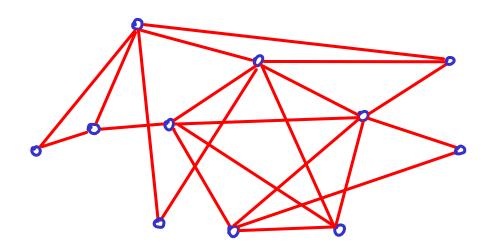
Game for 2 players:

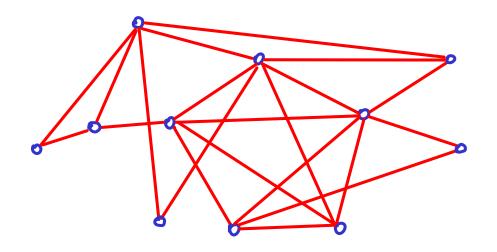
draw a complete graph, each player can draw edges using one color, take turns coloring one edge at a time.

Whoever completes a triangle first wins.

Is there always a winner?

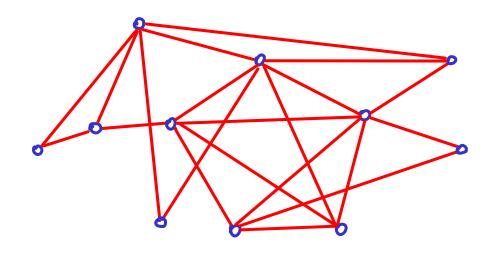


Given G, a subset S of V(G) is a clique if every $s_i, s_j \in S$ share an edge in G.



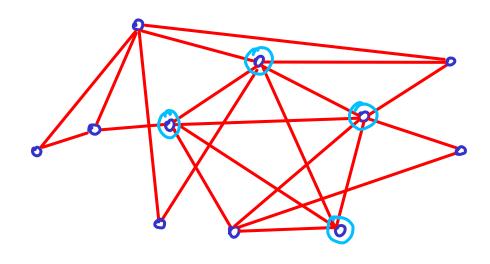
Given G, a subset S of V(G) is a clique if every $S_i, S_j \in S$ share an edge in G.

The induced subgraph obtained by removing all but S from V(G) is a complete graph (KS)



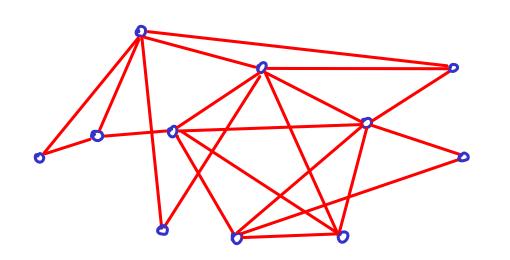
How many cliques? What is the largest clique? Given G, a subset S of V(G) is a clique if every $s_i, s_j \in S$ share an edge in G.

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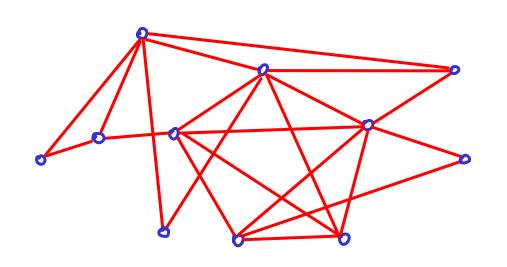


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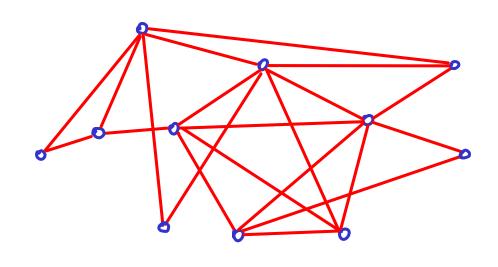


Given G, a subset S of V(G)is an independent set if no $s_i, s_i \in S$ share an edge in G.



