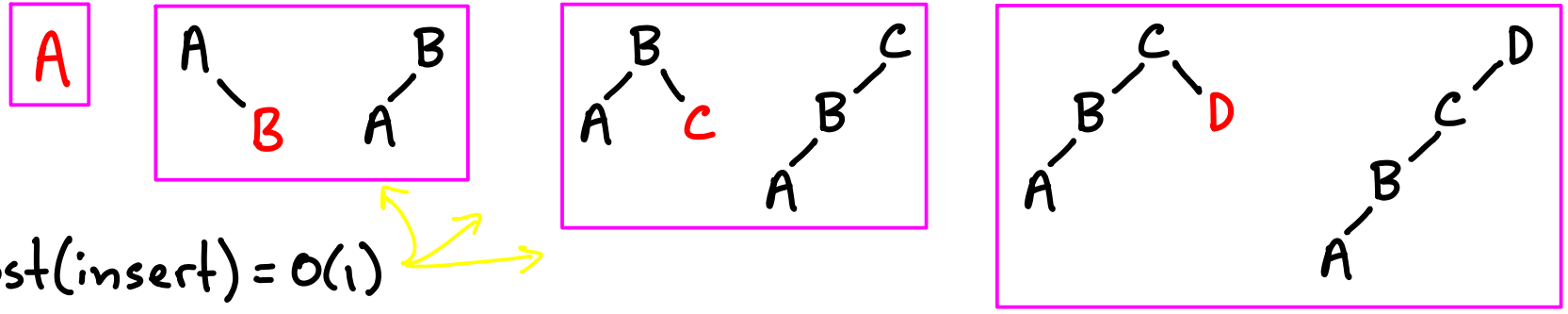
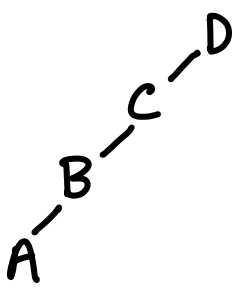
insert keys alphabetically



↳ depth = n but cost(insert) = O(1)

A splay tree can be awfully unbalanced but it will be cheap to create



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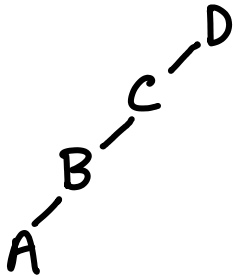

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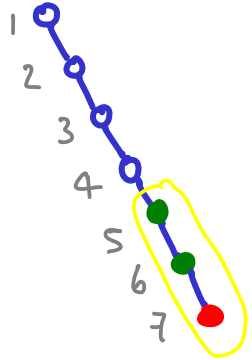

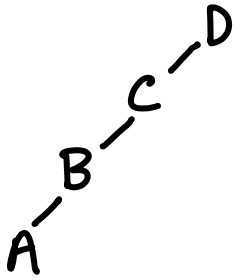

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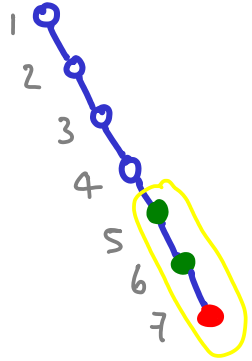

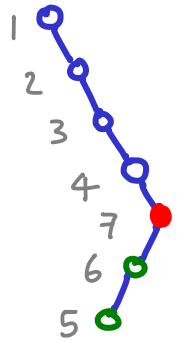

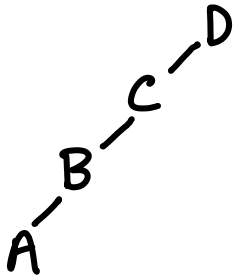

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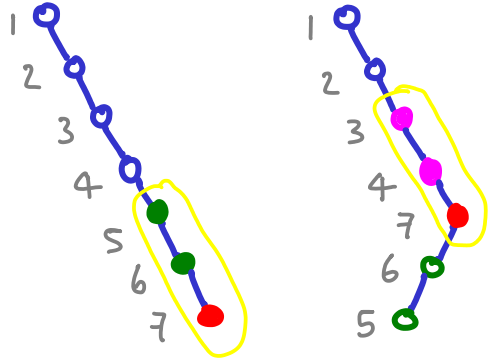

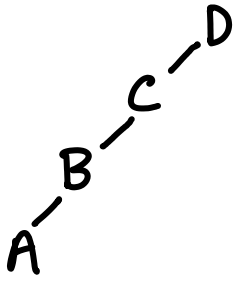

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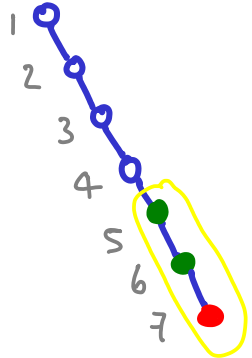

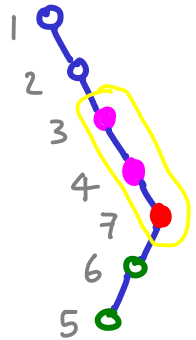

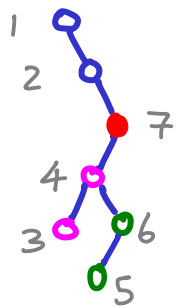

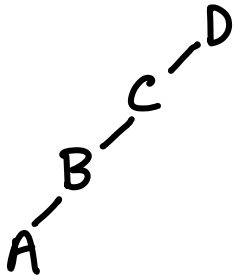

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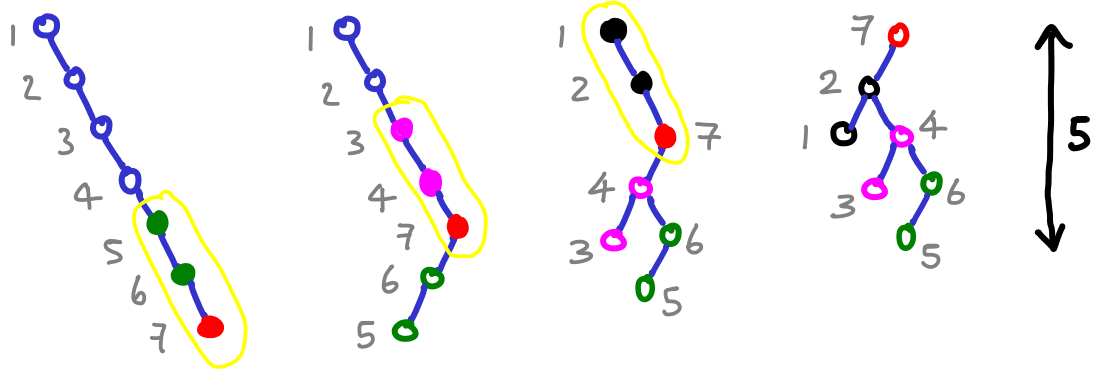

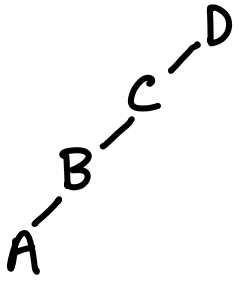

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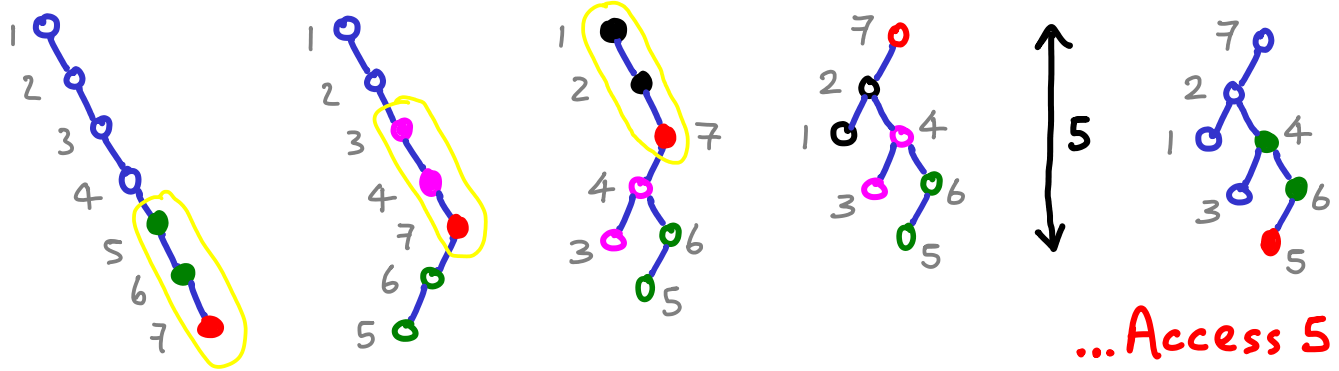

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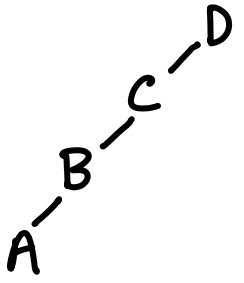

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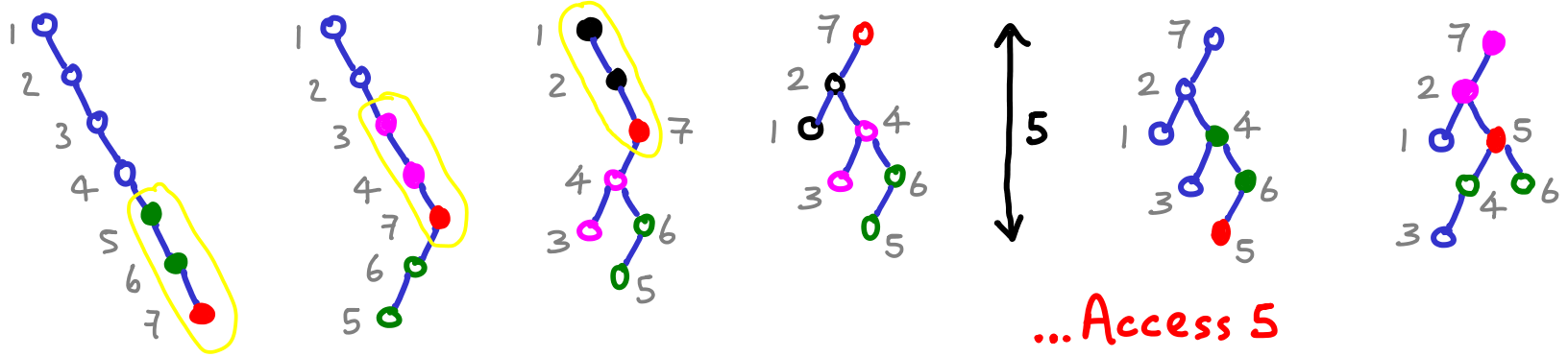

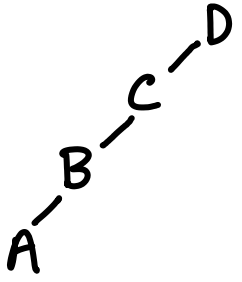

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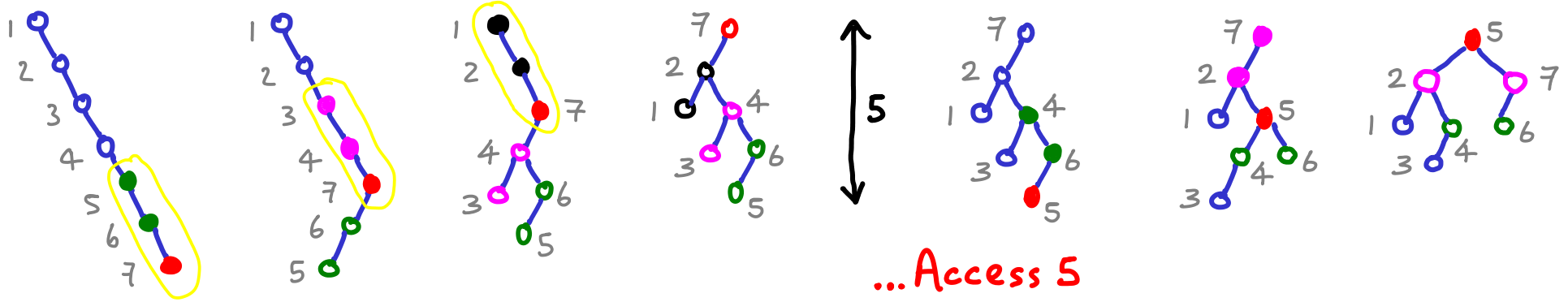

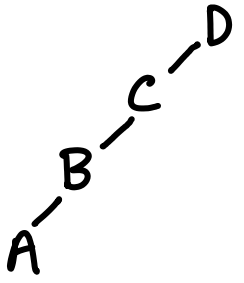

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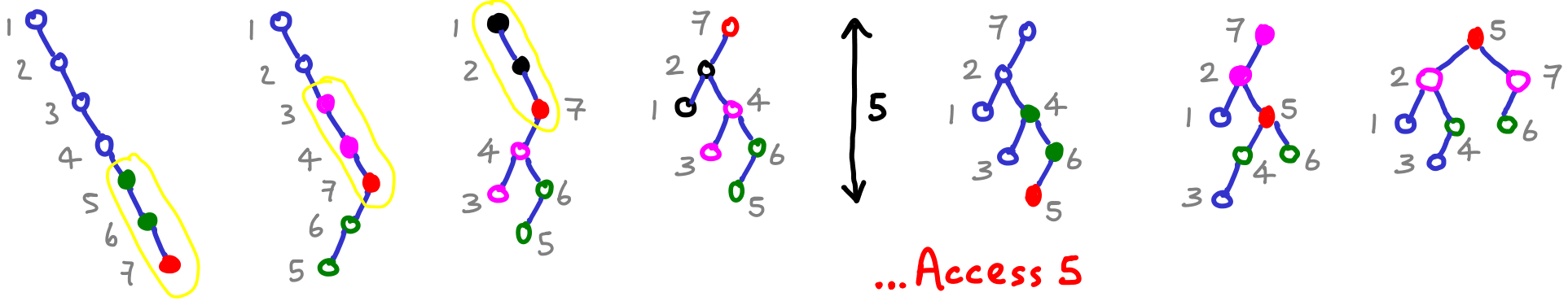

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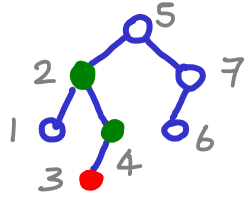

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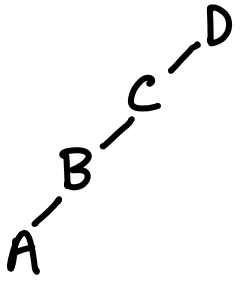

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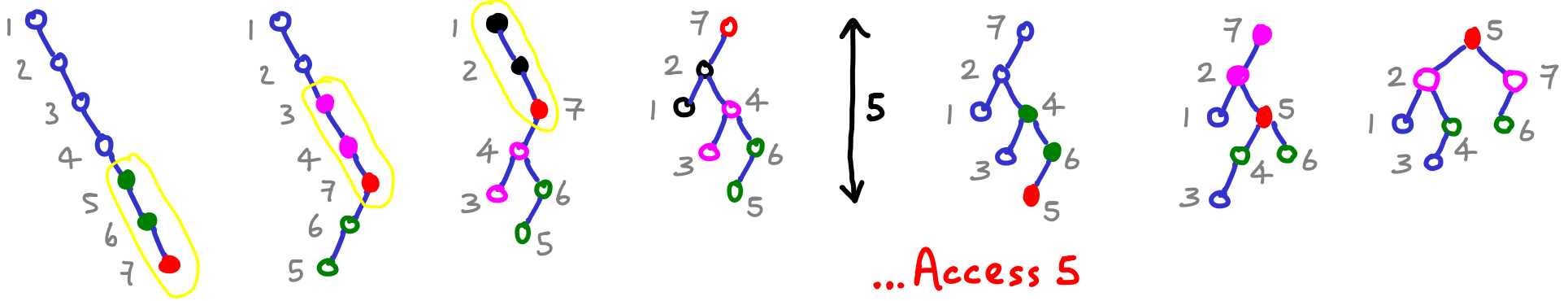

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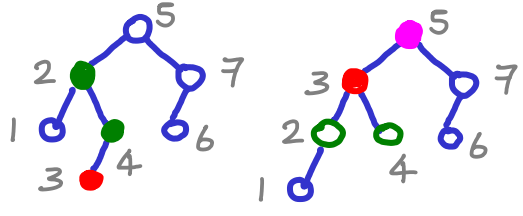

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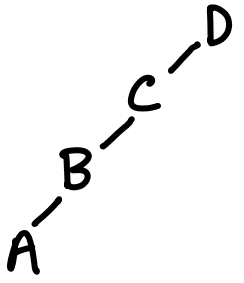

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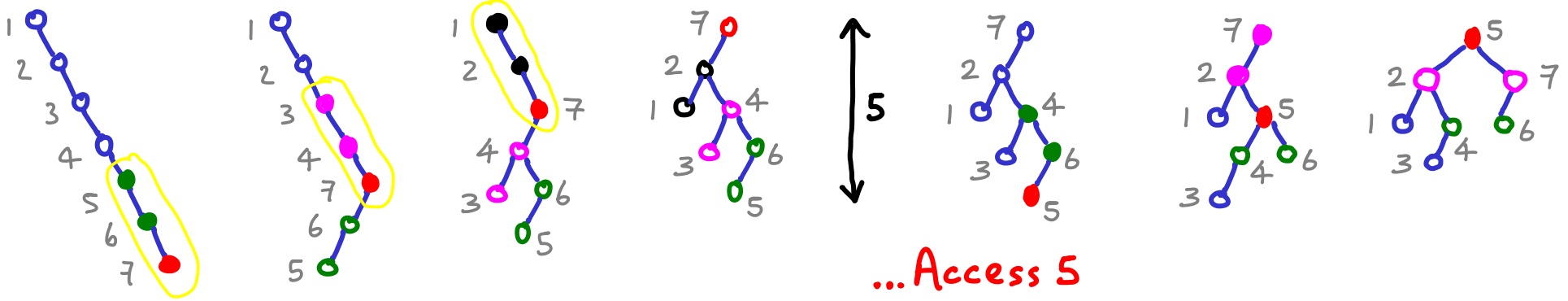

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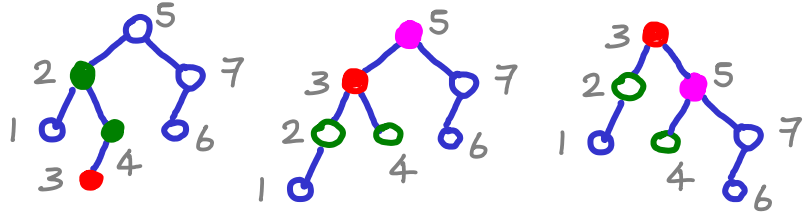

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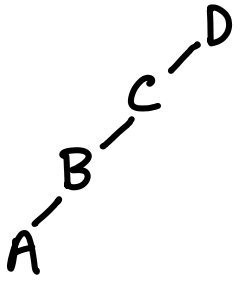

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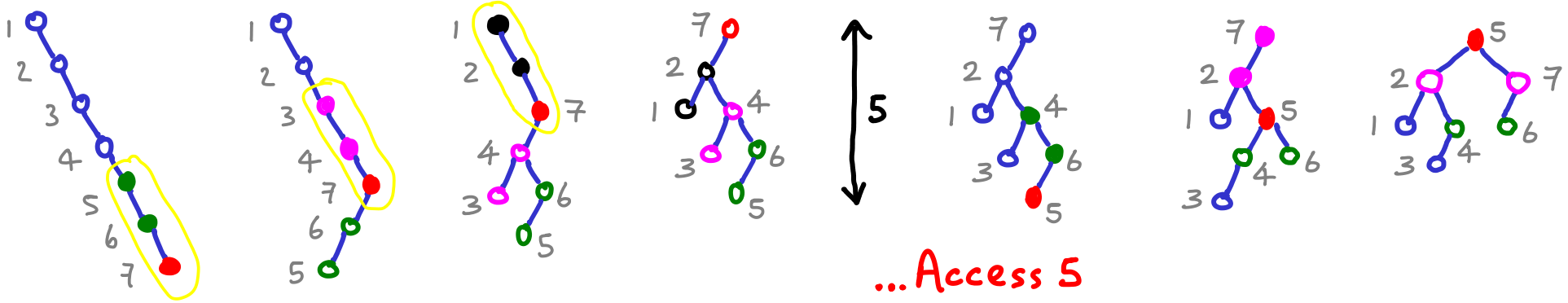

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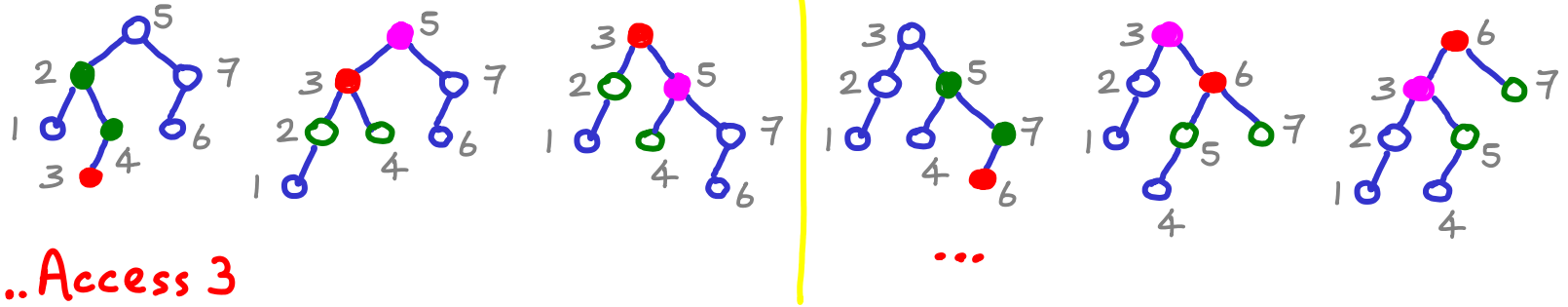

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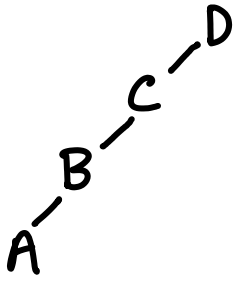

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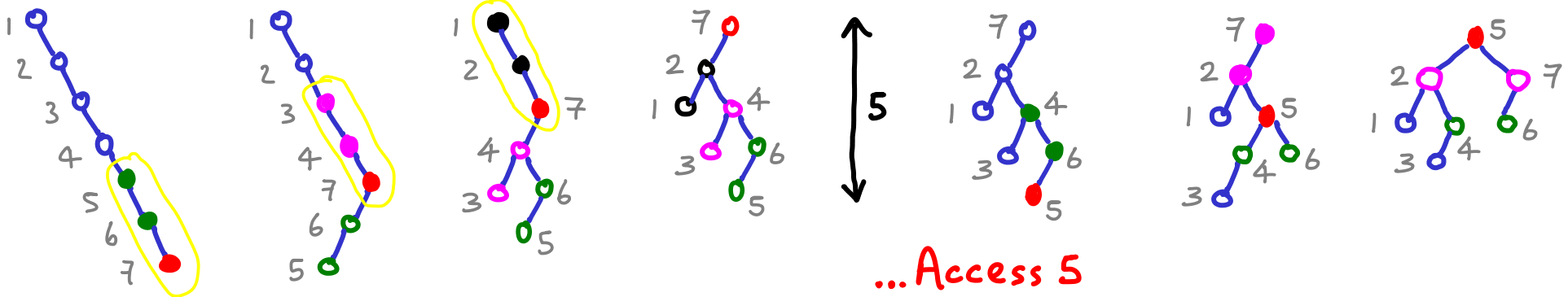

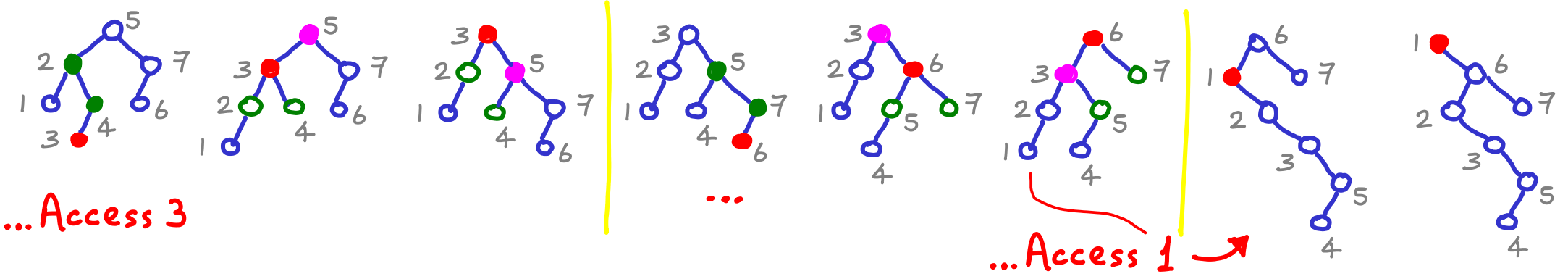

:?

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#nodes in subtree rooted at v = size(v)

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$s(v) = \log_2(\#\text{nodes in subtree rooted at } v) = \log \text{size}(v)$

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What happens when insert x ?