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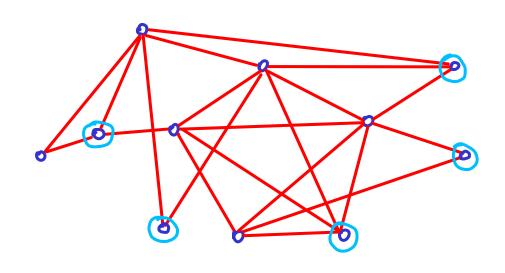
The induced subgraph obtained by removing all but S from V(G) is an edgeless graph.



Largest independent set?

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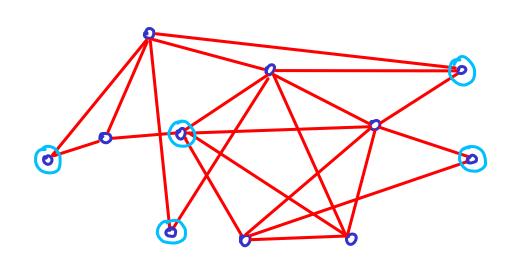
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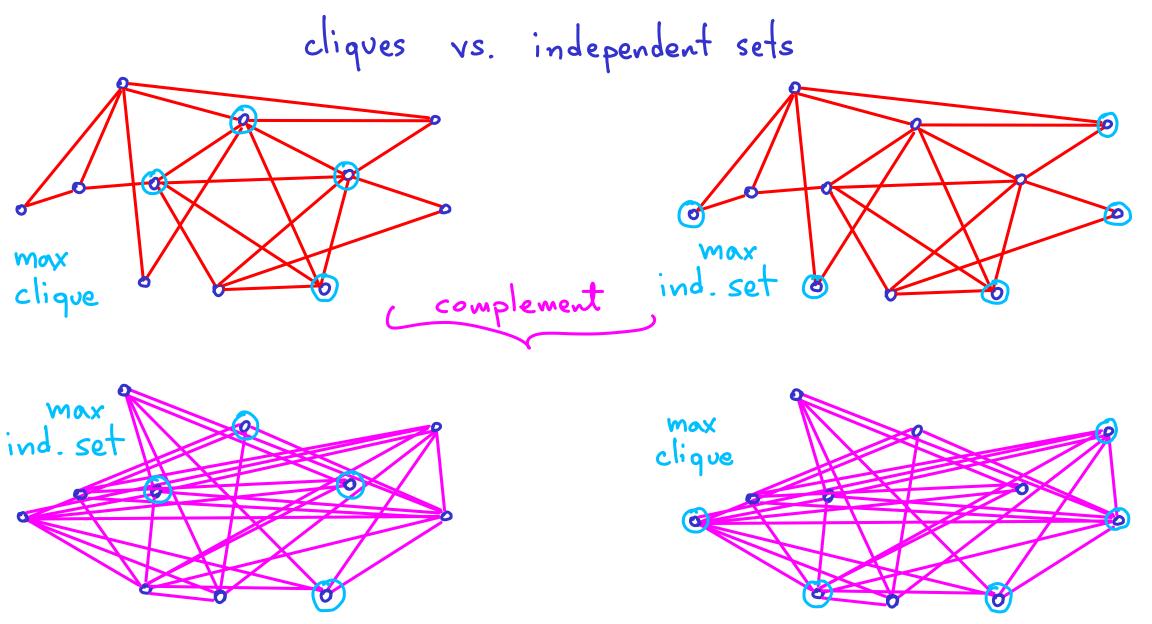
Largest independent set?