↑ to Φ

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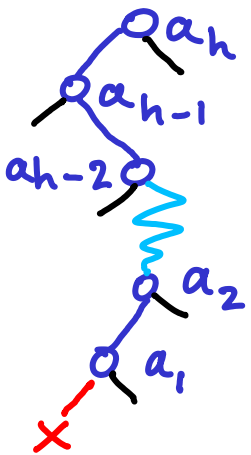

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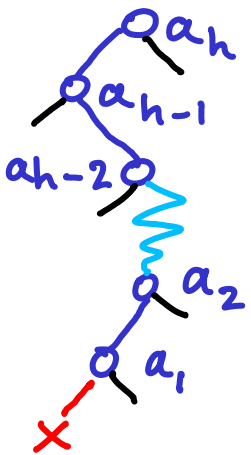
old old



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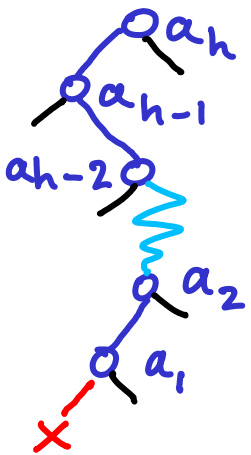

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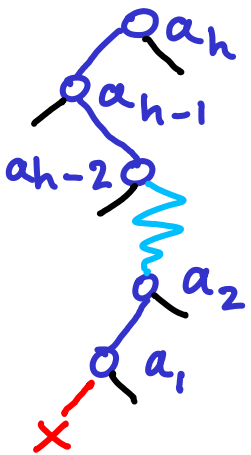

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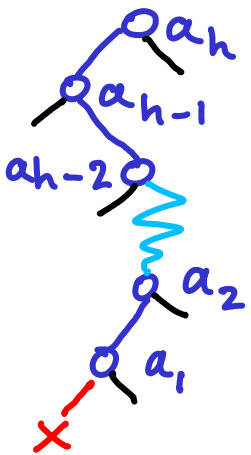

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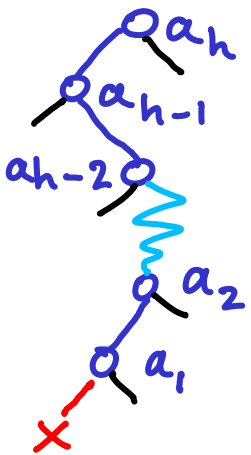

$$= \sum_{i=1}^{h} \log \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} = \log \prod_{i=1}^{h} \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} \leq \log \left[\prod_{i=1}^{h-1} \frac{\text{size}(a_{i+1})}{\text{size}(a_i)} \cdot \frac{1 + \text{size}(a_h)}{\text{size}(a_h)} \right]$$

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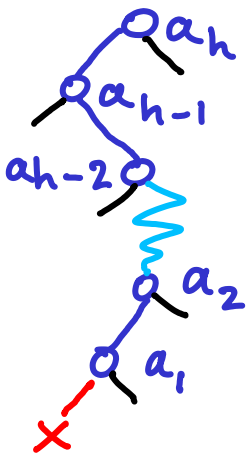
Amortize! $\Phi = \sum_{\text{all } v} s(v)$ $\quad s(v) = \log_2(\#\text{nodes in subtree rooted at } v) = \log \text{size}(v)$

What happens when insert x ?

↳ s( ) changes only for h nodes on path: each eventual ancestor $a_1 \ldots a_h$

$$\Delta\Phi = \sum_{i=1}^{h} \left[\log(\text{new size}(a_i)) - \log(\text{old size}(a_i))\right] = \sum_{i=1}^{h} \left[\log(1 + \text{size}(a_i)) - \log(\text{size}(a_i))\right]$$



$$= \sum_{i=1}^{h} \log \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} = \log \prod_{i=1}^{h} \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} \leq \log \left[\prod_{i=1}^{h-1} \frac{\text{size}(a_{i+1})}{\text{size}(a_i)} \cdot \frac{1 + \text{size}(a_h)}{\text{size}(a_h)} \right]$$

$$= \log \frac{1 + \text{size}(a_h)}{\text{size}(a_1)}$$

// fractions telescope

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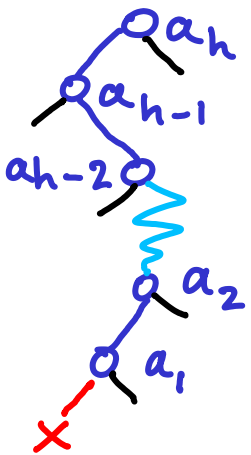

$$= \sum_{i=1}^{h} \log \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} = \log \prod_{i=1}^{h} \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} \leq \log \left[\prod_{i=1}^{h-1} \frac{\text{size}(a_{i+1})}{\text{size}(a_i)} \cdot \frac{1 + \text{size}(a_h)}{\text{size}(a_h)} \right]$$

$$= \log \frac{1 + \text{size}(a_h)}{\text{size}(a_1)} \leq \log\left[1 + \text{size}(a_h)\right]$$

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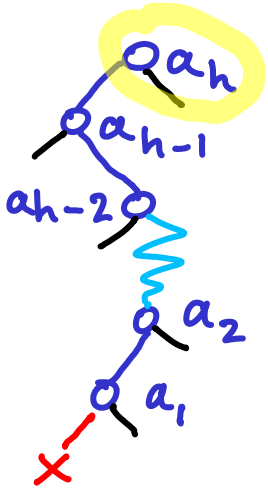

$$= \sum_{i=1}^{h} \log \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} = \log \prod_{i=1}^{h} \frac{1 + \text{size}(a_i)}{\text{size}(a_i)} \leq \log \left[\prod_{i=1}^{h-1} \frac{\text{size}(a_{i+1})}{\text{size}(a_i)} \cdot \frac{1 + \text{size}(a_h)}{\text{size}(a_h)} \right]$$

$$= \log \frac{1 + \text{size}(a_h)}{\text{size}(a_1)} \leq \log\left[1 + \text{size}(a_h)\right] = \log\left[1 + \text{size}(\text{root})\right]$$

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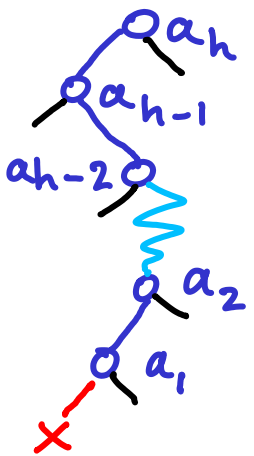

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$$= \log \frac{1 + \text{size}(a_h)}{\text{size}(a_1)} \leq \log\left[1 + \text{size}(a_h)\right] = \log\left[1 + \text{size}(\text{root})\right] = \log n$$

(old)

$\Phi = \sum_{\text{all } v} s(v)$ $\quad s(v) = \log_2(\#\text{nodes in subtree rooted at } v) = \log \text{size}(v)$

What happens when insert x ? $\quad \Delta\Phi \leq \log n$

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delete x ?

$\Phi = \sum_{\text{all } v} s(v)$ $\quad s(v) = \log_2(\#\text{ nodes in subtree rooted at } v) = \log \text{size}(v)$

What happens when insert x ? $\quad \Delta\Phi \leq \log n$

delete x ? $\quad \leq 0$ good news

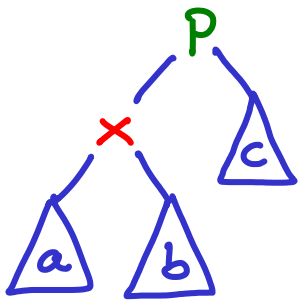
We must still consider $\Delta\Phi$ because of splaying

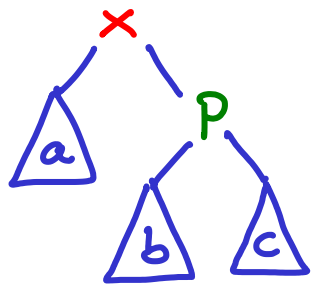
$\Phi = \sum_{\text{all } v} s(v)$

$s(v) = \log_2(\#\text{nodes in subtree rooted at } v)$

What happens when we splay x and there is no grandparent?

↳ s( ) changes only for x and p





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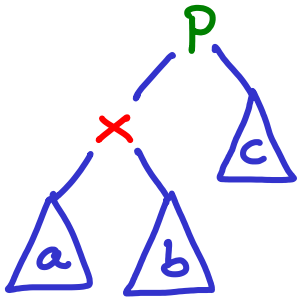

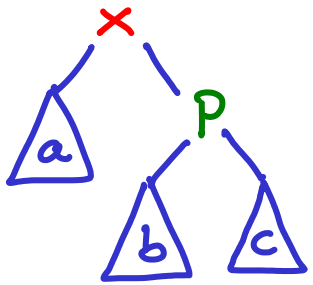

$\Phi = \sum_{\text{all } v} s(v)$ $\quad s(v) = \log_2(\#\text{nodes in subtree rooted at } v)$

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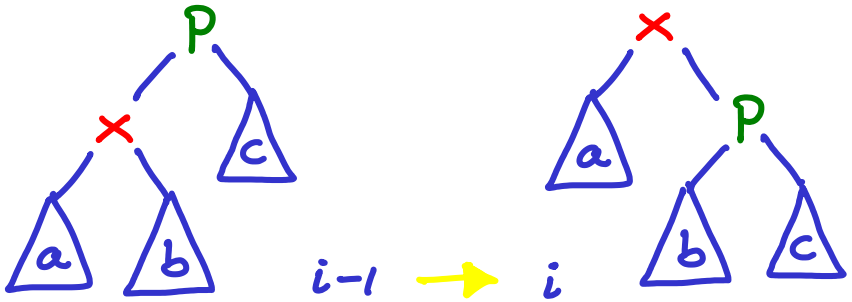
$$\Delta\Phi = \Delta s(p) + \Delta s(x)$$

$$= s_i(p) - s_{i-1}(p) + s_i(x) - s_{i-1}(x)$$



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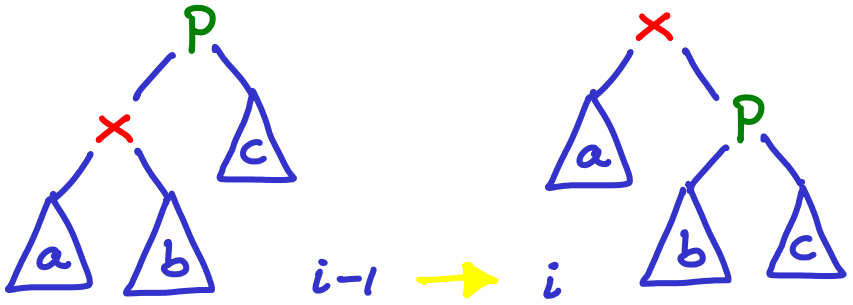

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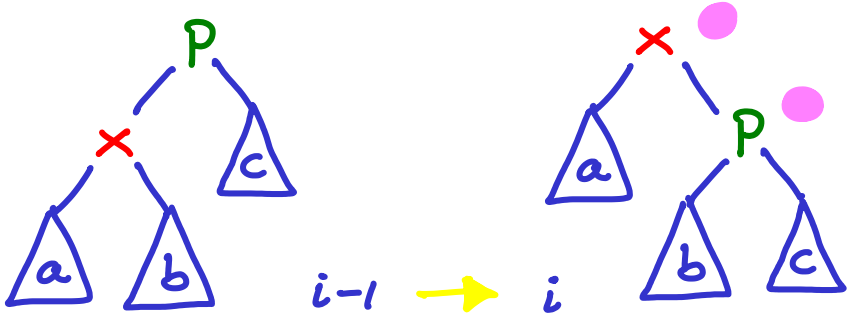

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$$= s_i(p) - s_{i-1}(p) + s_i(x) - s_{i-1}(x)$$

$$= s_i(p) - s_{i-1}(x) \quad // \quad s_i(p) < s_i(x)$$

$$< s_i(x) - s_{i-1}(x)$$

$\Phi = \sum_{\text{all } v} s(v)$ $\quad s(v) = \log_2(\#\text{nodes in subtree rooted at } v)$

What happens when we splay x and there is no grandparent?

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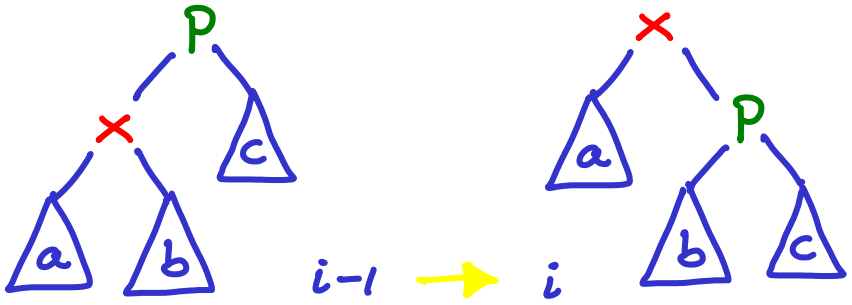

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$$= s_i(p) - s_{i-1}(x) \quad // \; s_i(p) < s_i(x)$$

$$< s_i(x) - s_{i-1}(x)$$

always positive

$\Delta\Phi$ could actually be $=0$, <0 or >0

Why are we happy with this?

$\Phi = \sum_{\text{all } v} s(v)$ $\quad s(v) = \log_2(\#\text{nodes in subtree rooted at } v)$

What happens when we splay x and there is no grandparent?