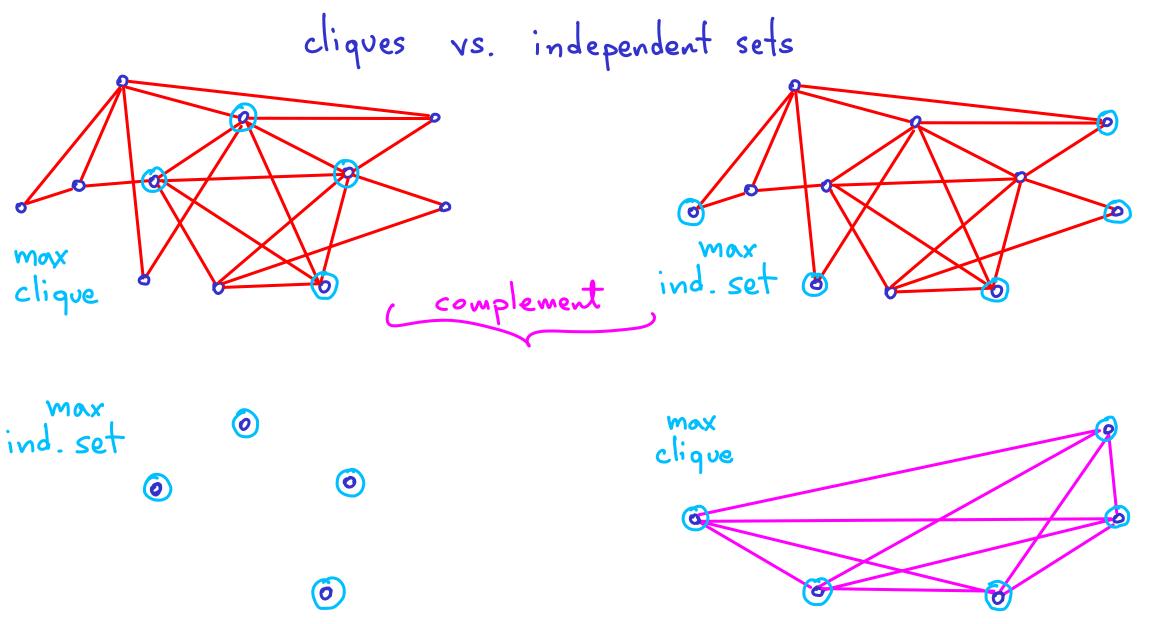


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The induced subgraph obtained by removing all but S from V(G) is an edgeless graph.

i.e. its complement is a complete graph





Claim: Every graph with |V| > 6 contains a triangle (clique of size 3) OR an independent set of size 3 Claim: Every graph with |V| > 6 contains a triangle (clique of size 3) OR an independent set of size 3 D 1 (0 1/1) a graph contains a triangle

Rephrase: (for 1>6) a graph contains a triangle or its complement does

Claim: Every graph with 14/26 contains a triangle (clique of size 3) OR an independent set of size 3 a graph contains a triangle or its complement does Rephrase: (for 1>6)

Proof: pick any vertex v. If d(v) ≥3 we have v Claim: Every graph with 11/26 contains a triangle (clique of size 3) OR an independent set of size 3 Rephrase: (for 1>6)

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Proof: pick any vertex v. If $d(v) \geqslant 3$ we have $v \rightleftharpoons v \end{Bmatrix}$ If xy or xz or yz:

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Proof: pick any vertex v.

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If $d(v) \le 2$, there are 3 vertices not neighboring 4. 3

Proof: pick any vertex v.

If $d(v) \geqslant 3$ we have $v \rightleftharpoons v \rightleftharpoons v \end{Bmatrix}$ Otherwise, x,y,z are an independent set.

If $d(v) \le 2$, there are $\gg 3$ vertices not neighboring $v. \rightarrow v.$ If ab, bc, ac are edges ...?

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Claim: Every graph with 14/26 contains a triangle (clique of size 3) OR an independent set of size 3 a graph contains a triangle or its complement does Rephrase: (for 1>6) Proof: pick any vertex v. If $d(v) \geqslant 3$ we have $\bigvee_{z} \int f(xy)$ or (xz) or (yz); we find a clique Δ or $(xz) \int f(xy)$ or $(xz) \int f(xy)$

Otherwise, x,y,z are an independent $d(v) \le 2$, there are $\gg 3$ vertices not neighboring $v. \rightarrow v.$ be all ab, bc, ac are edges, they are a clique Δ .
Otherwise ?