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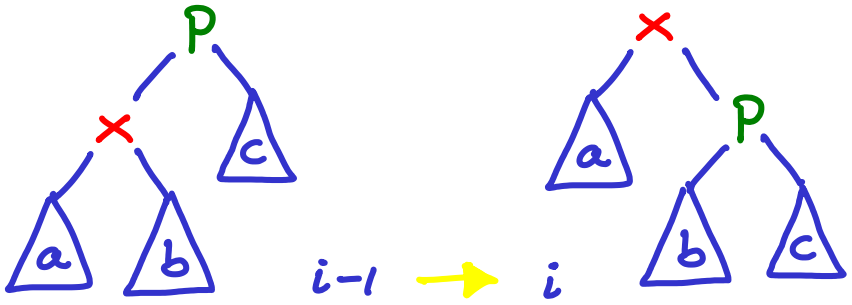

$\Delta\Phi = \Delta s(p) + \Delta s(x)$

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$= s_i(p) - s_{i-1}(x)$ // $s_i(p) < s_i(x)$

$< s_i(x) - s_{i-1}(x)$

always positive

We have $\Delta\Phi < s_i(x) = s(\text{root})$

$= O(\log n)$

$\Delta\Phi$ could actually be $=0$, <0 or >0

Why are we happy with this?

$\Phi = \sum_{\text{all } v} s(v)$ $\qquad s(v) = \log_2(\#\text{nodes in subtree rooted at } v)$

What happens when we splay x and there is no grandparent?

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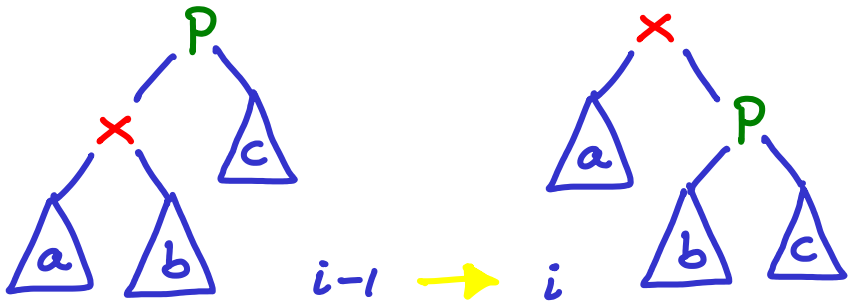


$$\Delta\Phi = \Delta s(p) + \Delta s(x)$$

$$= s_i(p) - s_{i-1}(p) + s_i(x) - s_{i-1}(x)$$

$$= s_i(p) - s_{i-1}(x) \quad // \; s_i(p) < s_i(x)$$

$$< s_i(x) - s_{i-1}(x)$$

always positive

We have $\Delta\Phi < s_i(x) = s(\text{root})$

$= O(\log n)$

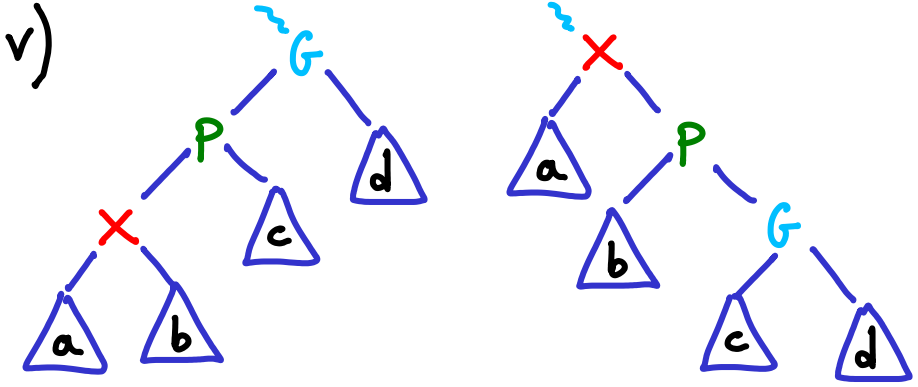
We would be happy even if this iterated (telescope)

$\Delta\Phi$ could actually be $=0$, <0 or >0

Why are we happy with this?

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$

What happens when we zig-zig x?



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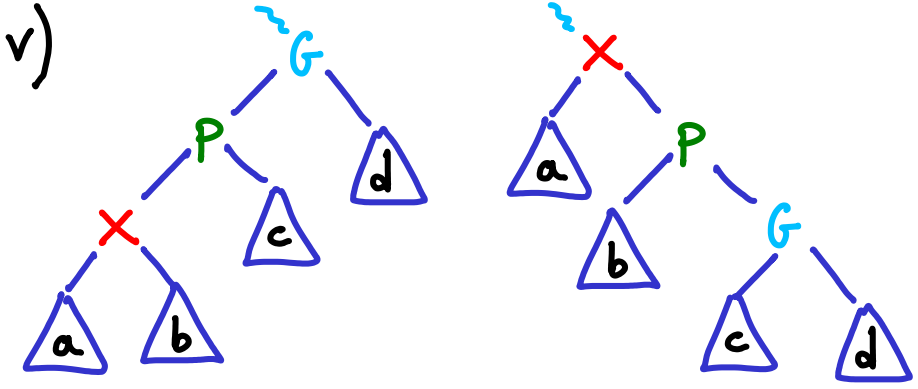
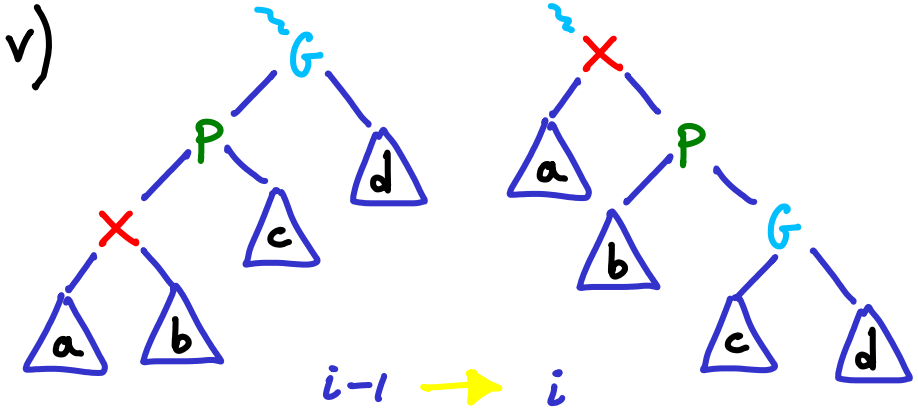


What happens when we zig-zig x?

$\Delta\Phi = \Delta s(G) + \Delta s(P) + \Delta s(x)$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$



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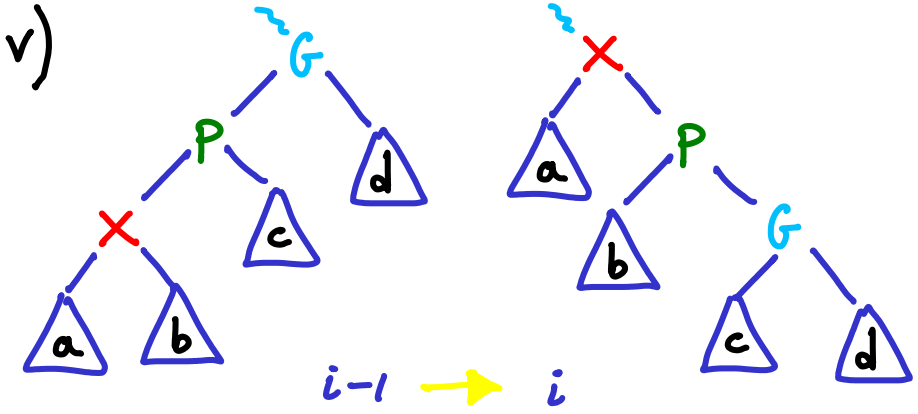
$$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(x) - s_{i-1}(x)$$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$



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$$\Delta\Phi = \Delta s(G) + \Delta s(P) + \Delta s(x)$$

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$$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(x)$$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$

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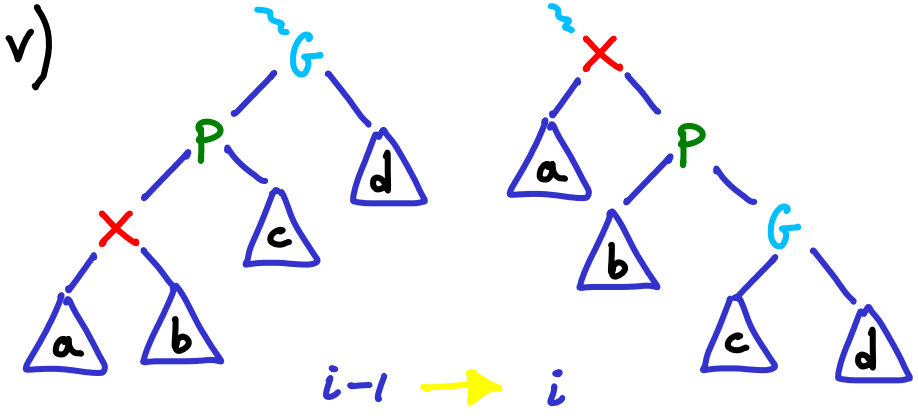


$\Delta\Phi = \Delta s(G) + \Delta s(P) + \Delta s(X)$

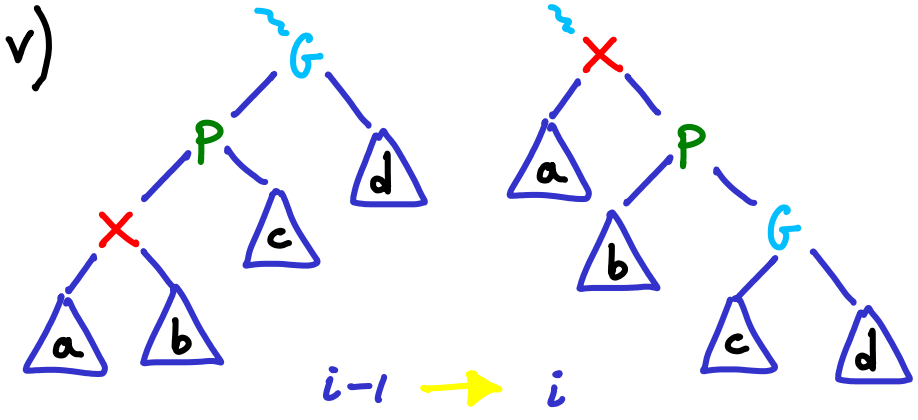
$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(X) - s_{i-1}(X)$

$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(X)$

$s_i(P) < s_i(X)$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$

What happens when we zig-zig x?



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$$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(x)$$

$$< s_i(G) + s_i(x) - s_{i-1}(P) - s_{i-1}(x)$$

$s_i(P) < s_i(x)$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$

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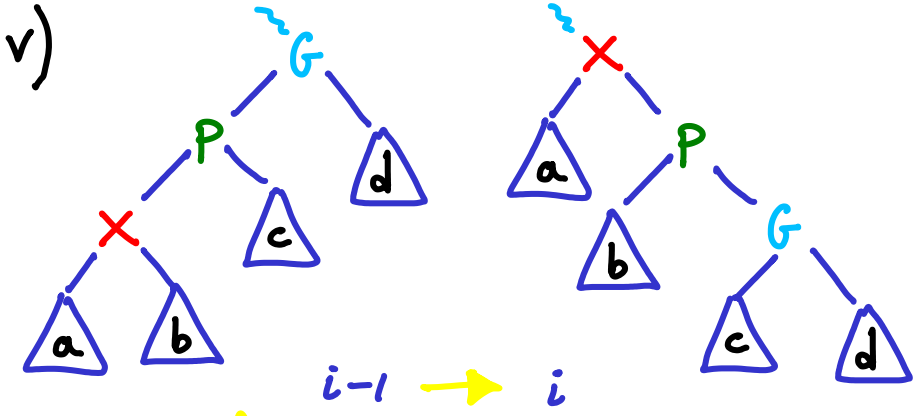


$$\Delta\Phi = \Delta s(G) + \Delta s(P) + \Delta s(x)$$

$$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(x) - s_{i-1}(x)$$

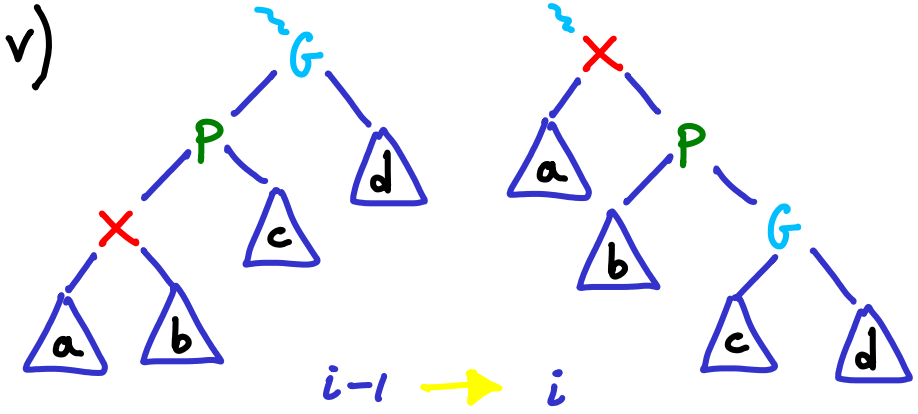
$$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(x)$$