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If $d(v) \geqslant 3$ we have $v \approx \frac{x}{z}$ Otherwise, x,y,z are an independent set.

Otherwise, x,y,z are an independent $d(v) \le 2$, there are >3 vertices not neighboring $v. \rightarrow v.$ It at, bc, ac are edges, they are a clique Δ .

Otherwise one edge is missing (w.l.o.g. ab) ...

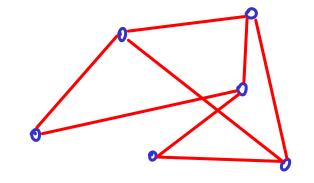
Proof: pick any vertex v.

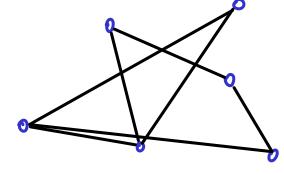
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Otherwise, x,y,z are an independent set $d(v) \le 2$, there are $\gg 3$ vertices not neighboring $v \to v$.

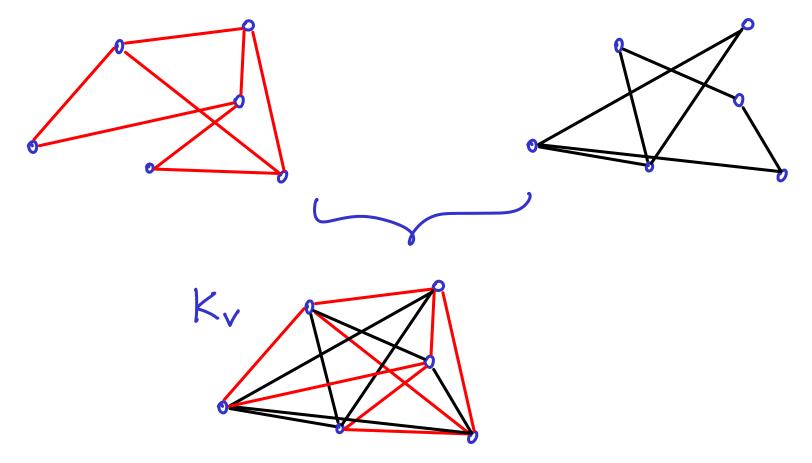
If at, bc, ac are edges, they are a clique A.
Otherwise one edge is missing (w.l.o.g. ab) ... so vab is an ind, set,

(for 1>6) a graph contains a triangle or its complement does

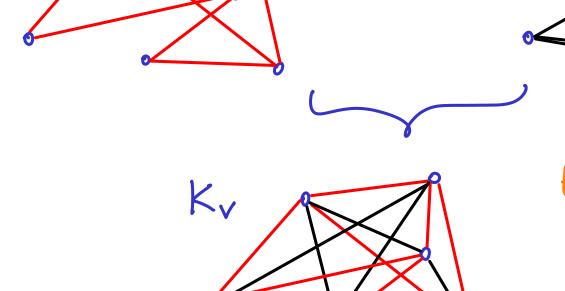




(for 1)6) a graph contains a triangle or its complement does



(for 1>6) a graph contains a triangle or its complement does



Equivalent statement

If we color each edge of Kv

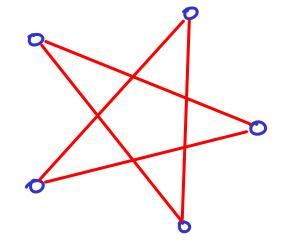
red or black, then we must get
a red triangle or
a black triangle

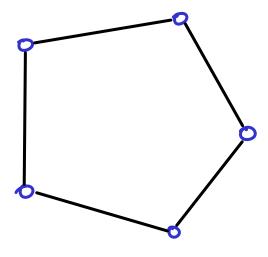
Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as IVI>6

|V| < 6 ?

Recap: if you want a clique or an independent set of size 3

then you'll be happy as long as IVI >6





Recap: if you want a clique or an independent set of size 3

then you'll be happy as long as IVI>6

+

(one direction to be shown)

Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as IVI>6

$$R(4,4)$$
 $R(5,5)$

We don't know.