$$< s_i(G) + s_i(x) - s_{i-1}(P) - s_{i-1}(x)$$

$s_i(P) < s_i(x)$

$s_{i-1}(P) > s_{i-1}(x)$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$



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$$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(X)$$

$$< s_i(G) + s_i(X) - s_{i-1}(P) - s_{i-1}(X) \qquad s_i(P) < s_i(X)$$

$$< s_i(G) + s_i(X) - s_{i-1}(X) - s_{i-1}(X) \qquad s_{i-1}(P) > s_{i-1}(X)$$

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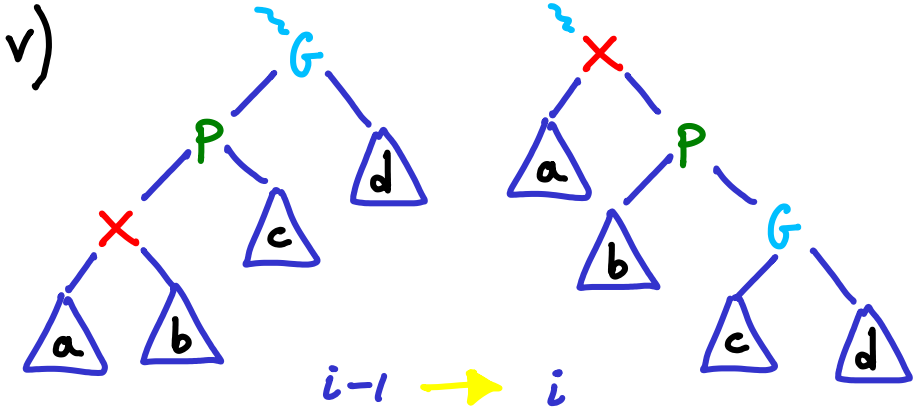

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$$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(X)$$

$$< s_i(G) + s_i(X) - s_{i-1}(P) - s_{i-1}(X) \quad (s_i(P) < s_i(X))$$

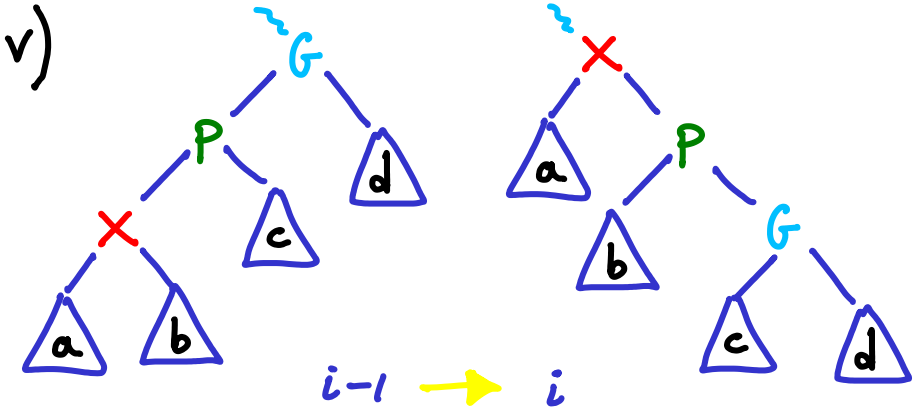
$$< s_i(G) + s_i(X) - s_{i-1}(X) - s_{i-1}(X) \quad (s_{i-1}(P) > s_{i-1}(X))$$

$$< s_i(G) + s_i(X) - 2s_{i-1}(X)$$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$

What happens when we zig-zig x?

$\Delta\Phi < s_i(G) + s_i(x) - 2s_{i-1}(x)$



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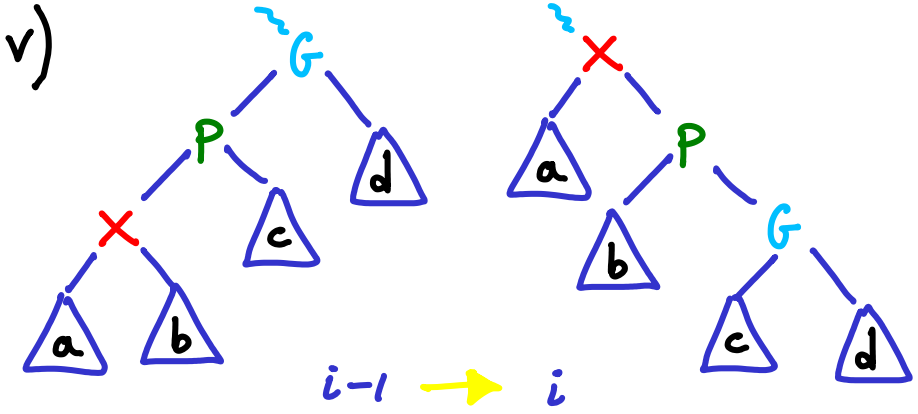
What happens when we zig-zig x?



$$\Delta\Phi < s_i(G) + s_i(x) - 2s_{i-1}(x)$$

$$= s_i(G) + s_{i-1}(x) + s_i(x) - 3s_{i-1}(x)$$

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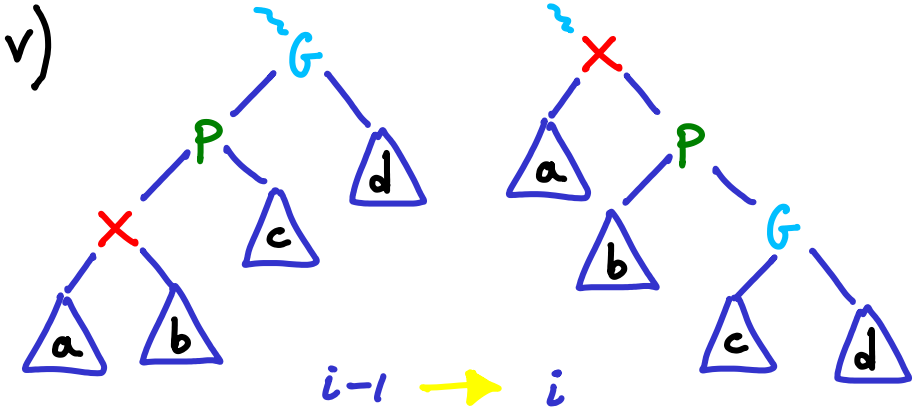

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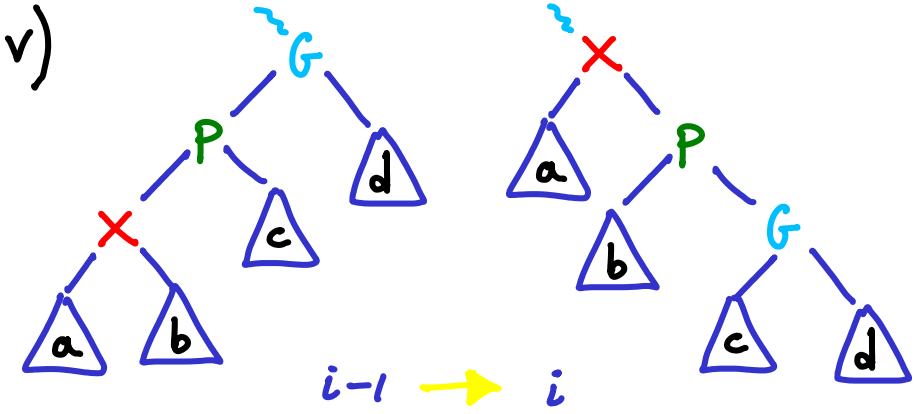
$s_i(G) + s_{i-1}(x) =$ ...

we will show: $s_i(G) + s_{i-1}(x) < 2s_i(x) - 2$

can then add to this

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$



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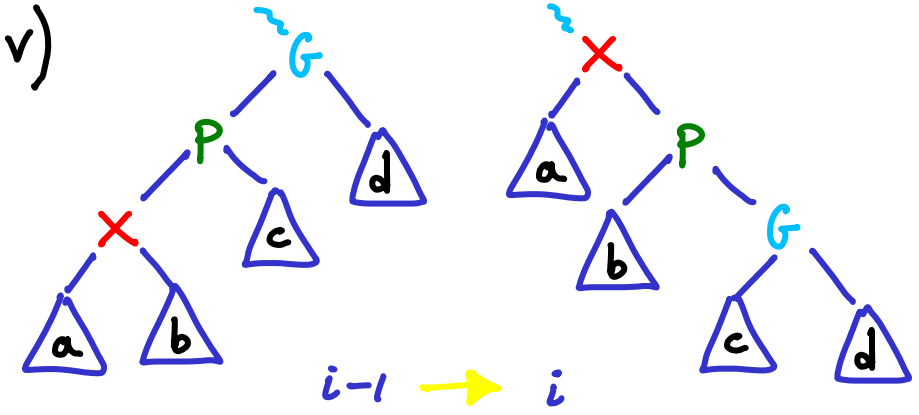