[43...49]

Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as IVI>6 > R(3,3) = 6R(n,n)R(S,S)R(4,4) we don't know.
[43...49]

~ exponential even for small values we will probably never know the exact answer Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as IVI>6 > R(3,3) = 6R(n,n)R(5,5) R(4,4) ~ exponential

we don't know!
[43...49]

even for small values
we will probably never
know the exact answer

For more, see Ramsey's Thm.

R(x,y): smallest number N such that any graph with >N vertices has a clique of size x or an independent set of size y

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$$R(4,2) = ?$$

R(x,y): smallest number N such that any graph with >N vertices has a clique of size x or an independent set of size y

Suppose |VI > 10

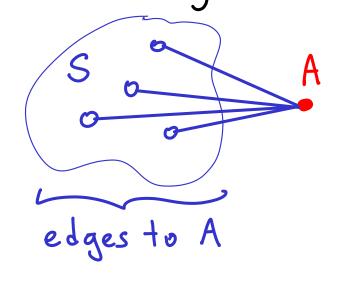
Pick any vertex, A. >9 vertices remain.

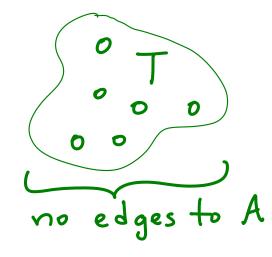
A

Suppose |VI > 10

Pick any vertex, A. >9 vertices remain.

Form 2 groups: S & T





Suppose |VI > 10 R(4,3)>9 vertices remain. Pick any vertex, A. Form 2 groups: S & T

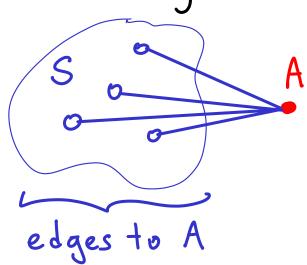
edges to A

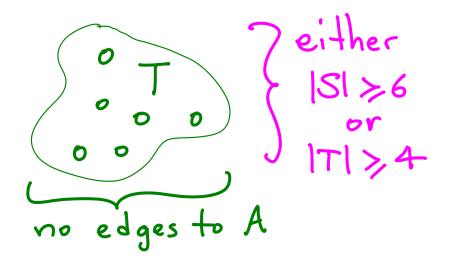
no edges to A

R(4,3) Sick of

Suppose |VI > 10

Pick any vertex, A. >9 vertices remain.



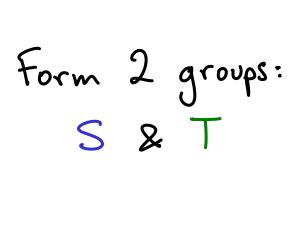


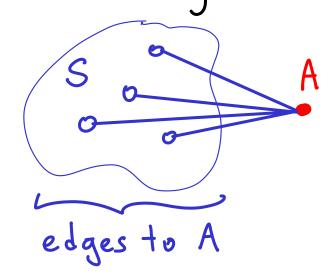
1f |S| >6, ...?

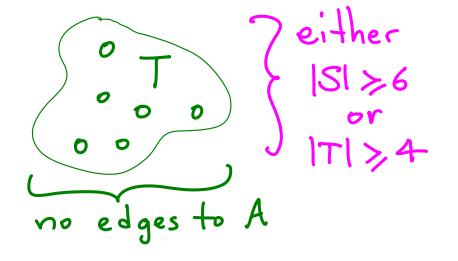
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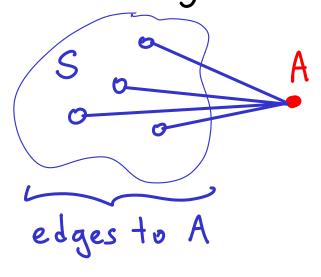
If |S| > 6, use R(3,3) = 6 ...?