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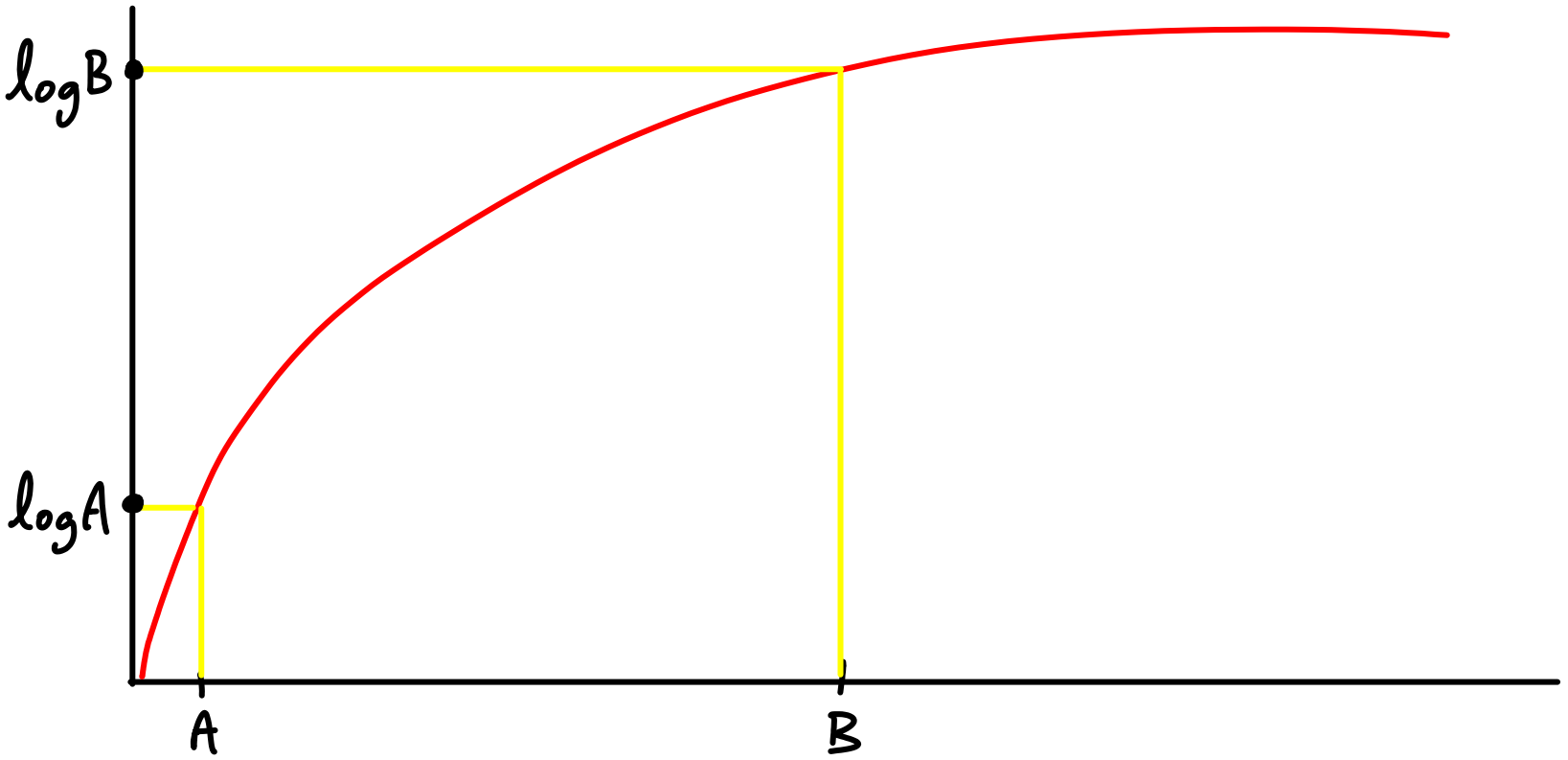
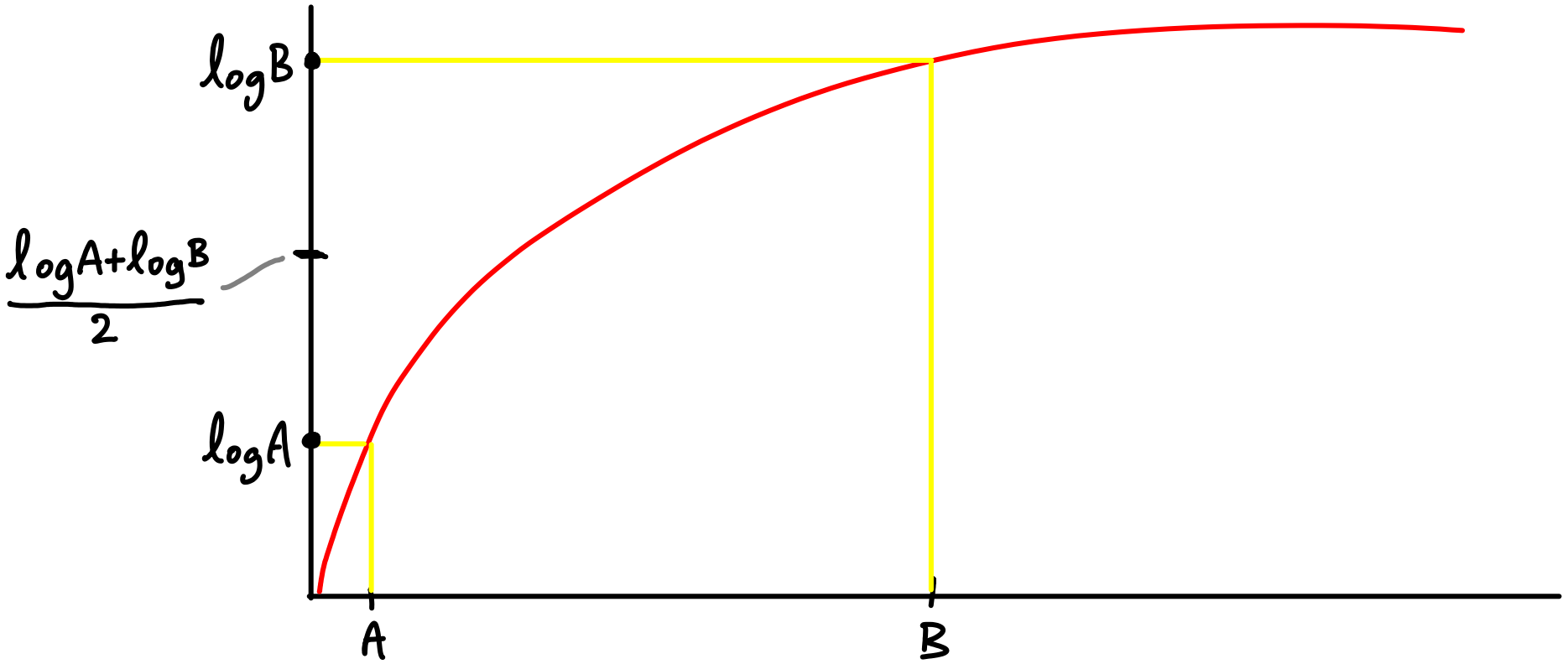
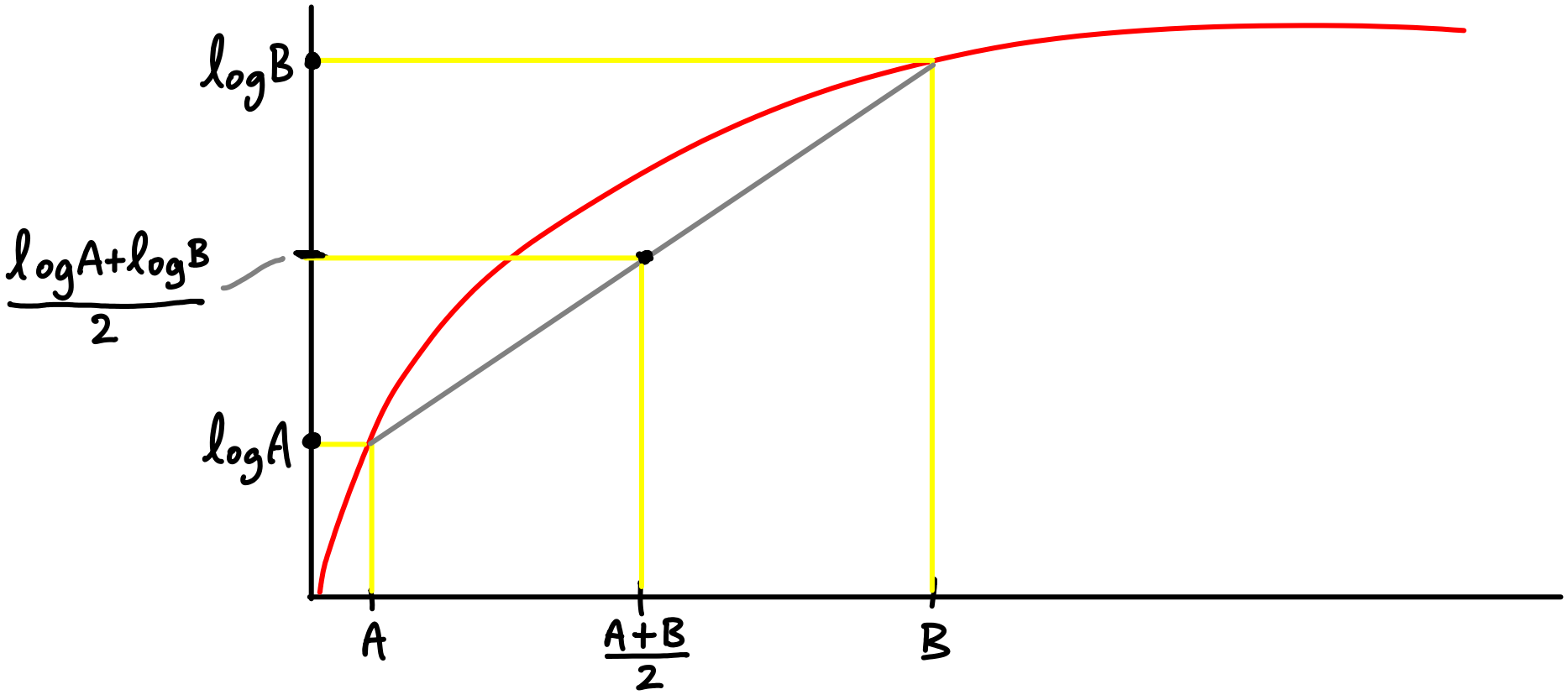
$= s_i(G) + s_{i-1}(x) + s_i(x) - 3s_{i-1}(x)$

$\log A + \log B$ / 2

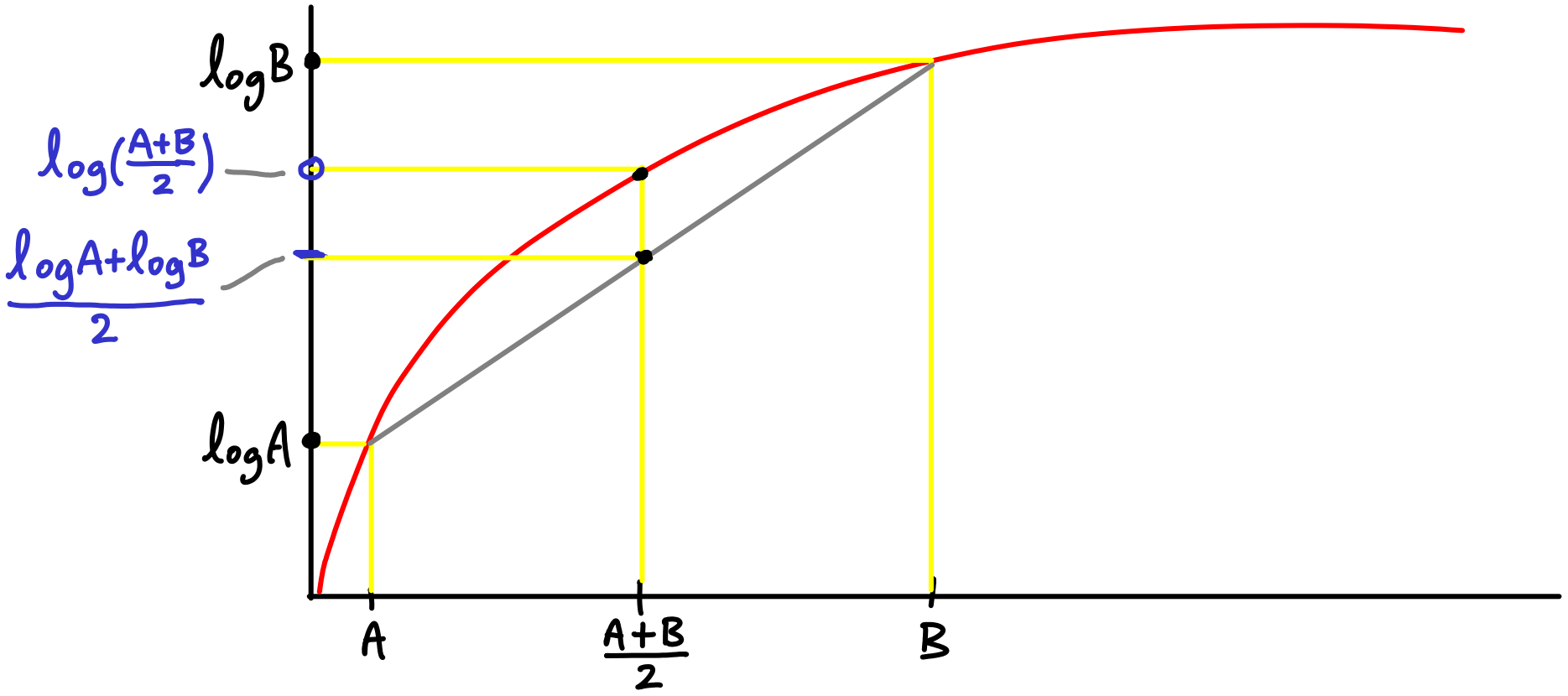
$$s_i(G) + s_{i-1}(x) = 2 \cdot \frac{\log(\text{size}_i(G)) + \log(\text{size}_{i-1}(x))}{2}$$


logB
logA
A
B


$\log B$
$\frac{\log A+\log B}{2}$
$\log A$
A
B


$\log B$
$\frac{\log A+\log B}{2}$
$\log A$
A
$\frac{A+B}{2}$
B

$$\frac{\log A + \log B}{2} \leq \log\left(\frac{A+B}{2}\right)$$



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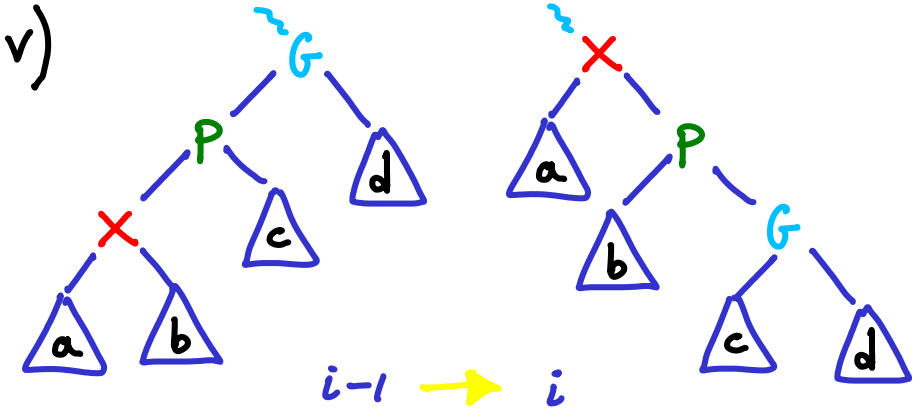

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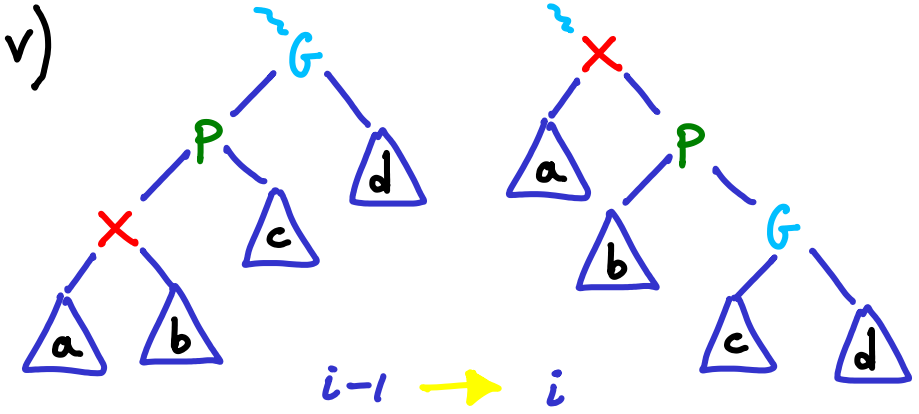