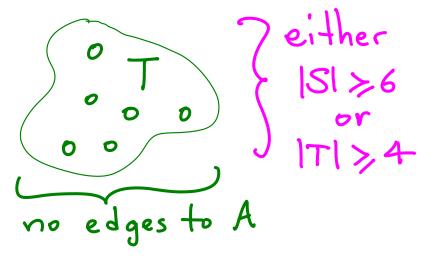
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Sold A of SISI > 6

Or

ITI > 4 Form 2 groups: S & T no edges to A edges to A If |S| > 6, use R(3,3) = 6: S has 3 independent vertices (done), or 5 has a 3-clique, so with A we get a 4-clique.

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Sold A OT SISI > 6

OT SISI > 6

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Suppose VI > 10

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Otherwise 3 a,b in T w/ no edge. Combine w/ A.

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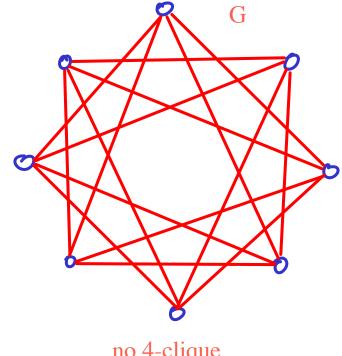
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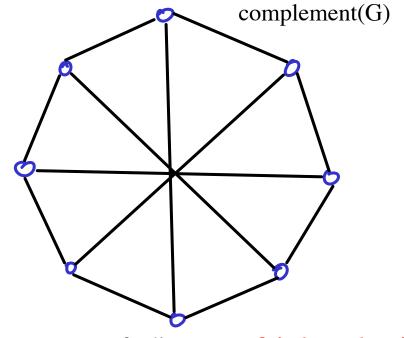
 $R(4,3) \leq 10$

Otherwise 3 a,b in T v/ no edge. Combine w/ A.

$$R(4,3) \le 10$$



no 4-clique



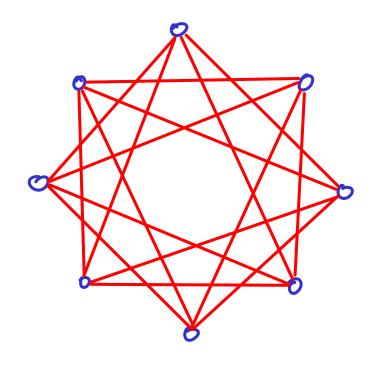
no 3-clique = no 3-independent in G

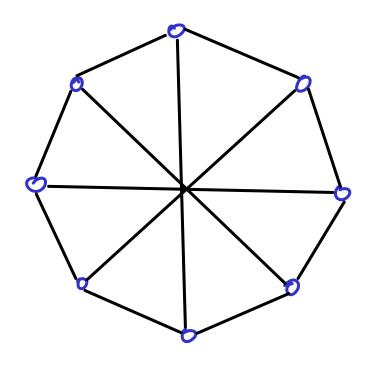
$$R(4,3) \le 10$$

R(4,3) > 8

... turns out R(4,3) = 9

4) not terribly hard 4) notice R(x,y) = R(y,x)





R(4,4)

R(4,4)

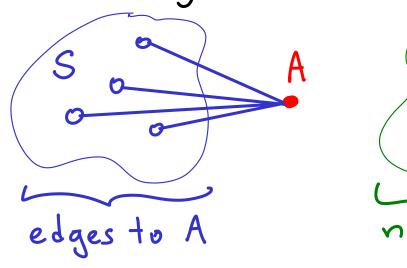
Suppose |VI > 18

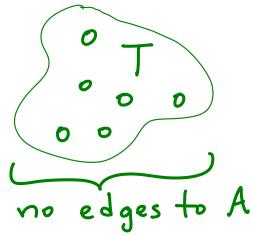
Pick any vertex, A. >17 vertices remain.