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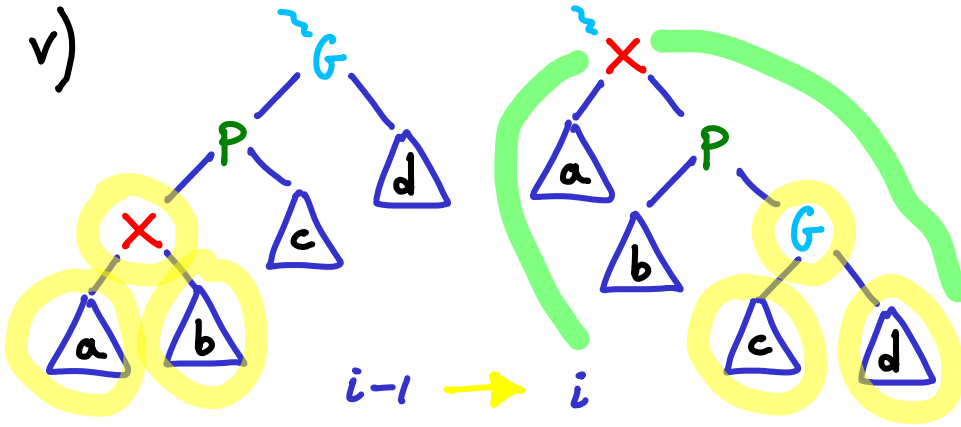

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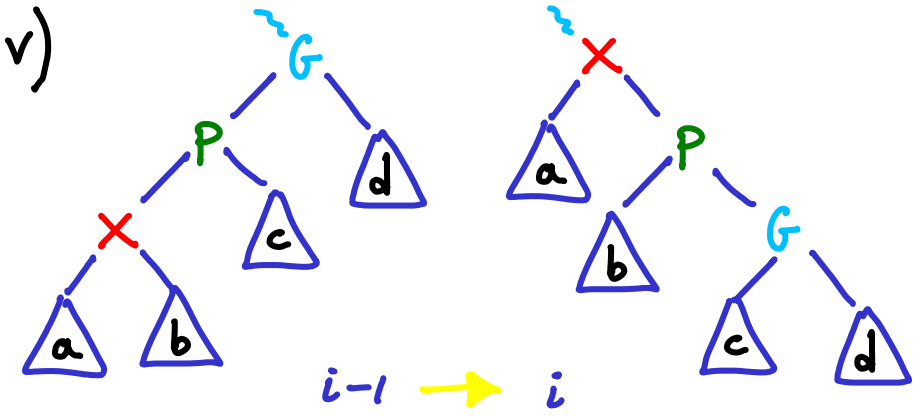

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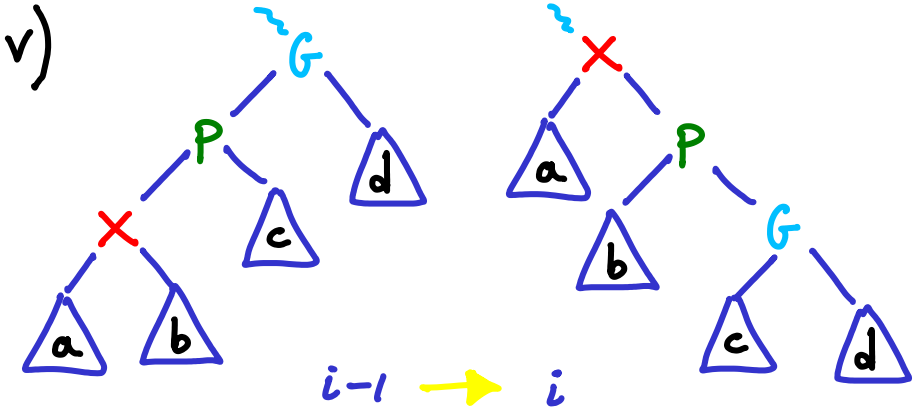

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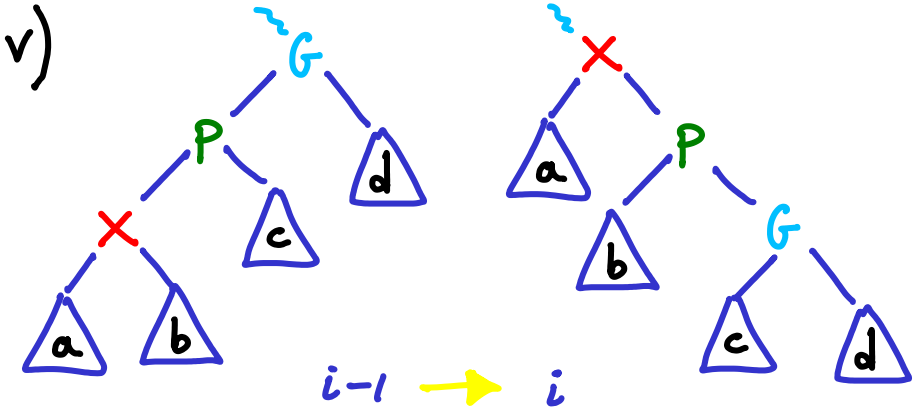

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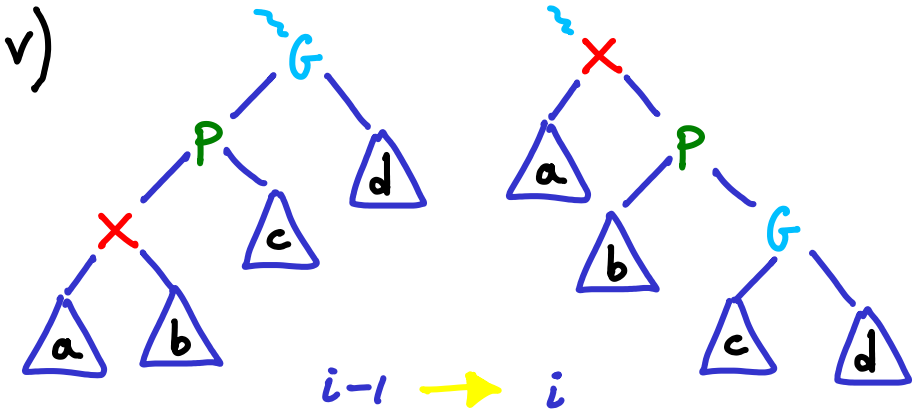

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$s_i(G) + s_{i-1}(x) < 2s_i(x) - 2$

$\Delta\Phi < [2s_i(x) - 2] + s_i(x) - 3s_{i-1}(x) = 3[s_i(x) - s_{i-1}(x)] - 2$

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

zig-zig: $\Delta\Phi < 3[s_i(x) - s_{i-1}(x)] - 2$ per step

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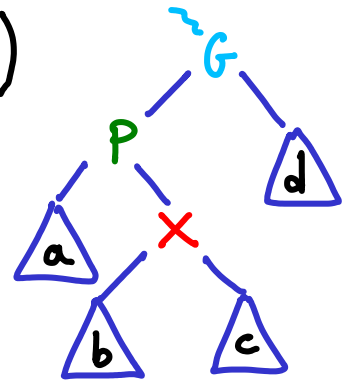

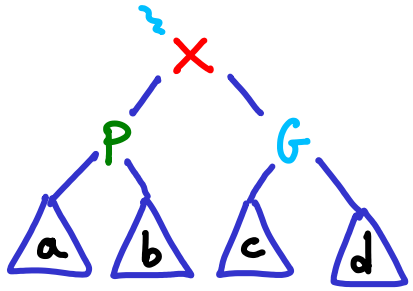

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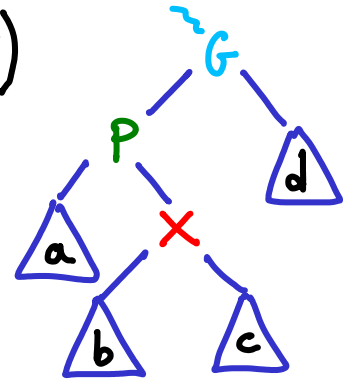

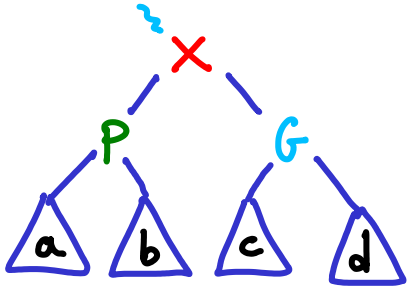

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$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(x) - s_{i-1}(x)$

$i-1 \rightarrow i$

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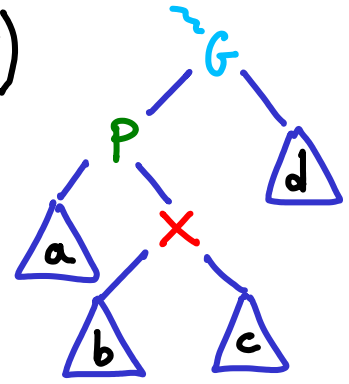

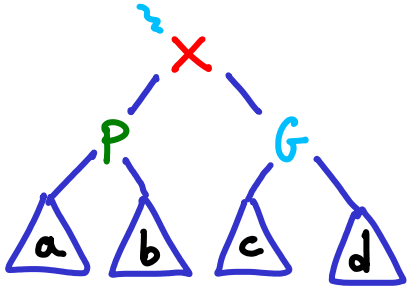

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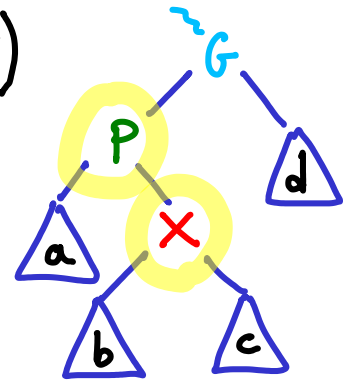

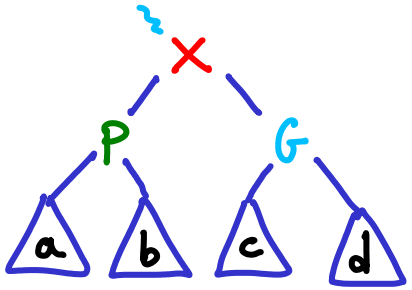

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$< s_i(G) + s_i(P) - 2\,s_{i-1}(x)$

$s_{i-1}(x) < s_{i-1}(P)$

$i-1 \rightarrow i$

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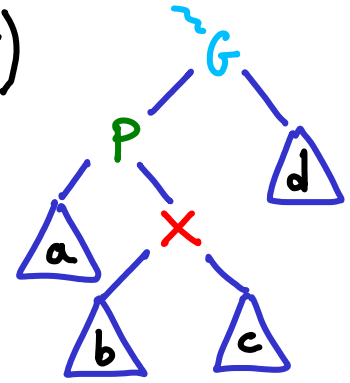

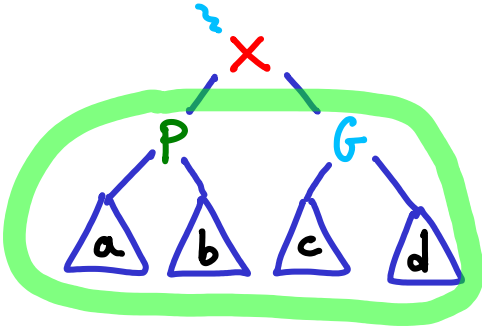

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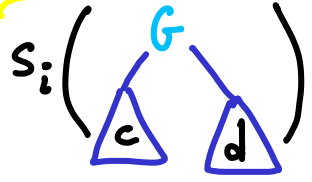
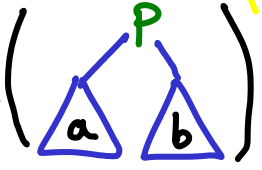
$i-1 \rightarrow i$

$s_{i-1}(x) < s_{i-1}(P)$

s_i(

) + s_i(

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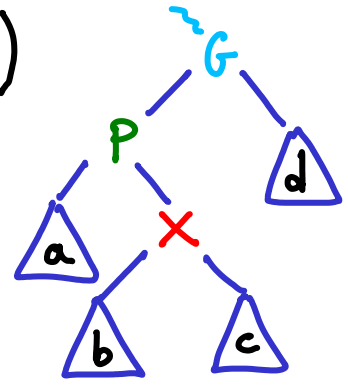

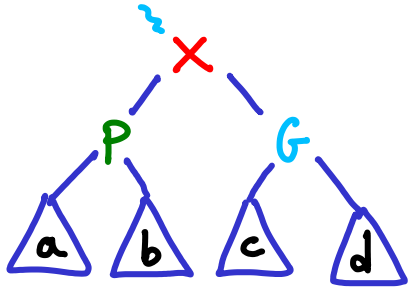

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$< s_i(G) + s_i(P) - 2\,s_{i-1}(x)$