A

R(4,4)Form 2 groups: S & T Suppose |VI > 18

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Suppose |VI > 18 R(4,4)Pick any vertex, A. >17 vertices remain. Form 2 groups: S & T no edges to A edges to A

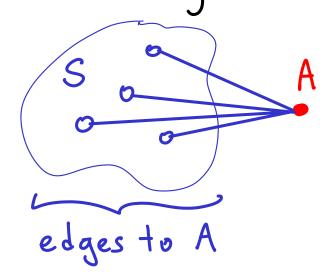
1f |S| >9 3

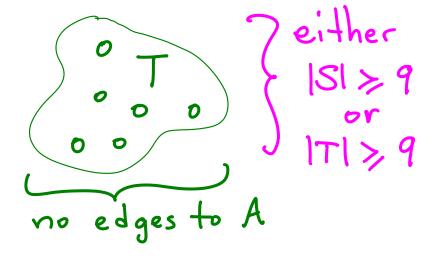
R(4,4)

Suppose |VI > 18

Pick any vertex, A. ≥17 vertices remain.

Form 2 groups: S&T





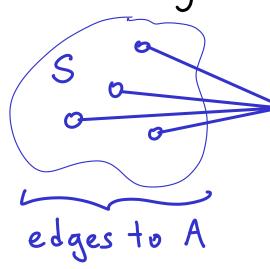
If |S| > 9, use R(3,4) = 9 ...?

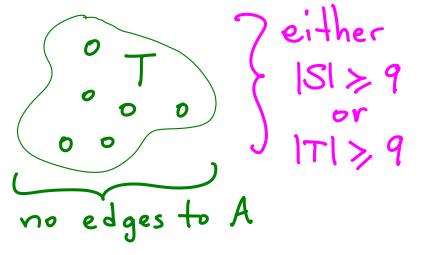
R(4,4)

Form 2 groups: S & T

Suppose |V| > 18

Pick any vertex, A. >17 vertices remain.





If |S| > 9, use R(3,4) = 9: S has 4 independent vertices or S has a 3-clique?

Solar Sither | Six 9 | Six 9 | Or | ITI > 9 Form 2 groups: S & T no edges to A edges to A If |S| > 9, use R(3,4) = 9: S has 4 independent vertices (done) or S has a 3-clique, so with A we get a 4-clique.

Suppose VI > 18

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R(4,4)

either ISI > 9

or

ITI > 9 Form 2 groups: S & T no edges to A edges to A If |S| > 9, use R(3,4) = 9: S has 4 independent vertices (done) or S has a 3-clique, so with A we get a 4-clique. 1f |T(>9

Suppose |V| > 18

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>17 vertices remain.

R(4,4)

Sold A of Sland Sl Form 2 groups: S & T no edges to A edges to A or S has a 3-clique, so with A we get a 4-clique. If |S| > 9, use R(3,4) = 9

Suppose VI > 18

Pick any vertex, A.

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R(4,4)

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Thas 3 independent vertices ...?

R(4,4)

Suppose VI > 18

Pick any vertex, A. >17 vertices remain. Solar Slaver Sla Form 2 groups: S & T no edges to A edges to A If |S| > 9, use R(3,4) = 9: S has 4 independent vertices (done) or S has a 3-clique, so with A we get a 4-clique. If |T| > 9, use R(4,3) = 9: T has a 4-clique, (done) or

R(4,4)

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Thas 3 independent vertices, so with A we have 4.

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 $R(4,4) \leq 18$

Suppose |V| > 18

Thas 3 independent vertices, so with A we have 4.

Notes: (i) if we only knew that $R(4,3) \le 10$ (instead of =9)

then ... ?

Notes:

(i) if we only knew that
$$R(4,3) \le 10$$
 (instead of = 9) we could have used $|V| > 20$ for $R(4,4)$

As you bound smaller R() values, you can get (loose) bounds for larger ones

Notes:

(1) if we only knew that
$$R(4,3) \le 10$$
 (instead of = 9) we could have used $|V| > 20$ for $R(4,4)$

(2) there is a graph
$$w/17$$
 vertices with no clique or independent set of size 4 $R(4,4) = 18$