$s_{i-1}(x) < s_{i-1}(P)$

$i-1 \to i$

s_i(G, P, c, d) + s_i(P, a, b) $\sim$ s_i(G, c, d) + s_{i-1}(X, a, b)

from zig-zig case

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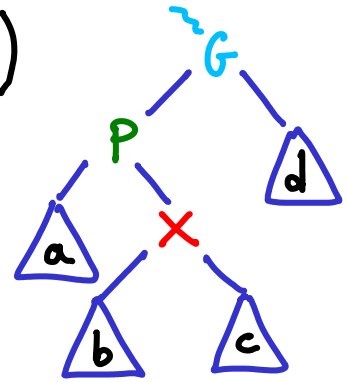

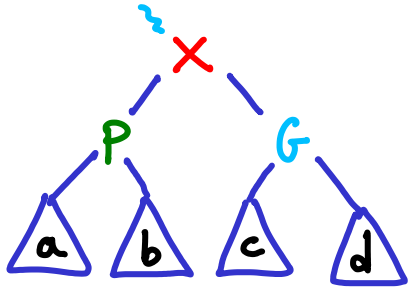

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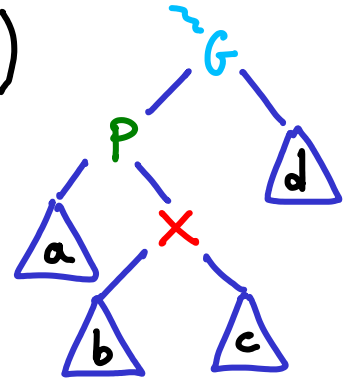
$s_{i-1}(x) < s_{i-1}(P)$

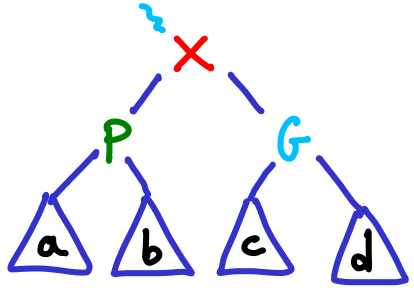
$i-1 \rightarrow i$

s_i(G: c, d) + s_i(P: a, b) $\sim s_i$(G: c, d) + s_{i-1}(X: a, b) from zig-zig case

$\Delta\Phi < 2s_i(x) - 2 - 2\,s_{i-1}(x) = 2[s_i(x)\ s_{i-1}(x)] - 2$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$





What happens when we zig-zag x ?

$\Delta\Phi = \Delta s(G) + \Delta s(P) + \Delta s(x)$

$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(x) - s_{i-1}(x)$

$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(x)$

$< s_i(G) + s_i(P) - 2\,s_{i-1}(x)$

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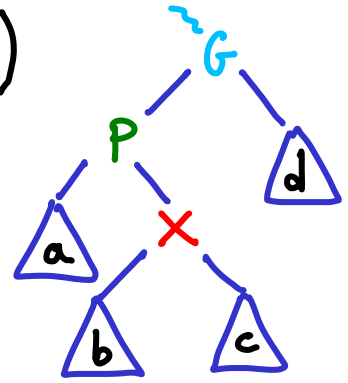

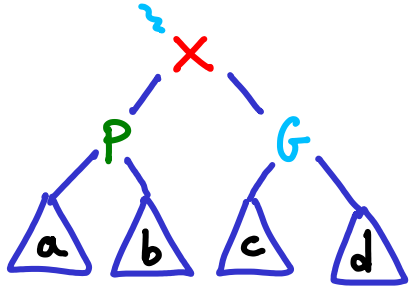

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$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(x) - s_{i-1}(x)$

$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(x)$

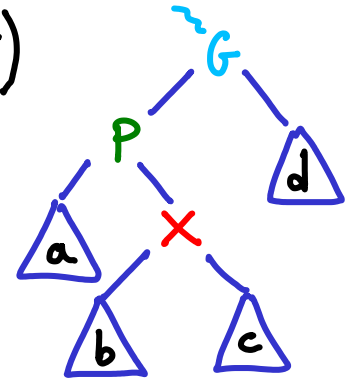
$< s_i(G) + s_i(P) - 2\,s_{i-1}(x)$

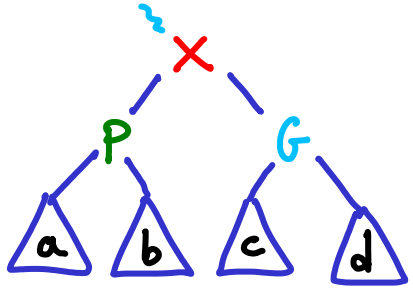
$= 2 \cdot \dfrac{\log(\text{size}_i(G)) + \log(\text{size}_i(P))}{2}$

$i-1 \rightarrow i$

$s_{i-1}(x) < s_{i-1}(P)$

$\Phi = \sum_{\text{all } v} s(v)$ // $s(v) = \log_2(\#\text{nodes in subtree of } v)$





What happens when we zig-zag x ?

$\Delta\Phi = \Delta s(G) + \Delta s(P) + \Delta s(x)$

$i-1 \rightarrow i$

$= s_i(G) - s_{i-1}(G) + s_i(P) - s_{i-1}(P) + s_i(x) - s_{i-1}(x)$

$= s_i(G) + s_i(P) - s_{i-1}(P) - s_{i-1}(x)$

$s_{i-1}(x) < s_{i-1}(P)$

$< s_i(G) + s_i(P) - 2\, s_{i-1}(x)$

$$= 2 \cdot \frac{\log(\text{size}_i(G)) + \log(\text{size}_i(P))}{2} \leq 2\log\left(\frac{\text{size}_i(G) + \text{size}_i(P)}{2}\right)$$

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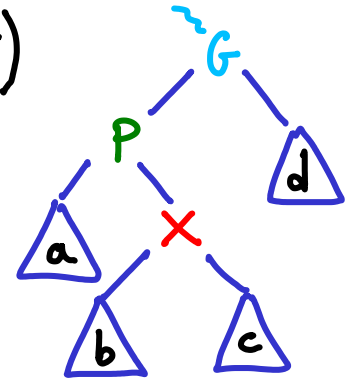

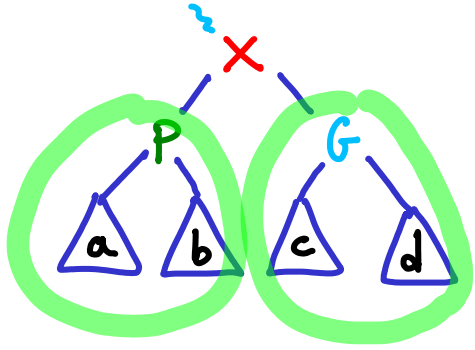

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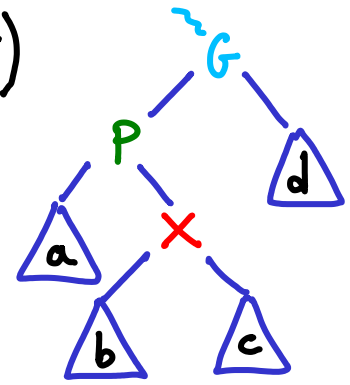

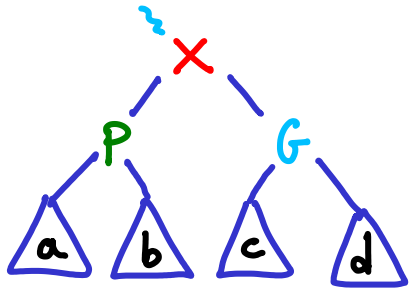

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$$= 2 \cdot \frac{\log(\text{size}_i(G)) + \log(\text{size}_i(P))}{2} \leq 2\log\left(\frac{\text{size}_i(G) + \text{size}_i(P)}{2}\right) < 2\log\left(\frac{\text{size}_i(x)}{2}\right)$$

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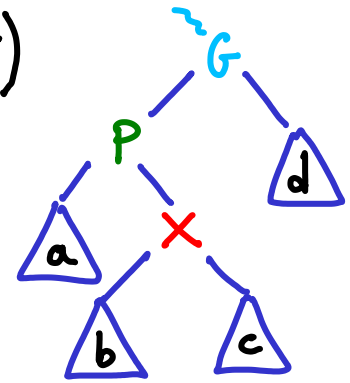

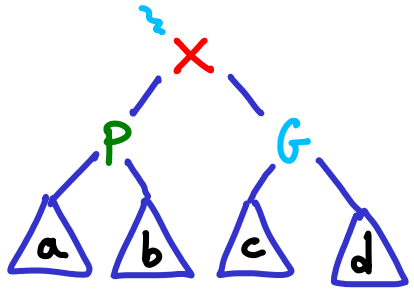

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$\Delta\Phi < 2s_i(x) - 2 - 2\,s_{i-1}(x) = 2[s_i(x) - s_{i-1}(x)] - 2$

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

zig-zig: $\Delta\Phi < 3[s_i(x) - s_{i-1}(x)] - 2$

zig-zag: $\Delta\Phi < 2[s_i(x) - s_{i-1}(x)] - 2$

} per step

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

zig-zig: $\Delta\Phi < 3[s_i(x) - s_{i-1}(x)] - 2$

zig-zag: $\Delta\Phi < 2[s_i(x) - s_{i-1}(x)] - 2$

← at most once per step

---

total $\Delta\Phi$ for one splay $< -2k + \sum 3[s_i(x) - s_{i-1}(x)]$ // k = # zig-zags & zig-zigs

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

zig-zig: $\Delta\Phi < 3[s_i(x) - s_{i-1}(x)] - 2$

zig-zag: $\Delta\Phi < 2[s_i(x) - s_{i-1}(x)] - 2$

} per step

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total $\Delta\Phi$ for one splay $< -2k + \sum 3[s_i(x) - s_{i-1}(x)]$ // k = # zig-zags & zig-zigs

telescope

$= 3[s(\text{root}) - s_0(x)] - 2k$

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

zig-zig: $\Delta\Phi < 3[s_i(x) - s_{i-1}(x)] - 2$

zig-zag: $\Delta\Phi < 2[s_i(x) - s_{i-1}(x)] - 2$

} per step

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$= 3[s(\text{root}) - s_0(x)] - 2k$

$\leq 3 \cdot s(\text{root}) - 2k$

$\leq 3 \cdot \log n - 2k$

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

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$\leq 3 \cdot s(\text{root}) - 2k$

$\leq 3 \cdot \log n - 2k$

Amortized cost $\hat{c} = c + \Delta\Phi$

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

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zig-zag: $\Delta\Phi < 2[s_i(x) - s_{i-1}(x)] - 2$

} per step

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telescope

$= 3[s(\text{root}) - s_0(x)] - 2k$

$\leq 3 \cdot s(\text{root}) - 2k$

$\leq 3 \cdot \log n - 2k$

Amortized cost $\hat{c} = c + \Delta\Phi \leq 1 + 2k + 3 \cdot \log n - 2k$

$\leq$ 1 rotation at top
+2 rotations for all other steps

no g: $\Delta\Phi < s_i(x) - s_{i-1}(x)$

zig-zig: $\Delta\Phi < 3[s_i(x) - s_{i-1}(x)] - 2$

zig-zag: $\Delta\Phi < 2[s_i(x) - s_{i-1}(x)] - 2$

} per step

---

total $\Delta\Phi$ for one splay $< -2k + \sum 3[s_i(x) - s_{i-1}(x)]$ // k = # zig-zags & zig-zigs

telescope

$= 3[s(\text{root}) - s_0(x)] - 2k$

$\leq 3 \cdot s(\text{root}) - 2k$

$\leq 3 \cdot \log n - 2k$

≤ 1 rotation at top
+2 rotations for all other steps

Amortized cost $\hat{c} = c + \Delta\Phi \leq 1 + 2k + 3 \cdot \log n - 2k = O(\log n)$

Summary: one splay (to top) costs $O(\log n)$ amortized

↳ $\Delta\Phi$ of splay pays for rotations during splay

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↳ can adjust (multiply) Φ so that splay pays for this cost too

splay path = search path

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Amortized cost $\sum_{j=1}^{m} \hat{c}_j$ of m operations on n elements $= O(m \log n)$

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If tree doesn't start empty,

each node could contribute $\leq \log n$ to Φ_0

could be $\lceil n \log n$

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If tree doesn't start empty,

each node could contribute $\leq \log n$ to $\Phi_n - \Phi_0$

(could be zero) (could be $n \log n$)

Summary: one splay (to top) costs $O(\log n)$ amortized

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(could be zero; could be $n \log n$)

$$\sum_{j=1}^{m} \hat{c}_j = \sum_{j=1}^{m} c + \Phi_n - \Phi_0$$

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↳ $\Delta\Phi$ of splay pays for rotations during splay

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If tree doesn't start empty,

each node could contribute $\leq \log n$ to $\Phi_n - \Phi_0$ so

(could be zero; could be $n \log n$)

$$\sum_{j=1}^{m} \hat{c}_j = \sum_{j=1}^{m} c + \Phi_n - \Phi_0 \Rightarrow \sum_{j=1}^{m} c = \sum_{j=1}^{m} \hat{c}_j + \Phi_0 - \Phi_n = O(m \log n) + O(n \log n)$$